Problem set 4

Due Wednesday, February 22

1. Let \( \{ e^{(i)}_\alpha \} \) and \( \{ e^{(i)}_{-\alpha} \} \) be dual bases of \( g_\alpha \) and \( g_{-\alpha} \), that is, suppose
\[
(e^{(i)}_\alpha, e^{(j)}_{-\alpha}) = \delta_{ij},
\]
with respect to the invariant symmetric form on a symmetrizable Kac-Moody Lie algebra \( g \). Show that
\[
\sum_i e^{(i)}_{-\alpha} \otimes [x, e^{(i)}_\alpha] = \sum_i [x, e^{(i)}_{-\beta}] e^{(i)}_\beta
\]
for any \( x \in g_{\beta-\alpha} \).

2. In the notation of Problem 1, show that
\[
\sum_i [e^{(i)}_{-\alpha}, [x, e^{(i)}_\alpha]] = \sum_i [x, e^{(i)}_{-\beta}] e^{(i)}_\beta \in g_{\beta-\alpha}
\]
and, similarly,
\[
\sum_i e^{(i)}_{-\alpha} [x, e^{(i)}_\alpha] = \sum_i [x, e^{(i)}_{-\beta}] e^{(i)}_\beta \in \mathcal{U}(g)_{\beta-\alpha}.
\]

3. Let \( M \) be a \( g_\alpha \)-module such that for any \( m \in M \),
\[
g_\alpha m = 0
\]
for all but finitely many positive \( \alpha \). (For example, if \( h \)-action on \( M \) is diagonalizable and all weights are bounded from above.) Show that the operator
\[
\Omega' = 2 \sum_{\alpha>0,i} e^{(i)}_{-\alpha} e^{(i)}_\alpha + \sum_i e^{(i)}_0 e^{(i)}_0
\]
is well-defined on \( M \) and commutes with \( h \).

4. Let \( M \) be as above and let \( \rho \in h \) be such that
\[
(\rho, h_i) = \frac{1}{2} a_{ii}.
\]
Show that the operator
\[ \Omega = \rho + \Omega' \]
commutes with the action of \( g \).

5. Let \( M \) be a Verma module for \( g \), that is a module freely generated by a vector \( |\lambda\rangle, \lambda \in \mathfrak{h}^* \), such that
\[ h|\lambda\rangle = \lambda(h)|\lambda\rangle \]
and
\[ e_i|\lambda\rangle = 0. \]
Show that \( \Omega \) acts by a scalar operator in \( M \) and compute that scalar.