Problem set 3

Due Wednesday, February 15

Let a vector space
$$V = \bigoplus_{d \in \mathbb{Z}^n} V_d$$
be graded by elements of $\mathbb{Z}^n$ with subspaces V_d of finite dimension. We define its Hilbert (or Poincaré) series by
$$h_V(x_1, \dots, x_n) = \sum_d x^d \dim V_d \,,$$
where $x^d = x_1^{d_1} \cdots x_n^{d_n}$.

1. Show that
$$h_{V_1 \oplus V_2} = h_{V_1} + h_{V_2}$$
and
$$h_{V_1 \otimes V_2} = h_{V_1} h_{V_2} \,.$$

2. If V is an algebra of polynomials and $\delta_1, \delta_2, \dots$ are the degrees of the generators then
$$h_{\text{polynomials}} = \prod \frac{1}{1 - x^{\delta_i}} \,,$$
while if V is a free associative algebra with generators of degree δ_i then
$$h_{\text{free associative}} = \frac{1}{1 - \sum x^{\delta_i}} \,.$$

3. If SV is a symmetric algebra of V, that is, an algebra of polynomials in a basis of V then
$$h_{SV} = \exp \sum_n \frac{h_V(x^n)}{n} \,,$$
where $h_V(x^n) = h_V(x_1^n, x_2^n, \dots)$. If V is a Lie algebra and $\mathcal{U}V$ is its universal enveloping Lie algebra then
$$h_{\mathcal{U}V} = h_{SV} \,.$$

4. Prove that the inverse of the relation in Problem 3 is

$$h_V = \sum_n \frac{\mu(n)}{n} \ln h_{SV}(x^n)$$

where $\mu(n)$ is the Möbius function, see wikipedia.

5. Prove that the universal enveloping of a free Lie algebra on some number of generators is the free associative Lie algebra on the same generators.

6. Let $F(n)$ be the free Lie algebra with generators of degree

$$\delta_i = (\dots, 0, 1, 0, \dots) \in \mathbb{Z}^n \,.$$

Prove that

$$\dim F(n)_d = \frac{1}{D} \sum_m \mu(m) \binom{D/m}{d_1/m, d_2/m, \dots} , \quad D = \sum d_i \,,$$

which is a very old theorem of Ernst Witt.

7. Let $\mathfrak{g}$ be Kac-Moody Lie algebra and let α_i, α_j be two simple roots of $\mathfrak{g}$. Prove that

$$\dim \mathfrak{g}_\alpha \leq 1$$

for

$$\alpha \in \{\alpha_i + m\alpha_i, 2\alpha_i + 2\alpha_j\} \,,$$

while

$$\dim \mathfrak{g}_{2\alpha_i+3\alpha_j} \leq 2 \,.$$

8. For which Cartan matrices do we have

$$\dim \mathfrak{g}_{2\alpha_i+3\alpha_j} < 2\,?$$