Alternating Sign Matrices and their various integrabilities

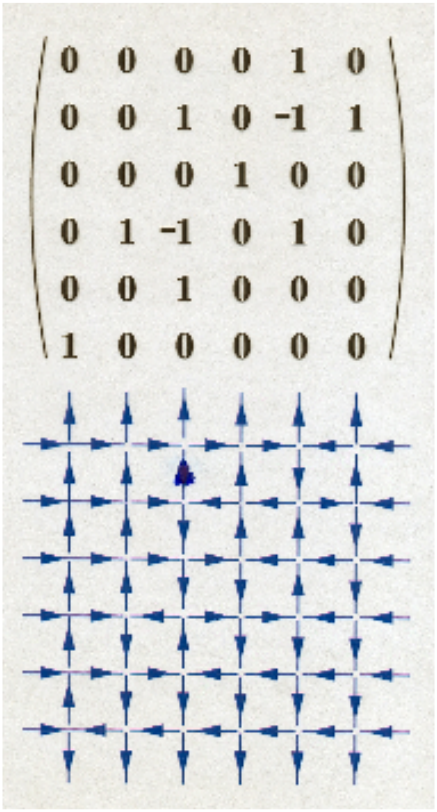
(PDF + Roger Behrend + Paul Zinn-Justin + E. Gutter + C. Kristjansen)

- Quantum Integrability
- Descending Plane Partitions
- Alternating Sign Matrices

identity between refined enumerations

$$\det(DPP) = \det(ASM)$$

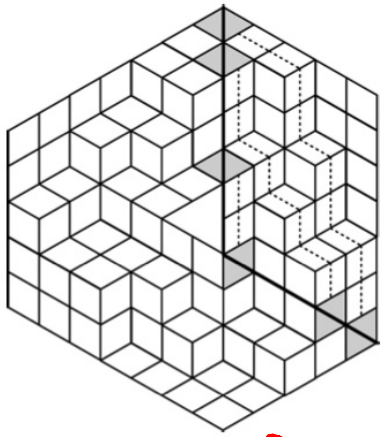
6V, ASM, [redacted], [redacted] and LOOP MODEL



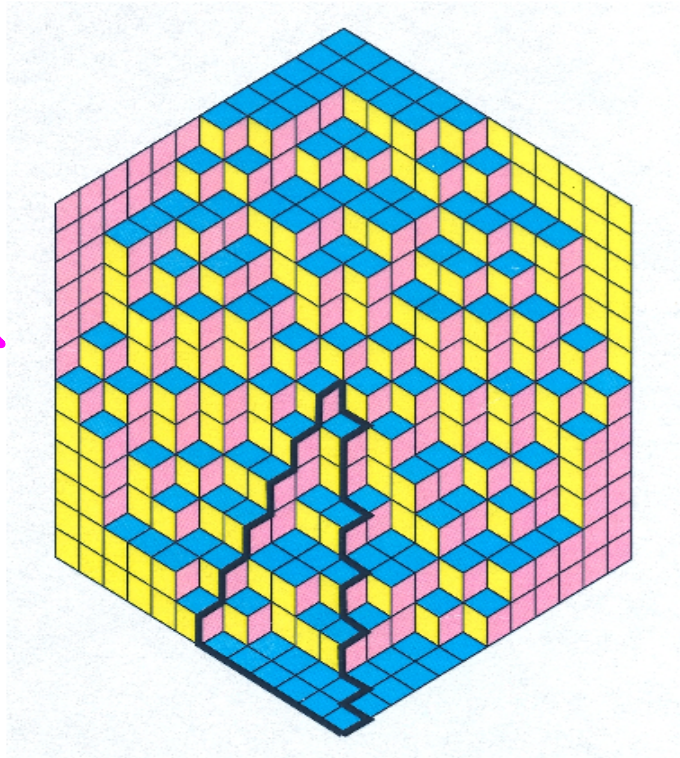
Alternating sign matrices

six vertex model

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

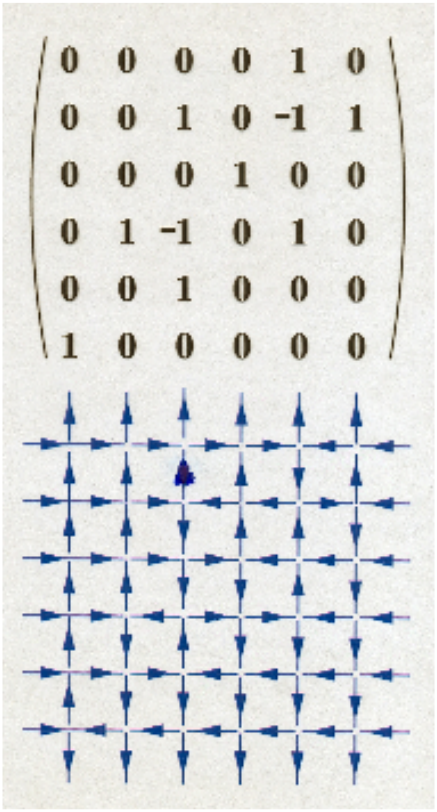


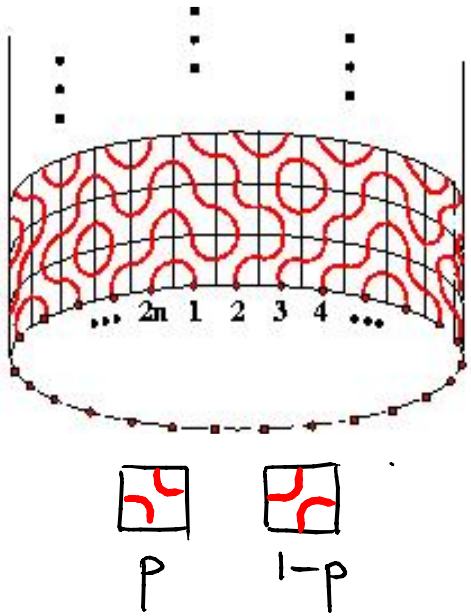
Descending Plane partitions

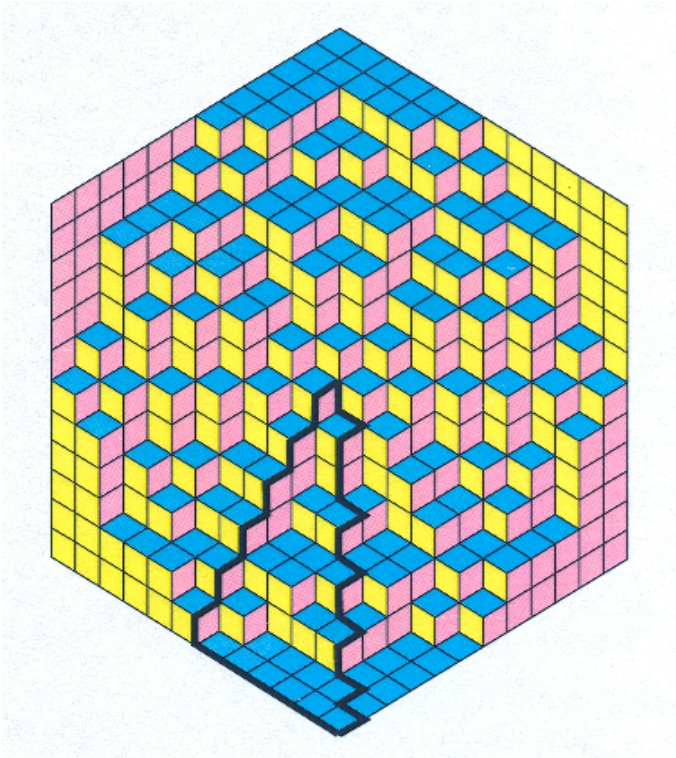


Totally Symmetric Self-Complementary Plane Partition

6V, ASM, [redacted], [redacted] and [redacted]







Alternating sign matrices (82)

. six vertex model

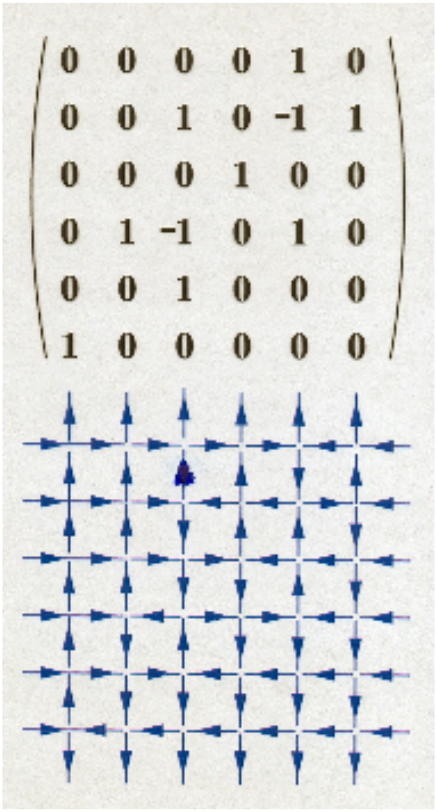
loop model

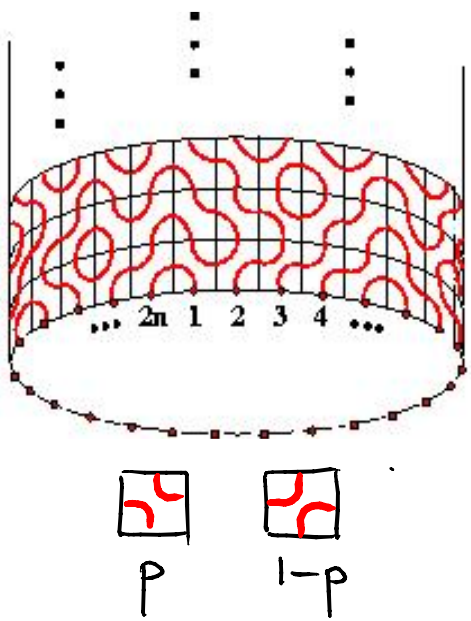
q KZ solution

[Nienhuis de Gier Razumov Stroganov DF+Zinn-Justin]

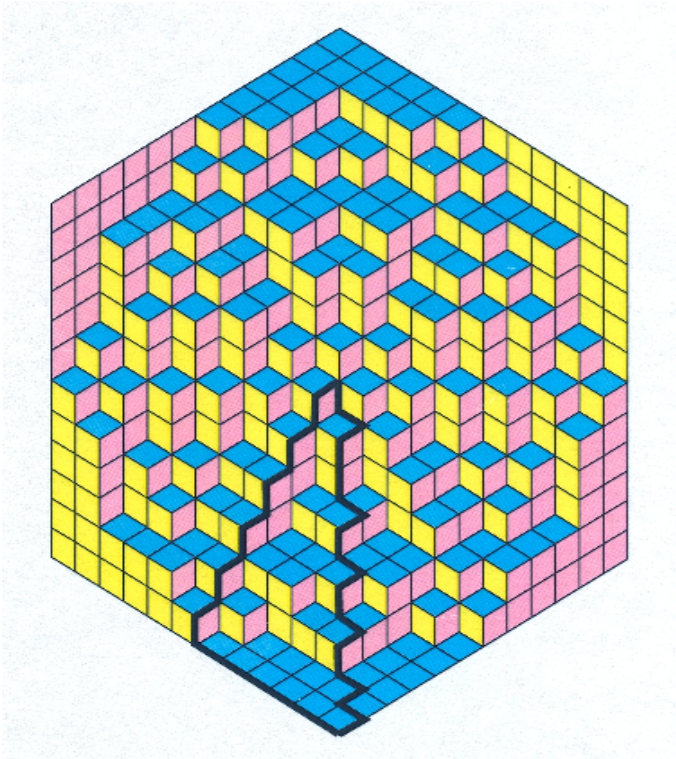
Totally Symmetric Self-Complementary Plane Partition (70)

6V, ASM, [redacted], [redacted] and [redacted]





Loop model



$V = \{ M \in \mathcal{M}_{2n}(\mathbb{C}),\ M^2 = 0$ and M strictly upper triangular$\}$

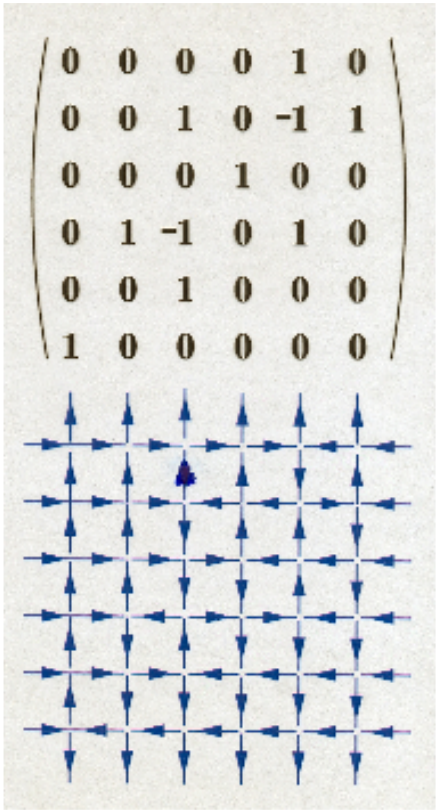
[DF-Zinn Justin 04-08]

Alternating sign matrices (82)

six vertex model

Totally Symmetric Self-Complementary Plane Partition (70)

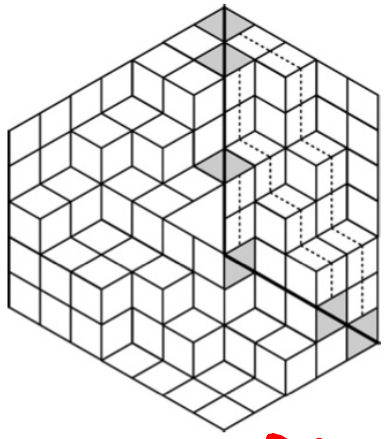
6V, ASM



Alternating sign matrices (82)

six vertex model

DPP



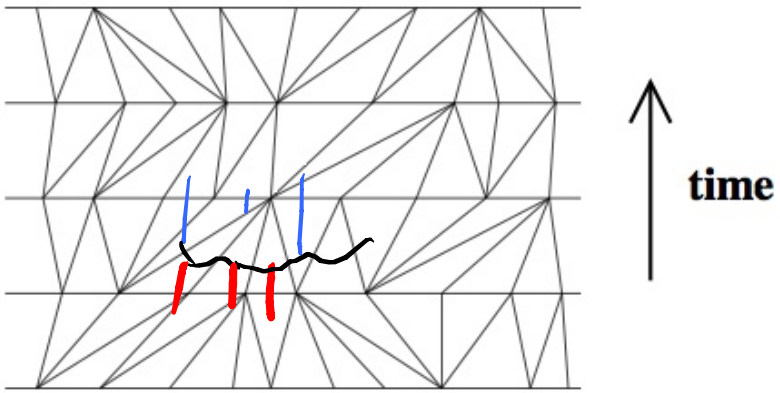
Descending Plane partitions

[DF Behrend Zinn Justin '11-12]

Refined Enumerations Coincide (Proof)

PREAMBLE : 1+1 Dimensional Lorentzian quantum gravity

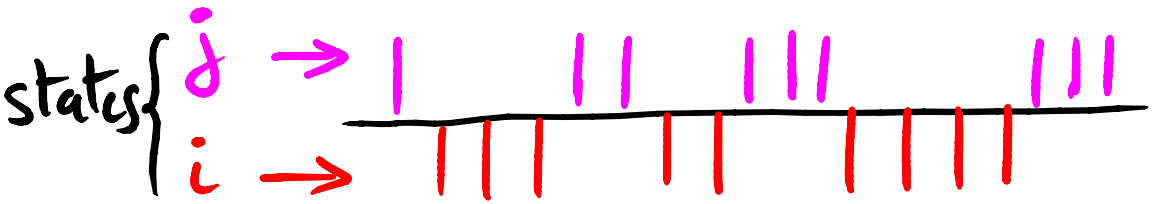
(PDF + Emmanuel Guilter + Charlotte Kristjansen '99)



Triangulations that are

1. Random in space direction
2. Regular in time direction

$\Rightarrow$ TRANSFER MATRIX



$$T_{ij} = \binom{i+j}{i}$$

$(i, j \in \mathbb{Z}_+)$

Include { curvature weight a / ⫫ or ⊓
area weight g / ⊥ or ⊤

Then

$$T_{i,j}(g,a) = (ag)^{i+j} \sum_{k=0}^{Min(i,j)} \binom{i}{k}\binom{j}{k} a^{-2k}$$

Generating Function

$$\sum_{i,j\geq 0} z^i w^j T_{i,j}(g,a) = \frac{1}{1-ga(z+w)-g^2(1-a^2)zw}$$

INFINITE MATRIX !



INTEGRABILITY

$$\left[T(g,a) \, , \, T(g',a') \right] = 0$$

$$\Leftrightarrow \quad \varphi(g,a) = \varphi(g',a')$$

$$\varphi(g,a) = \frac{1 - g^2(1-a^2)}{ag} \qquad (= q + q^{-1})$$

- diagonalization of T
- correlation functions

DIAGONALIZATION of $T(g,a)$

$$\frac{1-g^2(1-a^2)}{ga} = q+q^{-1} \Rightarrow a = \frac{q}{1-q^2}\,\frac{1-\lambda}{\sqrt{\lambda}}\,;\; g = \sqrt{\lambda}\,\frac{1-q^2}{1-\lambda q^2}$$

$$\sum_{i,j\geq 0} T(g,a)_{ij}\, z^i w^j = \frac{1-\lambda q^2}{(1-qz)(1-qw)-\lambda(q-z)(q-w)}$$

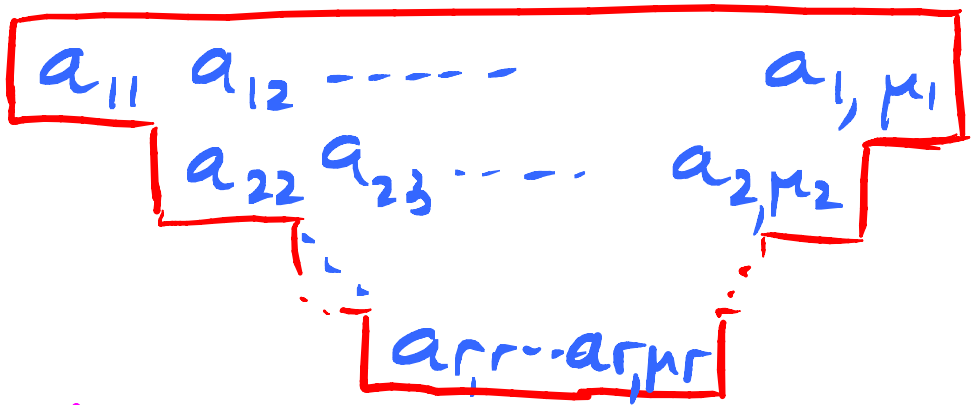
$$= \sum_{m\geq 0} \sqrt{1-q^2}\,\frac{(q-z)^m}{(1-qz)^{m+1}}\;\underbrace{\frac{1-\lambda q^2}{1-q^2}}\;\lambda^m\;\sqrt{1-q^2}\,\frac{(q-w)^m}{(1-qw)^{m+1}}$$

eigenvector $\sum_{j\geq 0} z^j\, v_j^{(m)}$ — eigenvalue λ_m — eigenvector $\sum_{i\geq 0} w^i\, v_i^{(m)}$

$$\Longleftrightarrow \boxed{T(g,a)\, v^{(m)} = \lambda_m\, v^{(m)}}$$

END OF PREAMBLE

DPP = Arrays of positive integers



1. $a \geq b$

2. a over b

3. $\lambda_i = \mu_i - i + 1$ = # parts in raw i

$\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary

- a_{ij} = part
- $a_{ij} \leq j-i$ = special part
- order n = $a_{ij} \leq n$ $\forall ij$.

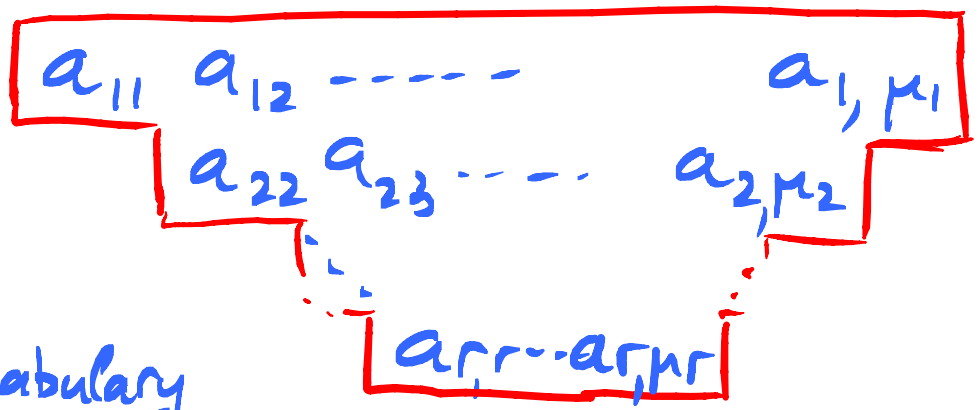
OBSERVABLES

parts = n
special parts
non-special parts

$n=3$ — 7 DPP's

$\varnothing$ | 2 | 3 | 33 | 32 | 31 | 33 / 2

DPP = Arrays of positive integers



1. $a \geq b$
2. a over b
3. $\lambda_i = \mu_i - i + 1$ = # parts in raw i

$\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary
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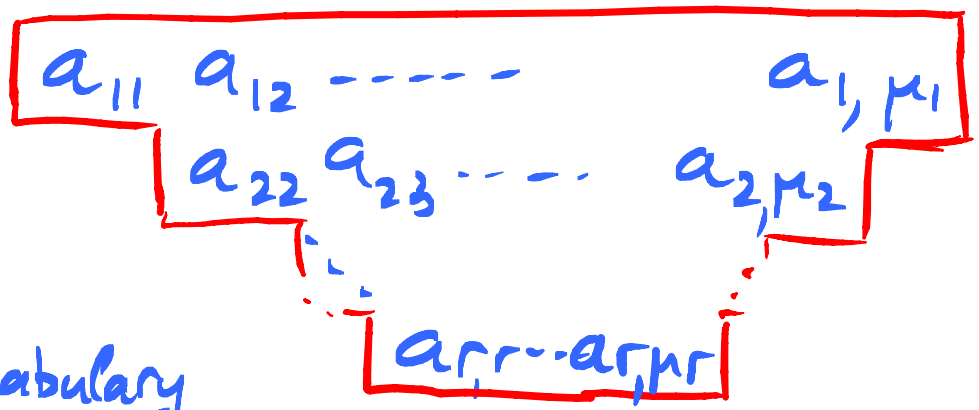
OBSERVABLES

parts = n → (3)
special parts
non-special parts

$n=3$: 7 DPP's

∅	2	3	33	32	31	33 / 2
0	0	1	2	1	1	2

DPP = Arrays of positive integers



1. $a \geq b$
2. a over b
3. $\lambda_i = \mu_i - i + 1$ = # parts in raw i

$\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary

- a_{ij} = part
- $a_{ij} \leq j-i$ = special part
- order n = $a_{ij} \leq n$ $\forall ij$.

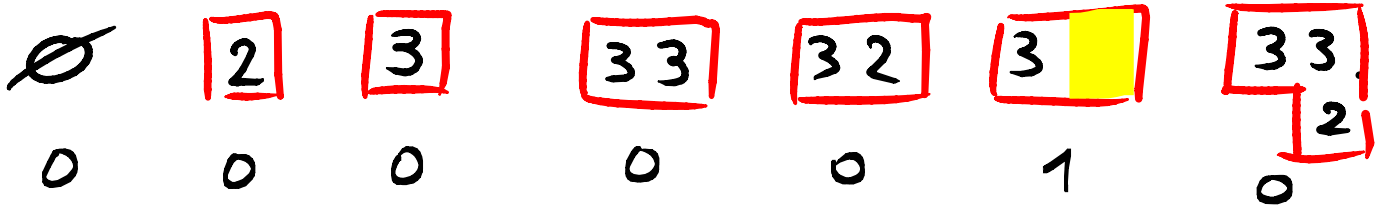
OBSERVABLES

parts = n → 3

special parts →

non-special parts

$n=3$: 7 DPP's



ASM

- $n \times n$ matrices with elements $0, \pm 1$
- ± 1 alternate along rows and columns
- row and column sums $= 1$

Generalize Permutation matrices

Arise from the Laurent phenomenon for the octahedron equation (Fomin-Zelevinsky '99)

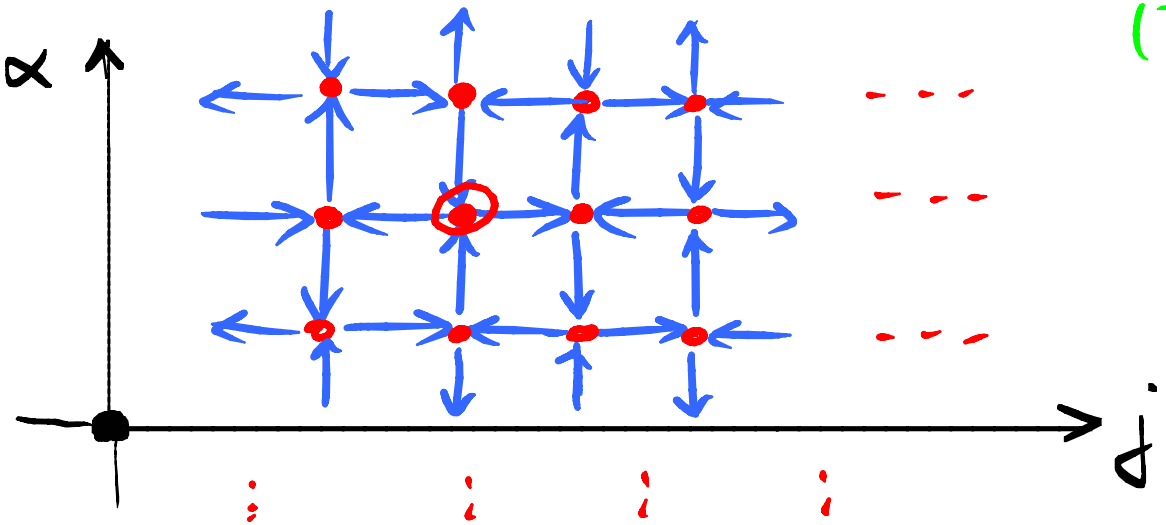
$$T_{ijk+1}\, T_{ijk-1} = T_{ij+1k}\, T_{ij-1k} + T_{i+1jk}\, T_{i-1jk}$$

$T_{ijk}\left(\{T_{ijb}; T_{i'j'b}\}_{i',j' \in \mathbb{Z}}\right) \equiv$ Laurent Polynomial.

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:



Cluster :

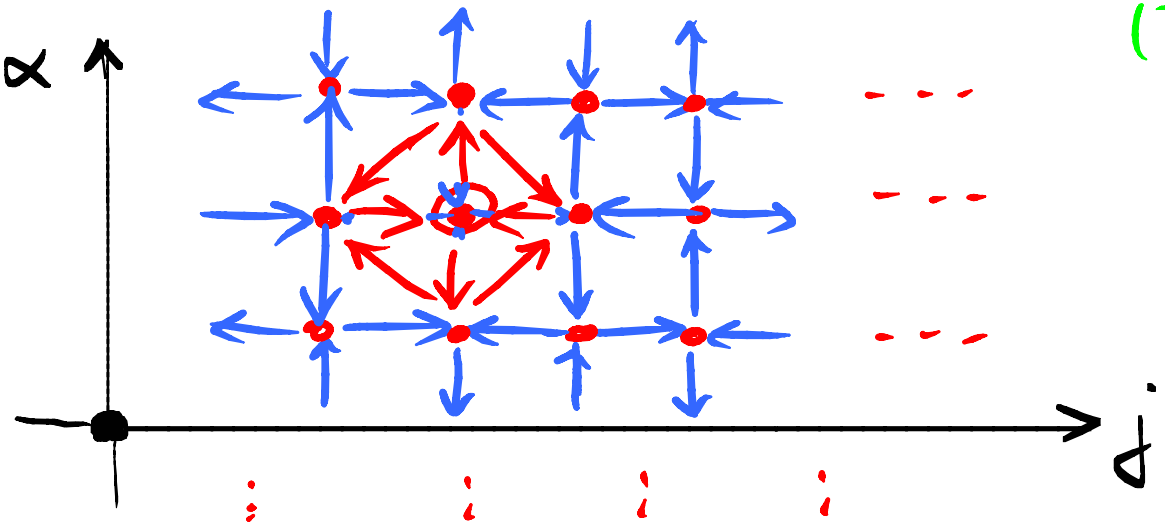
$$\begin{array}{cccccc} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & T_{1-10} & T_{101} & T_{110} & T_{121} & \cdots \\ \cdots & T_{0-11} & T_{000} & T_{011} & T_{020} & \cdots \\ \cdots & T_{-1-10} & T_{-101} & T_{-110} & T_{-121} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{array}$$

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:

mutation



Cluster :

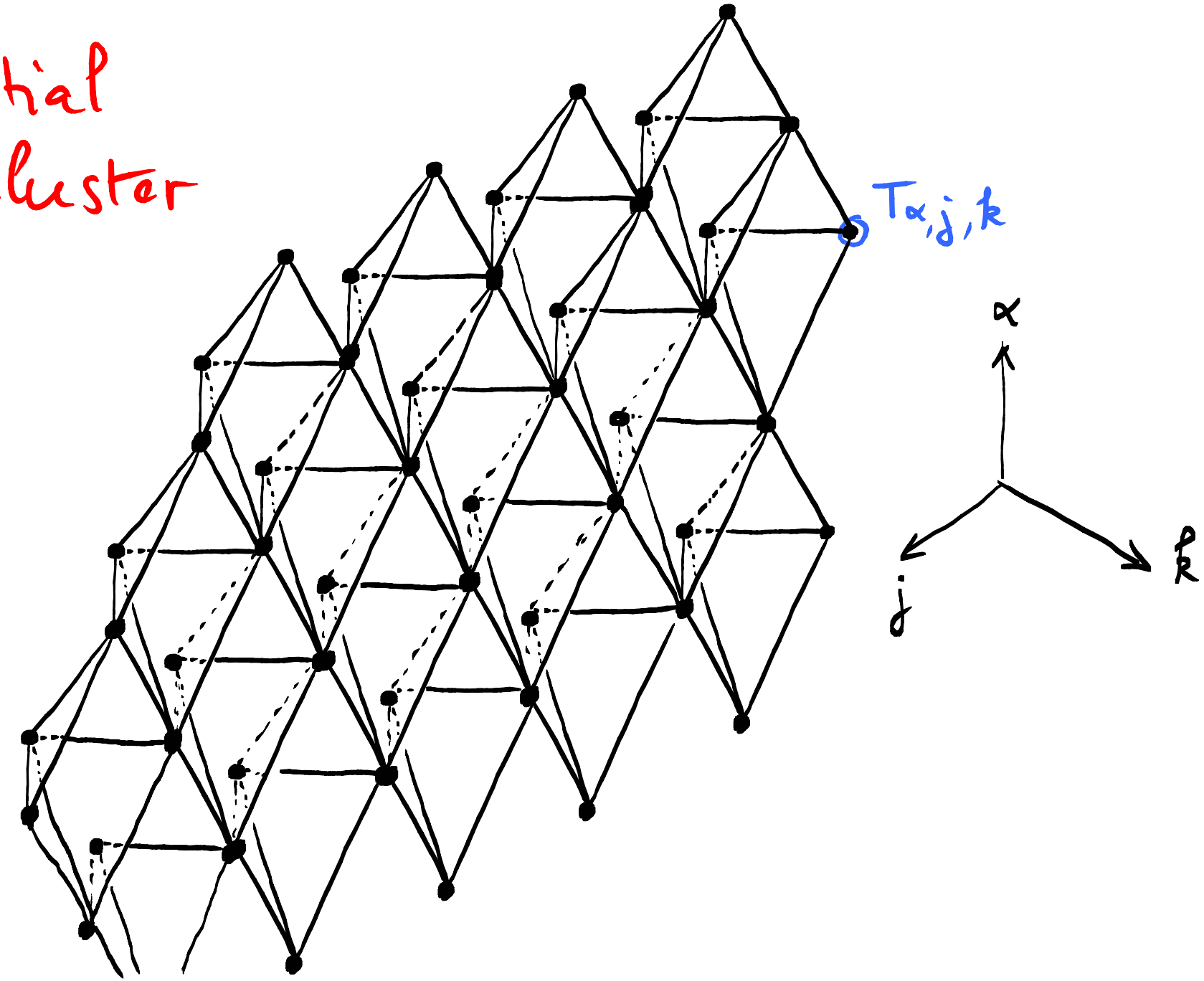
$$\frac{T_{0-11}T_{011}+T_{101}T_{-101}}{T_{000}}$$

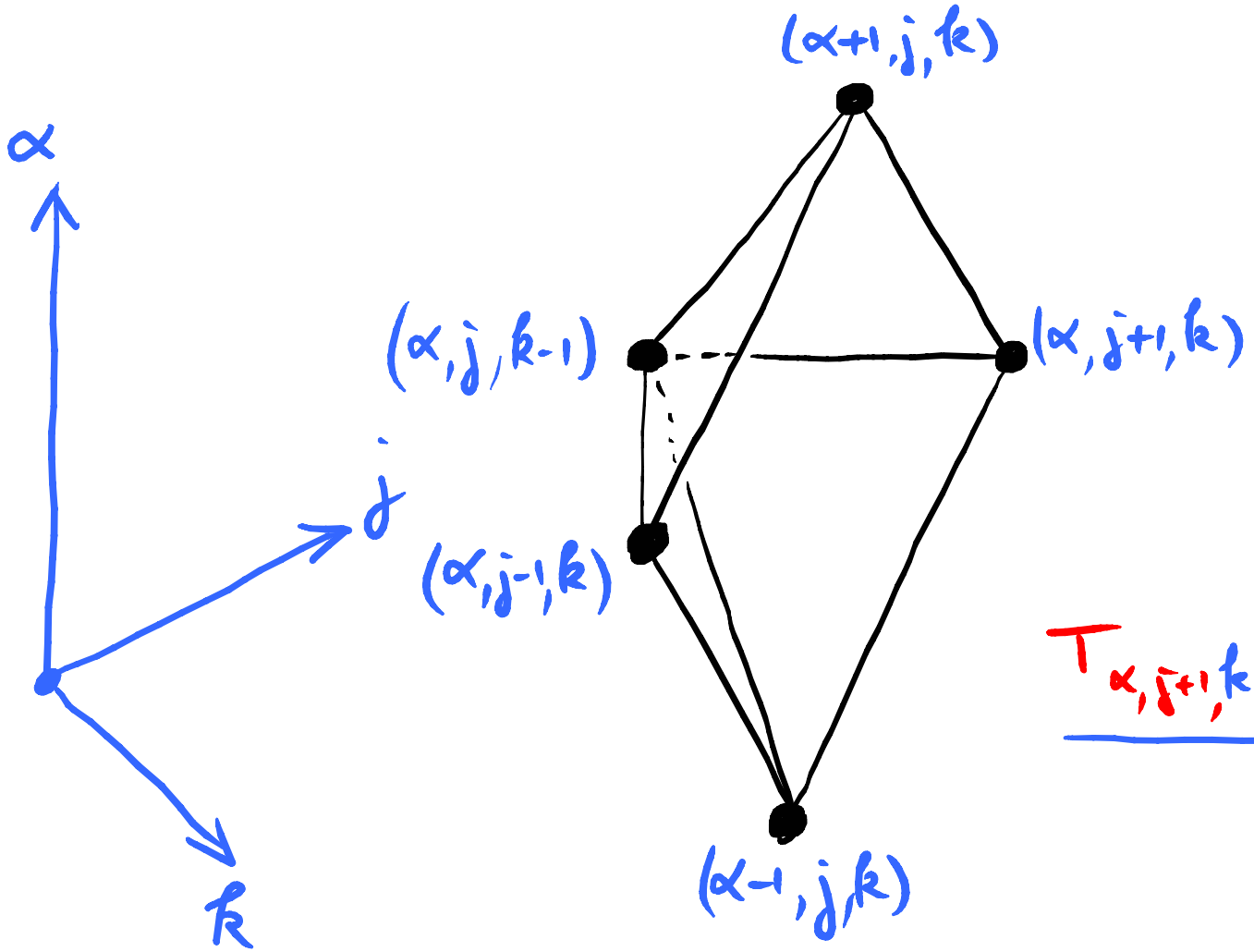
$$\cdots\ T_{1-10}\ \ T_{101}\ \ T_{110}\ \ T_{121}\ \cdots$$

$$\cdots\ T_{0-11}\ \ T_{011}\ \ T_{020}\ \cdots$$

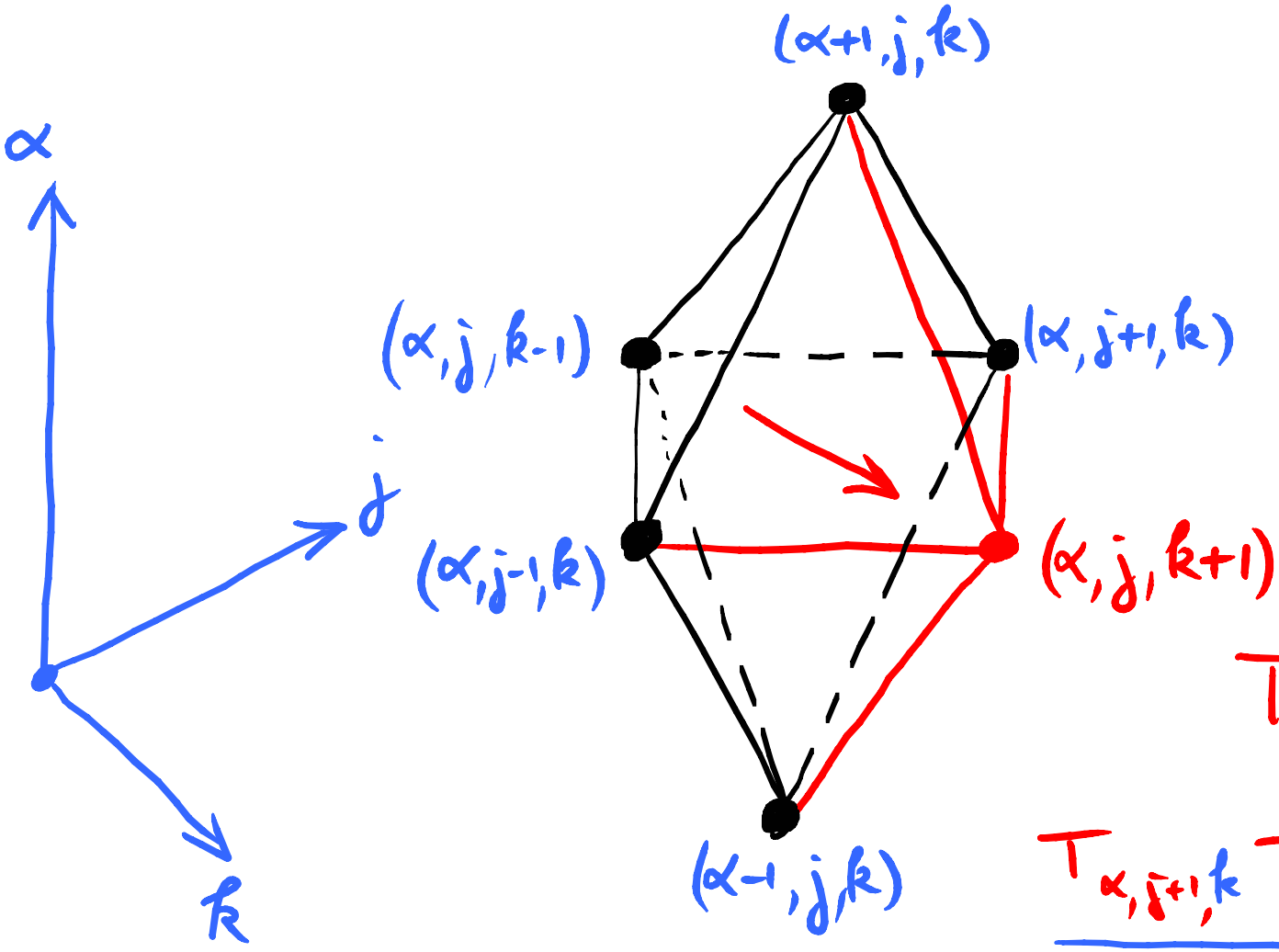
$$\cdots\ T_{-1-10}\ \ T_{-101}\ \ T_{-110}\ \ T_{-121}\ \cdots$$

initial cluster

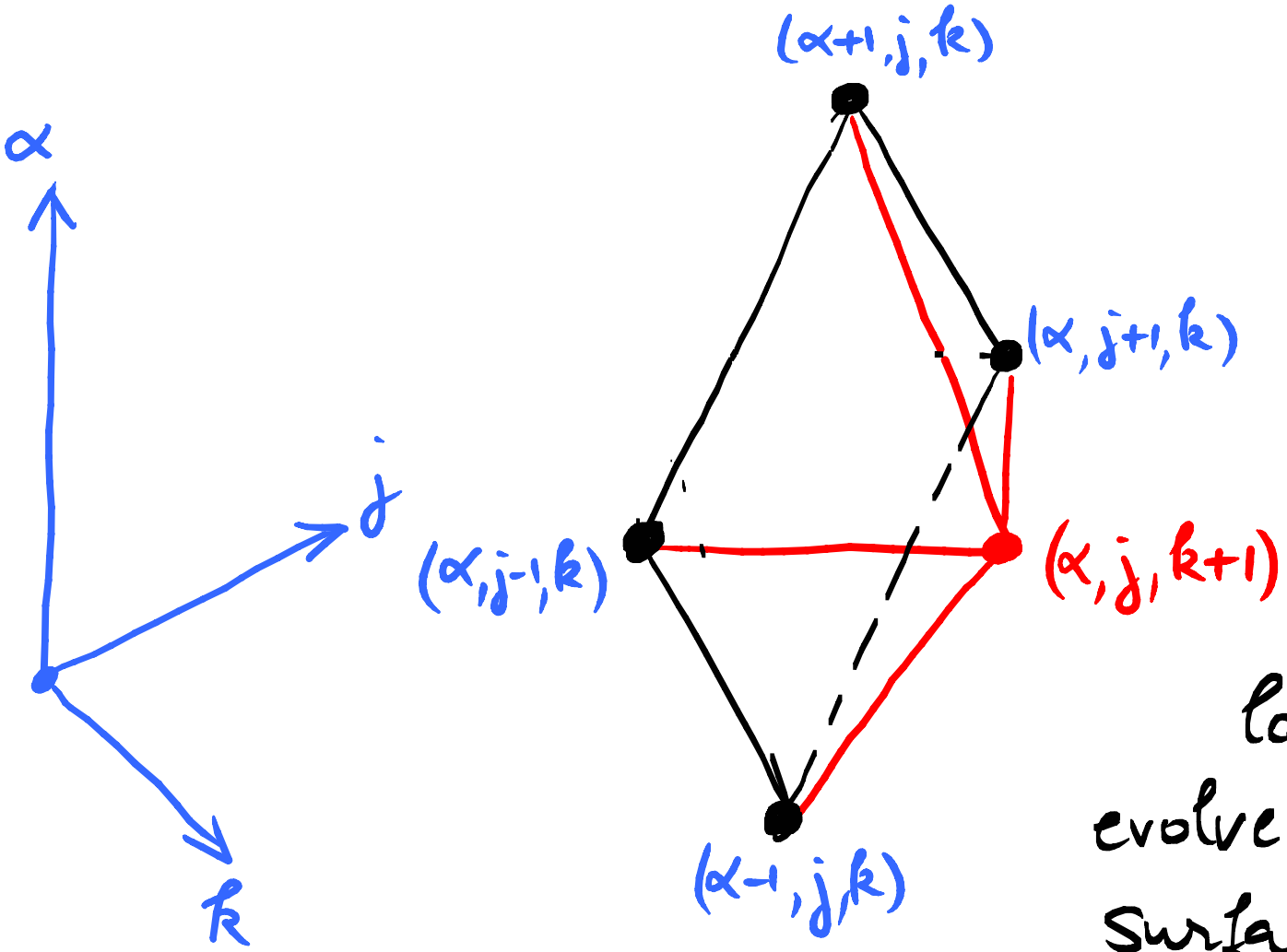




$$\frac{T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k}}{T_{\alpha,j,k-1}}$$

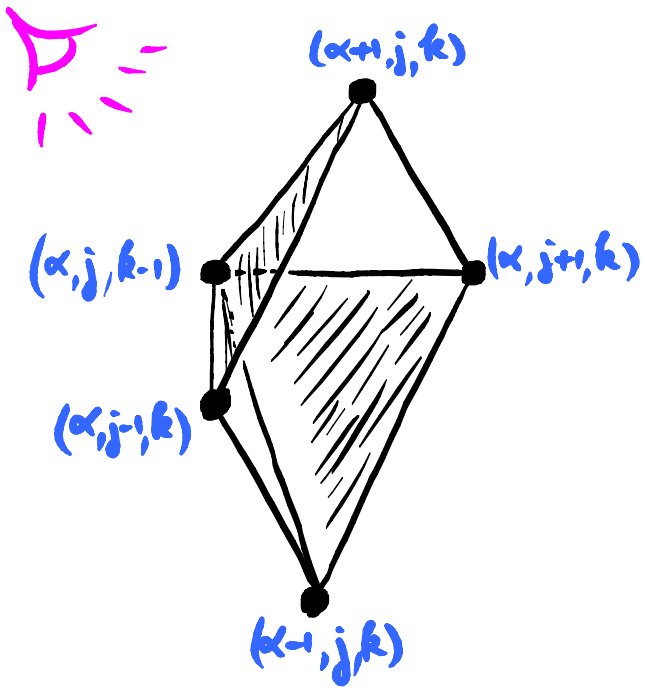


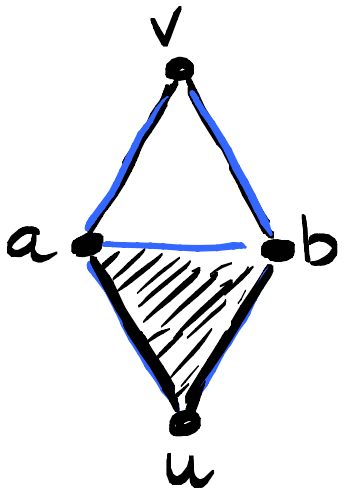
$$T_{\alpha,j,k+1} = \frac{T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k}}{T_{\alpha,j,k-1}}$$



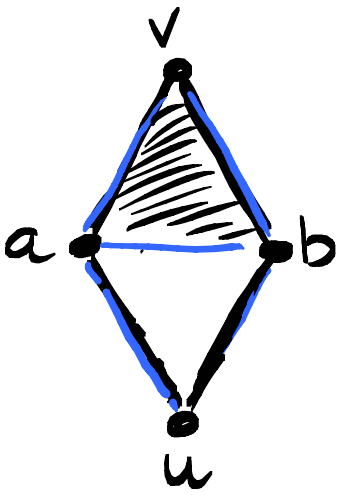
local move that evolves the stepped surface by "adding" an octahedron

DISCRETE INTEGRABILITY

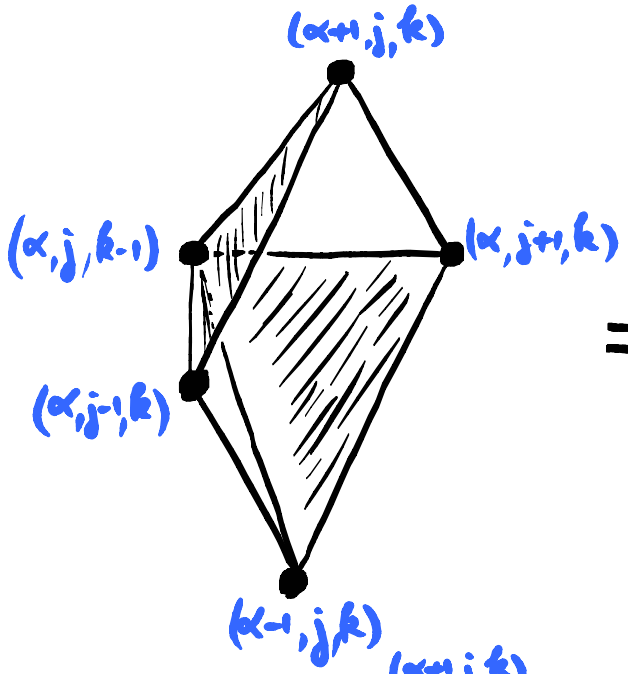


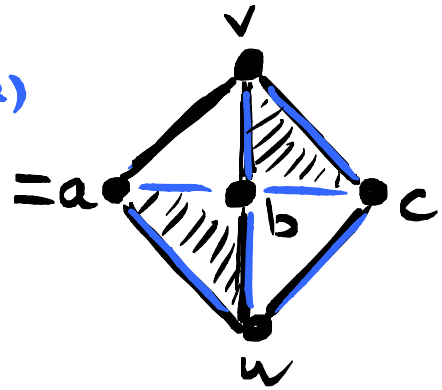


$$\rightarrow \quad D(u,a,b)=\begin{pmatrix} \frac{a}{b} & \frac{u}{b} \\ 0 & 1 \end{pmatrix}$$

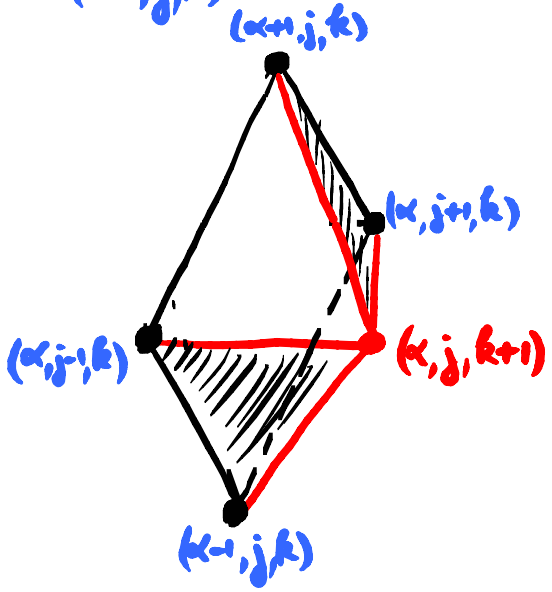


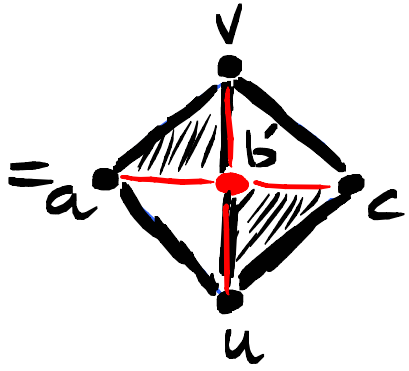
$$\rightarrow \quad U(a,b,v)=\begin{pmatrix} 1 & 0 \\ \frac{v}{b} & \frac{a}{b} \end{pmatrix}$$



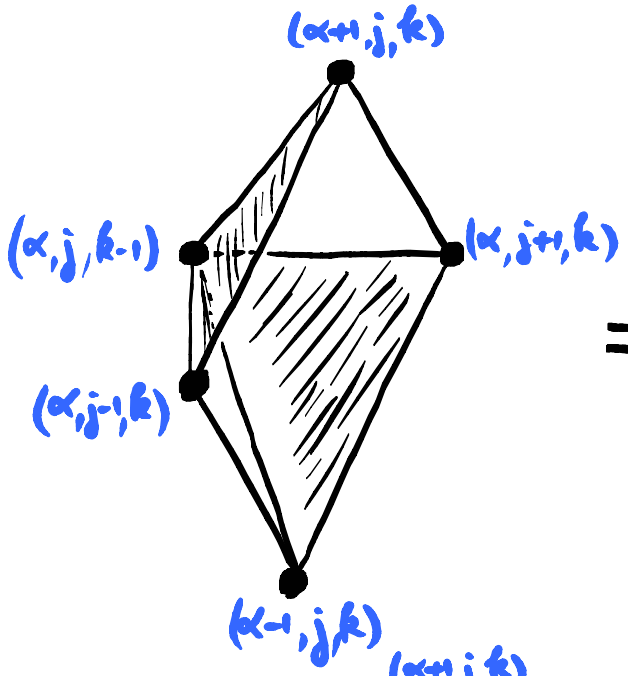


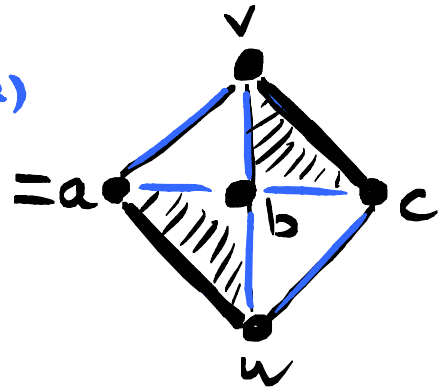
$= D(u,ab)\,U(b,c,v)$





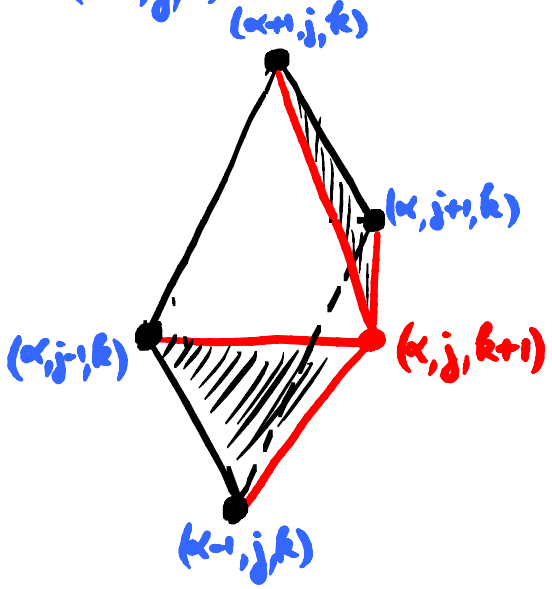
$= U(a,b',v)\,D(u,b'c)$

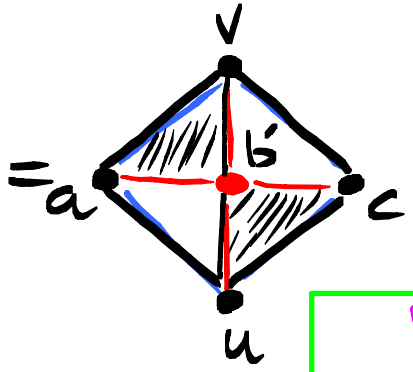




$= D(u,ab)\,U(b,c,v)$

$\Leftrightarrow \boxed{bb' = ac + uv}$

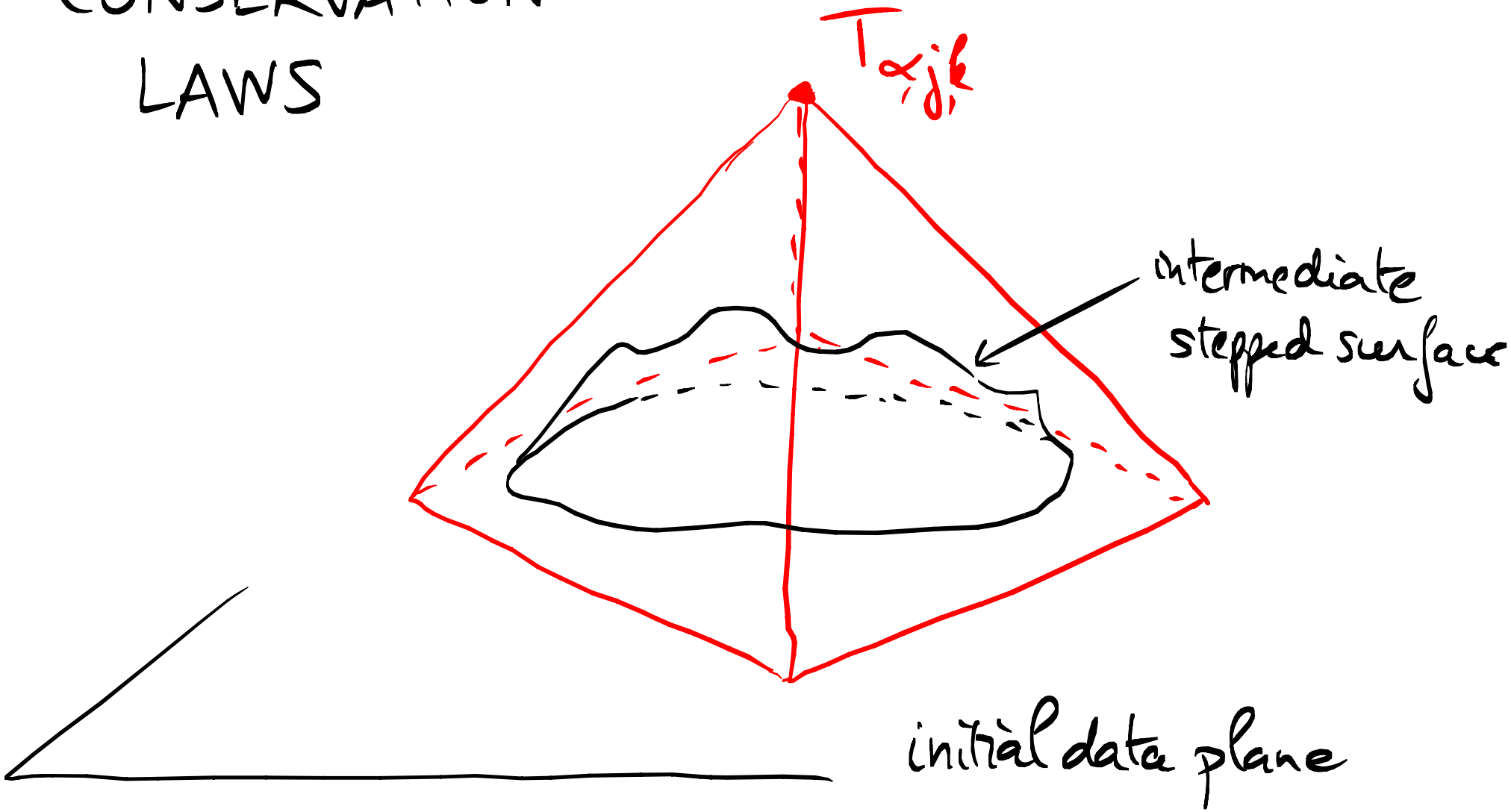




$= U(a,b',v)\,D(u,b'c)$

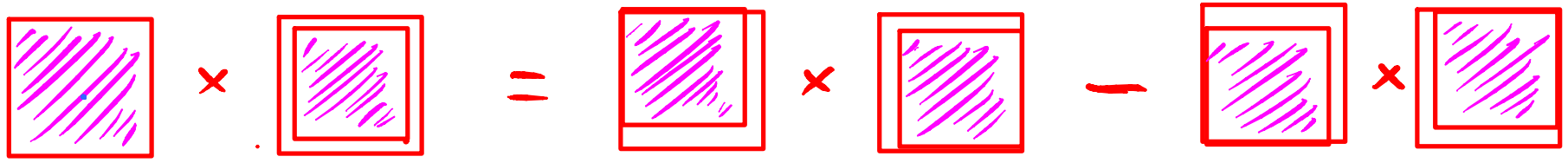
"Flat connection"
(related to Yang-Baxter eq)

CONSERVATION LAWS



Robbins-Rumsey (82): **λ-determinant**

use Desnanot-Jacobi relation for minors of $i+1 \times i+1$ matrix



size: $i+1$ $i-1$ i i i i

→ Lewis Carroll algorithm for computing determinant

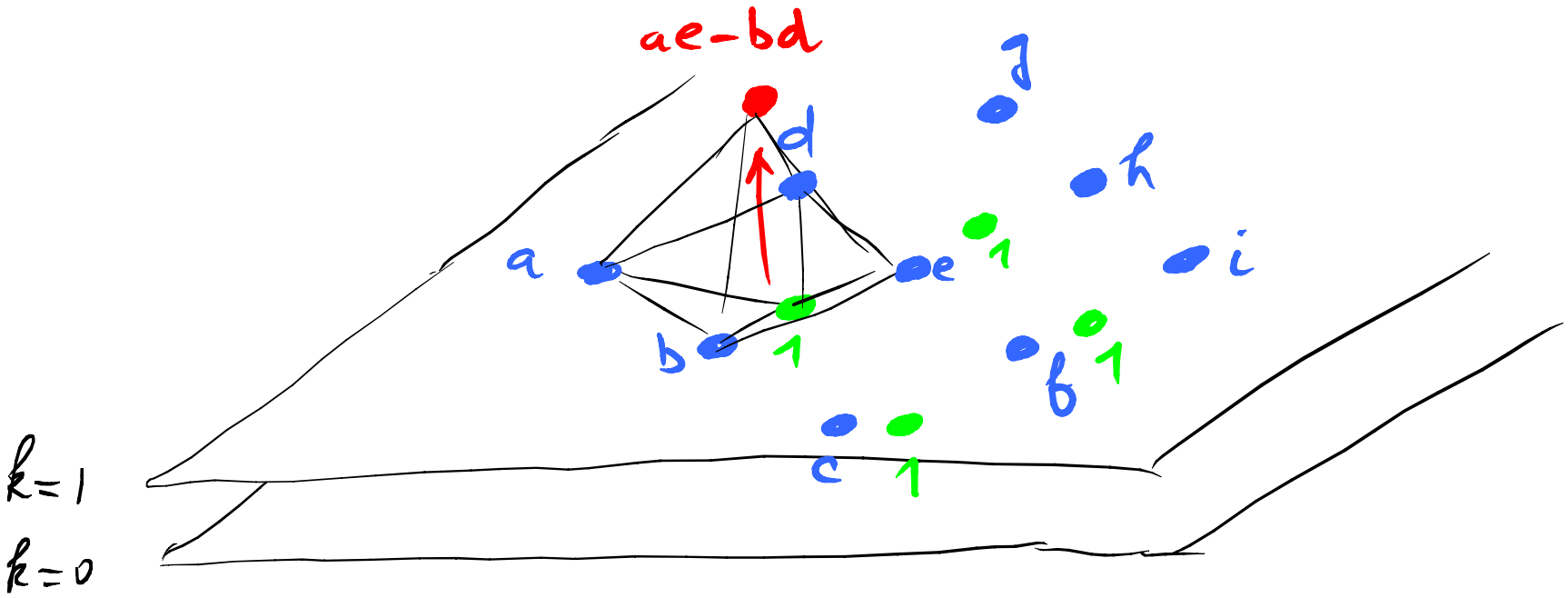
$$T_{ij0} = 1 \; ; \; T_{ij1} = a_{\frac{j-i}{2}, \frac{i+j}{2}} \Rightarrow T_{0,k+1,k} = \det_{k \times k}(A)$$

$(\mu = 1 \, , \lambda = -1)$

$$\det_{k \times k}(A) = \sum_{\substack{\text{permutation} \\ \text{matrices } P_{ij} \\ k \times k}} (-1)^{\text{inv}(P)} \prod_{ij} a_{ij}^{P_{ij}}$$

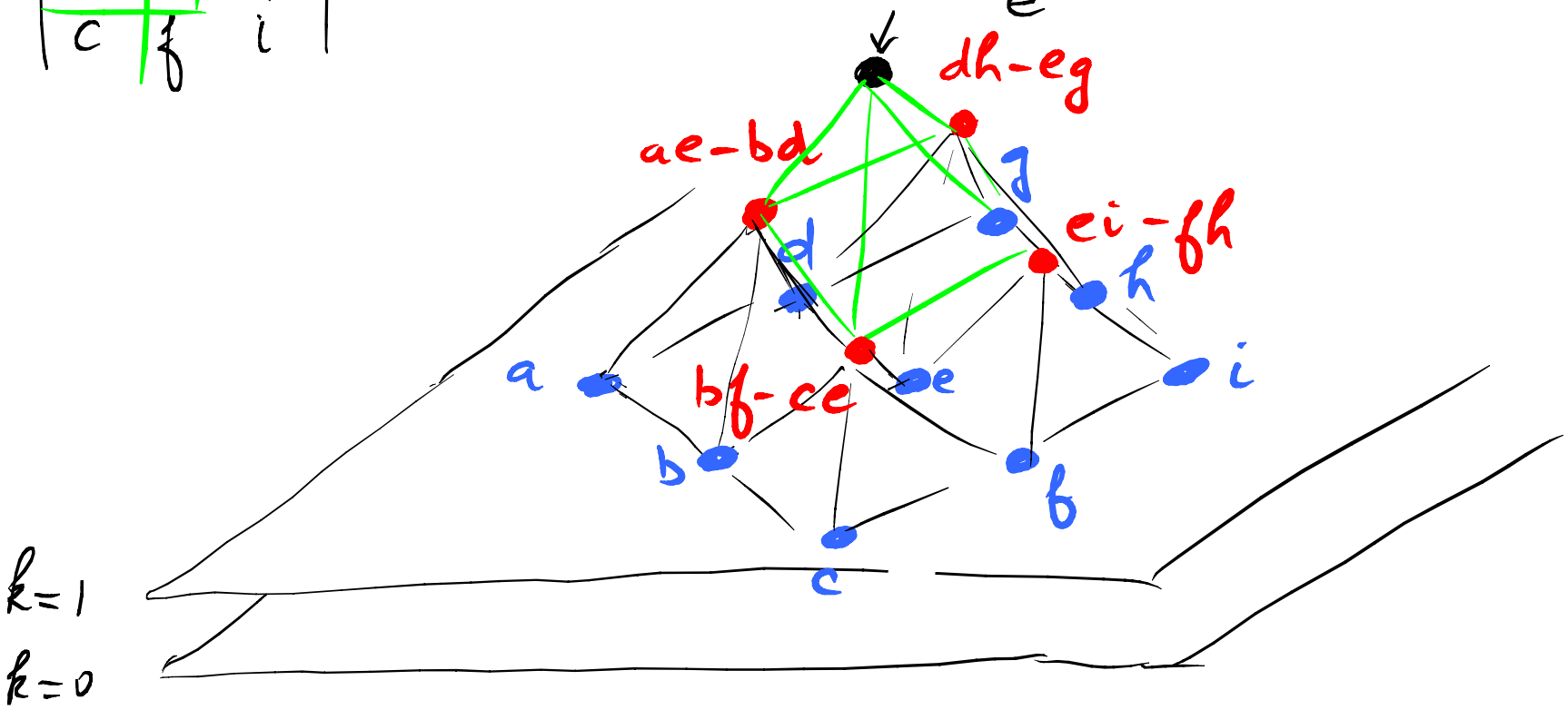
Computing determinants is not more difficult than constructing pyramids





$$\begin{vmatrix} a & d \\ b & e \end{vmatrix} \times 1 = a \times e - b \times d$$

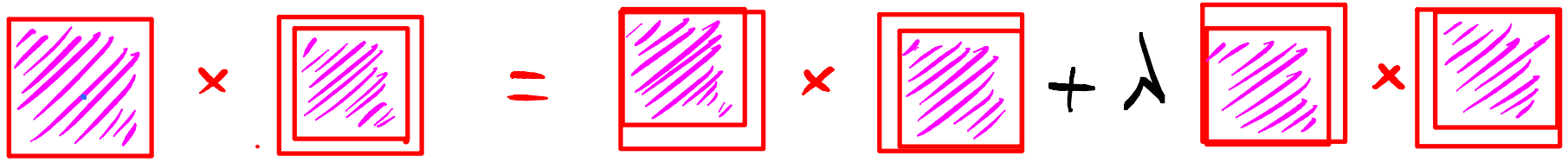
$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \frac{(ae-bd)(ei-fh) - (dh-eg)(bf-ce)}{e}$$



$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \times |e| = \begin{vmatrix} a & d \\ b & e \end{vmatrix} \times \begin{vmatrix} e & h \\ f & i \end{vmatrix} - \begin{vmatrix} b & e \\ c & f \end{vmatrix} \times \begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

Robbins-Rumsey (82): **λ-determinant**

use Desnanot-Jacobi relation for minors of $i+1 \times i+1$ matrix



size: $i+1$ $i-1$ i i i i

→ Lewis Carroll algorithm for computing determinant

$T_{ij0} = 1$; $T_{ij1} = a_{\frac{j-i}{2}, \frac{i+j}{2}}$ ⇒ $T_{0,k+1,k} = |A|_\lambda$

$(\mu = 1, \lambda = \lambda)$

THM (MRR)

$$|A|_\lambda = \sum_{ASM_{k\times k}} \lambda^{Inv(B)} (1+\lambda^{-1})^{\#(-1)_B} \prod_{ij} a_{ij}^{B_{ij}}$$

THM (MRR)

$$|A|_\lambda = \sum_{\substack{ASM \\ k\times k}} \lambda^{Inv(B)} (1+\lambda^{-1})^{\#(-1)_B} \prod_{ij} a_{ij}^{B_{ij}}$$

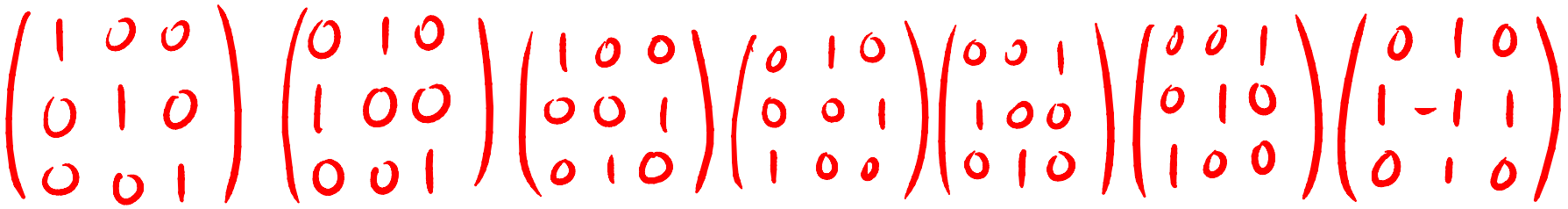
EXAMPLE:

$n=3$ 7 ASM's

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

OBSERVABLES FOR ASMs

- position of the 1 in the first row



0 1 0 1 2 2 1

OBSERVABLES FOR ASMs

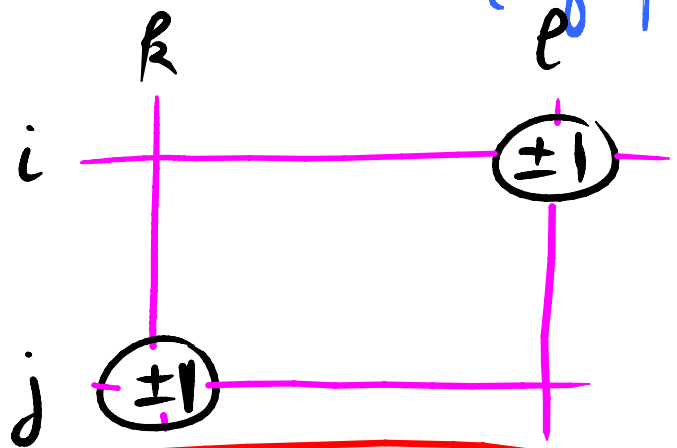
- position of the 1 in the first row
- #(−1) number of −1's

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

0 0 0 0 0 0 1

OBSERVABLES FOR ASMs

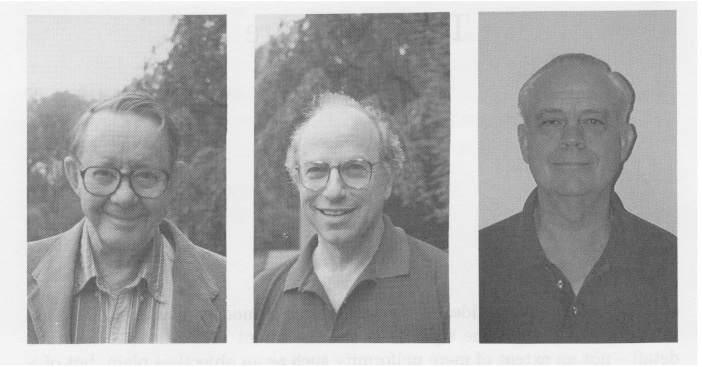
- position of the 1 in the first row
- #(−1) number of −1's
- inversion number (of permutations)



$$\mathrm{inv}(B_\cdot) = \sum_{i<j} \sum_{k<\ell} B_{i\ell} B_{jk}$$

THE ASM-DPP CONJECTURE

JOURNAL OF COMBINATORIAL THEORY, Series A 34, 340–359 (1983)



Alternating Sign Matrices and Descending Plane Partitions

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Institute for Defense Analyses, Princeton, New Jersey 08540-3699

Communicated by the Managing Editors

Received March 15, 1982

Conjecture 3. Suppose that n, k, m, p are nonnegative integers, $1 \leqslant k \leqslant n$. Let $\mathscr{A}(n, k, m, p)$ be the set of alternating sign matrices such that

(i) the size of the matrix is $n \times n$,

(ii) the 1 in the top row occurs in position k,

(iii) the number of -1's in the matrix is m,

(iv) the number of inversions in the matrix is p.

On the other hand, let $\mathscr{D}(n, k, m, p)$ be the set of descending plane partitions such that

(I) no parts exceed n,

(II) there are exactly $k-1$ parts equal to n,

(III) there are exactly m special parts,

(IV) there are a total of p parts.

Then $\mathscr{A}(n, k, m, p)$ and $\mathscr{D}(n, k, m, p)$ have the same cardinality.

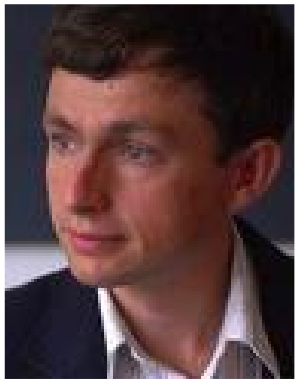
ASM (A(n))
- n = size
- position of top 1
- # −1's
- # inversions

DPP (D(n))
- n = order
- # parts = n
- # special parts
- # parts

Counting







- DPP : Andrews '79 → formula D(n)
- ASM : Zeilberger '96 A(n) = TSSCPP(n) = D(n)
  Kuperberg '96 6vertex model

Izergin-Korepin integrable lattice model

Refinements ?

Computation of the refined numbers

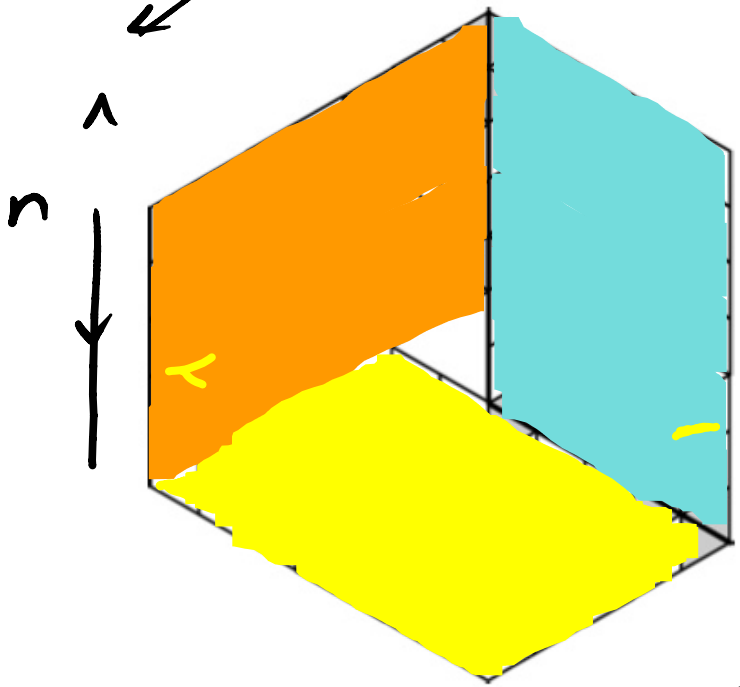
Strategy

1. reformulate in terms of known (and manageable objects)
   - DPP → lattice paths (lattice fermions)
   - ASM → 6vertex model (integrable lattice model)
2. write refined generating functions as determinants
3. Prove identity between determinants

DPP

DPP as Lattice Paths

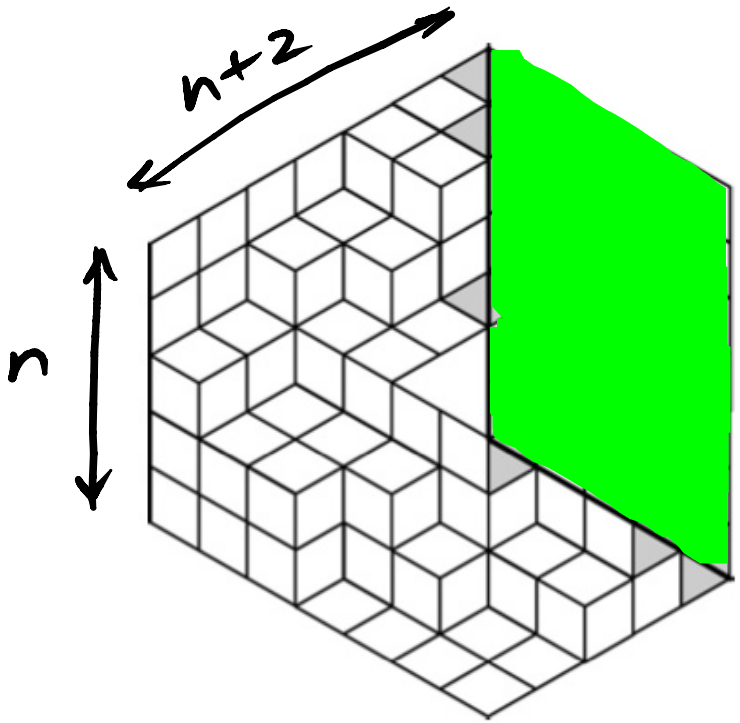
Lalonde '03
(Krattenthaler '06)

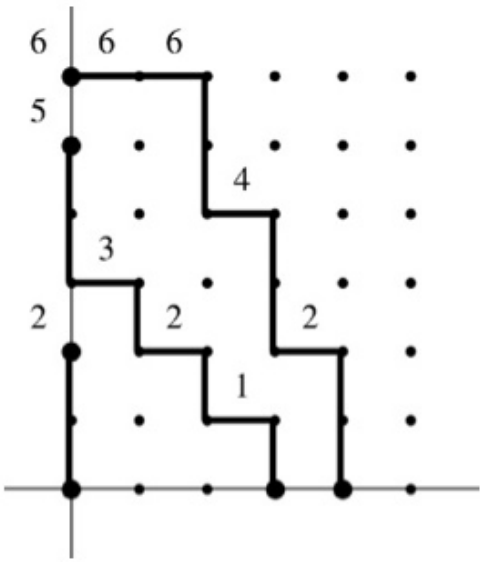


Cyclically symmetric Rhombus tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole $\leftrightarrow$ Lattice Paths

$2\times2\times2$

DPP as Lattice Paths

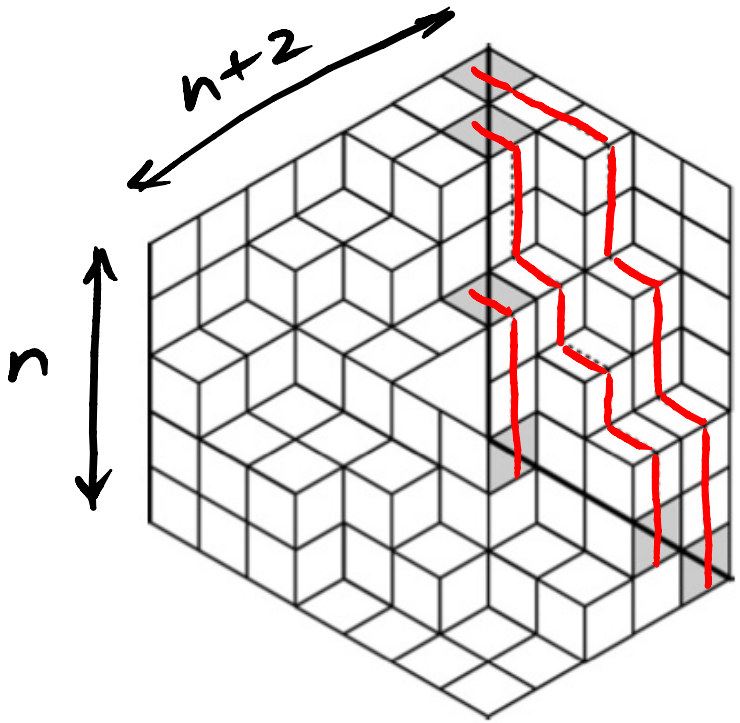


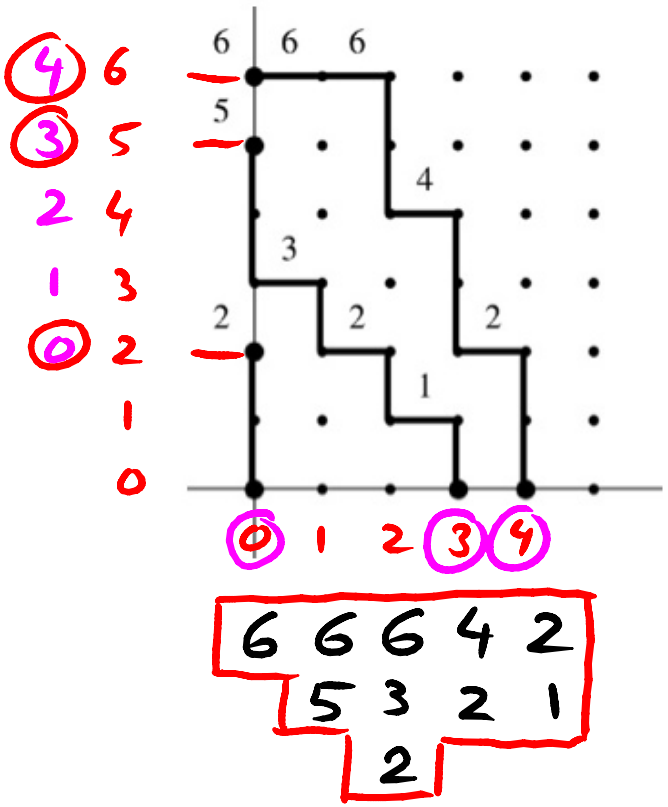


Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\underset{2\times2\times2}{\triangle}$ hole $\leftrightarrow$ Lattice Paths

DPP as Lattice Paths

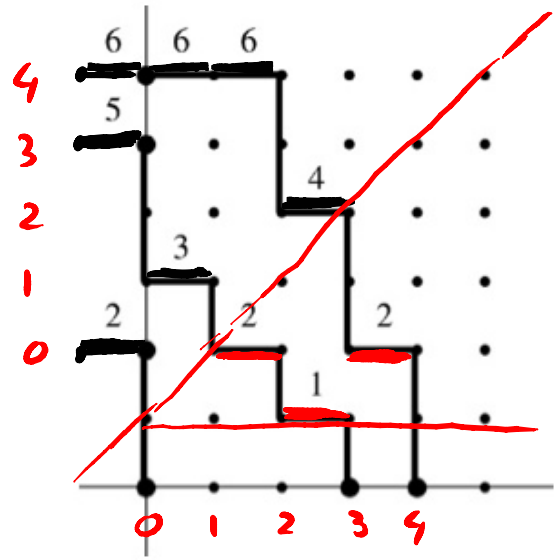
(Krattenthaler '06)





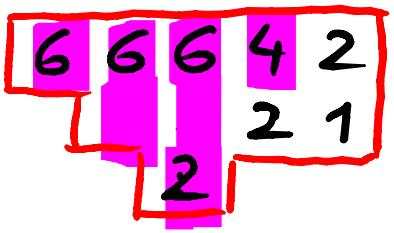
Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\underset{2\times2\times2}{\triangle}$ hole $\leftrightarrow$ Lattice Paths

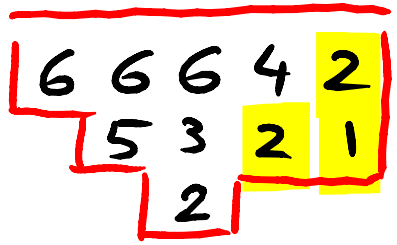
horizontal steps — = non-special parts



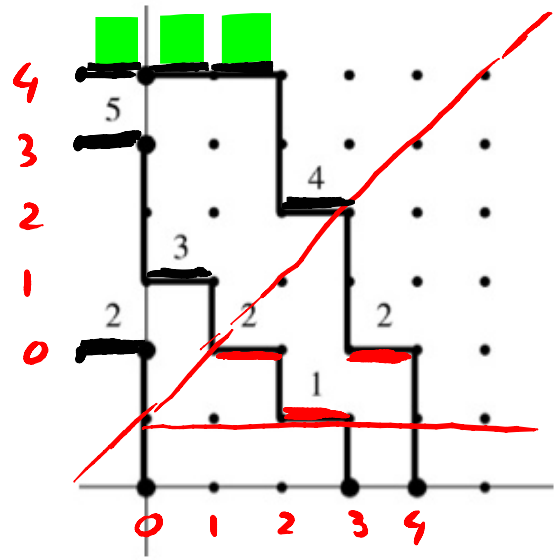
horizontal steps — = special parts

horizontal steps here do not count





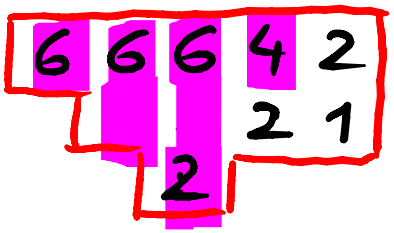
horizontal steps — = non-special parts

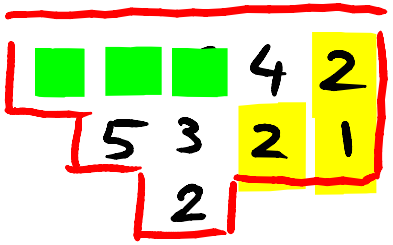


■ = parts equal to n

horizontal steps — = special parts

} horizontal steps here do not count

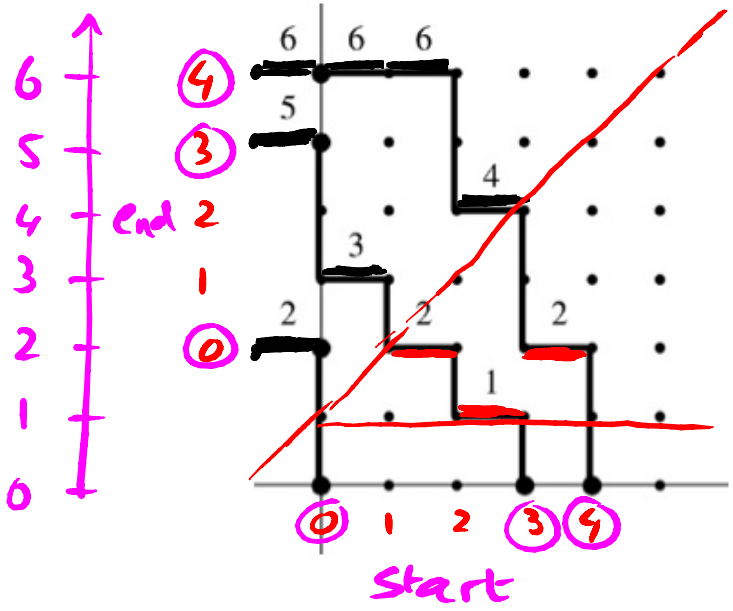




DPP : NON-INTERSECTING LATTICE PATHS

horizontal steps — = non-special parts

weight y



horizontal steps — = special parts

weight x

} horizontal steps here do not count

6	6	6	4	2
	5	3	2	1
		2		

6	6	6	4	2
	5	3	2	1
		2		

Pb = count these families of non-intersecting paths

- M_{ij} = Partition Function for 1 path : $(i,0) \to (0,j)$

$$M_{ij} = \sum_{\text{paths } (i,0)\to(0,j)} x^{\#(\text{—})} \, y^{\#(\text{—})}$$

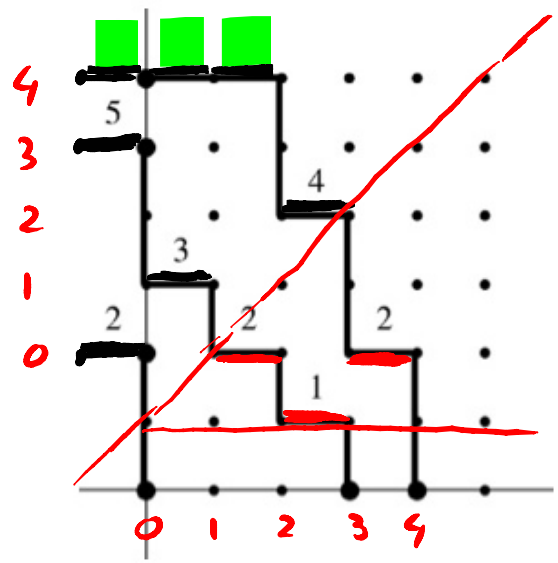
horizontal steps : lower wedge, upper wedge

- By Gessel–Viennot theorem



$\det\left(M^{i_1 \cdots i_k}_{j_1 \cdots j_k}\right)$ = Partition fctn for families of k non-intersecting paths starting at $(i_1,0)\cdots(i_k,0)$, ending at $(0,j_1)\cdots(0,j_k)$

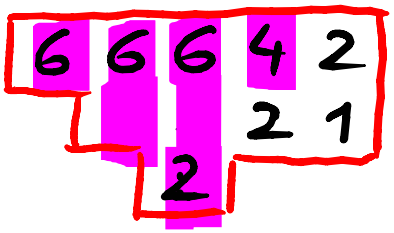
horizontal steps — = non-special parts

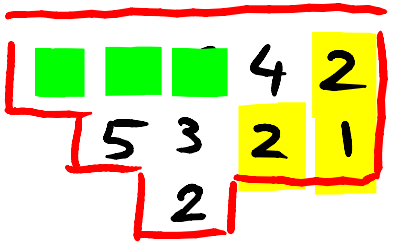


■ = parts equal to n

horizontal steps — = special parts

} horizontal steps here do not count

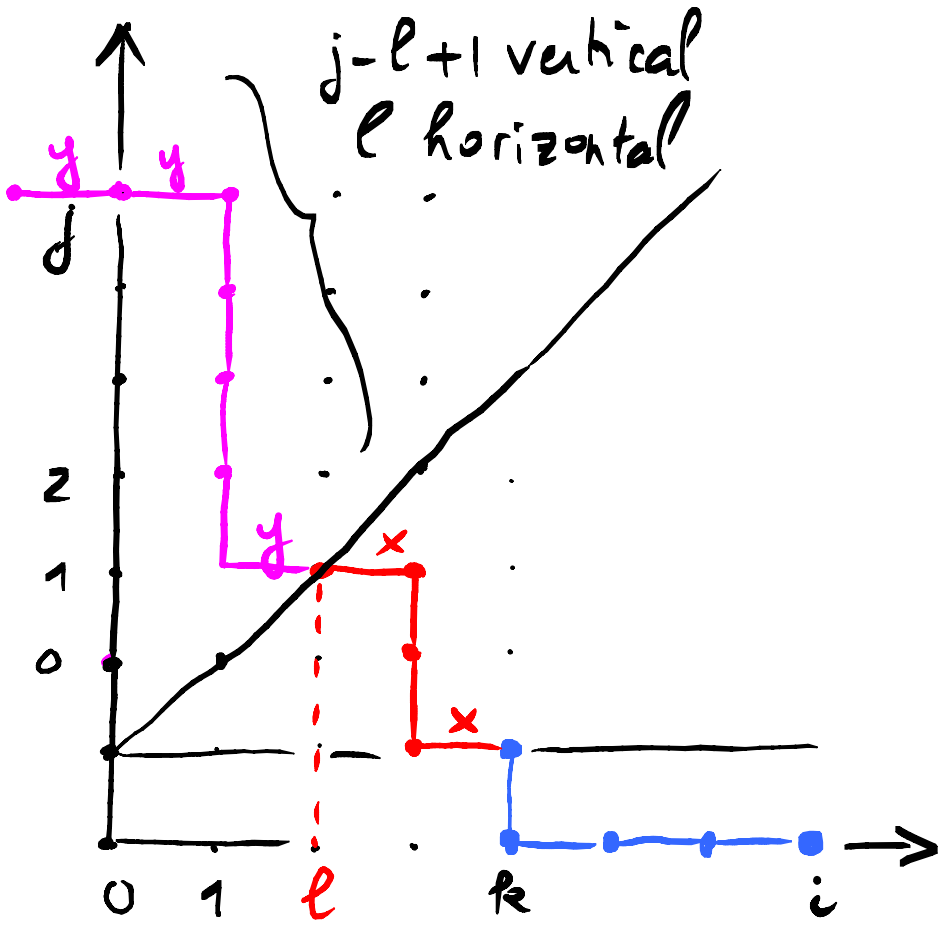




Lemma $\det_{n\times n}(I+M) = \sum_{1\le i_1<\cdots<i_k\le n} \det\left(M^{i_1\cdots i_k}_{i_1\cdots i_k}\right)$

$M^{i_1\cdots i_k}_{i_1\cdots i_k}$: M's minor with rows $i_1\cdots i_k$, cols $i_1\cdots i_k$

Here: M_{ij} = Part. fctn (path $(i,0)\to(0,j)$)



$$Z^{(n)}_{DPP}(x,y) = \det(I+M)$$

↑ per special part ↑ per non-special part

$$M_{i,j} = \sum_{k=0}^{i} \sum_{\ell \geq 0} \binom{k}{\ell} x^{k-\ell} \binom{j+1}{\ell} y^{\ell+1}$$

Generating function :

$$f_{DPP}(z,w) = \sum_{i,j \geq 0} z^i w^j \quad (I+M)_{i,j}$$

THM

$$f_{DPP}(z,w) = \frac{1}{1-zw} + \frac{1}{1-z} \frac{yz}{1-xz-w-(y-x)zw}$$

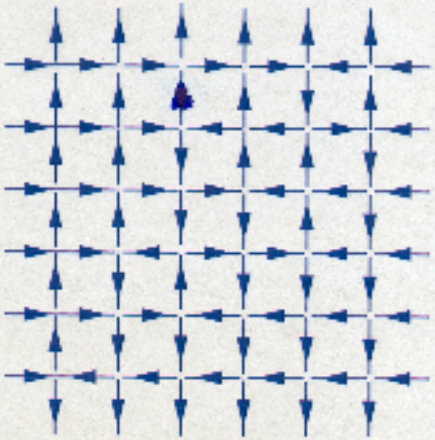
weights : x / special part y / non-special part

ASM

From ASM to 6 Vertex model with Domain Wall Boundary conditions (Kupeborg)

nxn ASM

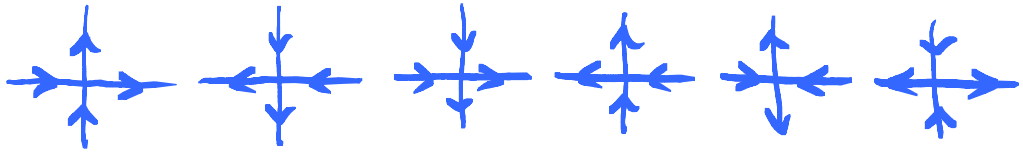
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



6V+ DWBC on nxn grid

Bijection :

0 0 0 0 1 -1

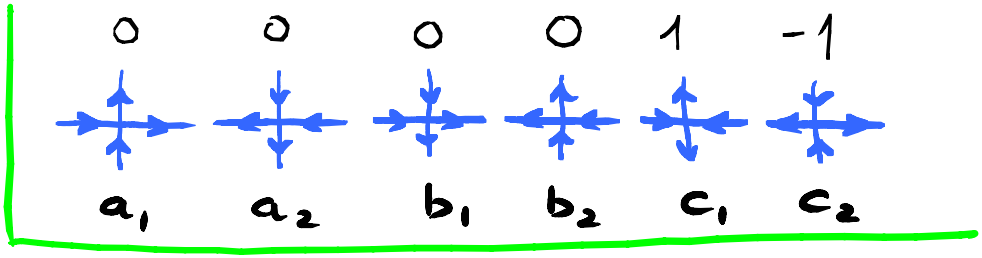


a_1 a_2 b_1 b_2 c_1 c_2

$qz - q^{-1}w$ $q^{-1}z - qw$ $(q^2 - q^{-2})\sqrt{zw}$

(integrable weights)

Refinements :

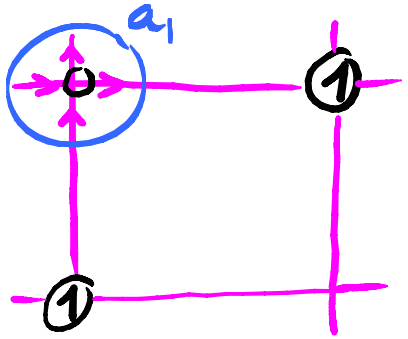


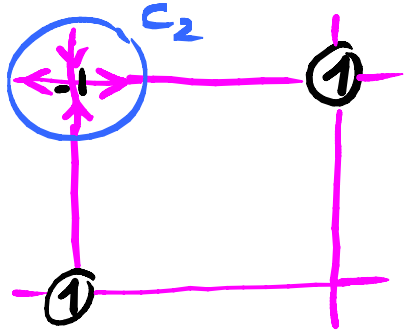
by symmetry :

$$\begin{cases} N_{a_1} = N_{a_2} = \frac{N_a}{2} & N_{b_1} = N_{b_2} = \frac{N_b}{2} \\ N_{c_1} = N_{c_2} + n & N_c = N_{c_1} + N_{c_2} \end{cases}$$

- $\#(-1) = N_{c_2} = \frac{N_c - n}{2}$
- $\mathrm{Inv}(A) = N_{a_1} + N_{c_2}$

$$\Rightarrow \mathrm{Inv}(A) - \#(-1) = N_{a_1} = \frac{N_a}{2}$$





Partition function

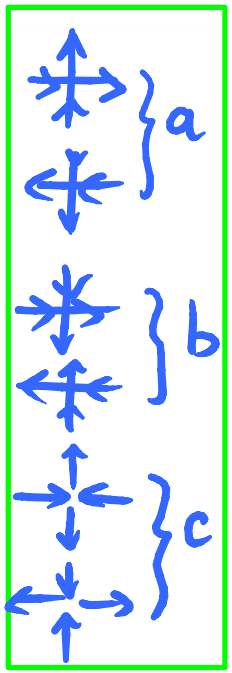
$$Z_{ASM}^{(n)}(x,y,z) = \sum_{\substack{\text{configs of } n\times n \\ 6V\,DWBC}} x^{\frac{N_c-n}{2}}\, y^{\frac{N_a}{2}}\, z^{N_{a'}}$$

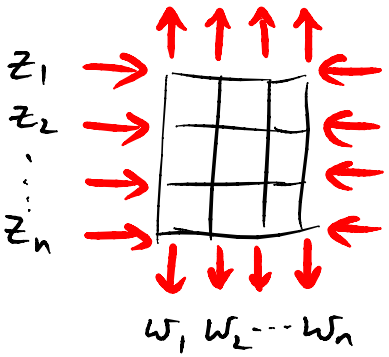
$z = 1$

usually, one considers $Z_{6V}^{(n)}(a,b,c) = \sum_{\substack{\text{configs} \\ 6V\,DWBC}} a^{N_a} b^{N_b} c^{N_c} \times \underbrace{\left(\frac{a'}{a}\frac{b}{b'}\right)^{N_{a'}}}_{1}$

$$Z_{6V}^{(n)}(a,b,c) = b^{n^2}\left(\frac{c}{b}\right)^{n} \sum \left(\frac{c}{b}\right)^{N_c-n} \left(\frac{a}{b}\right)^{N_a}$$

$$\Leftrightarrow \boxed{x = \left(\frac{c}{b}\right)^2 \quad y = \left(\frac{a}{b}\right)^2}$$





Partition function of 6V+DWBC

$$Z_n = \sum_{\substack{\text{configs} \\ \text{on grid}}} \prod_{\substack{\text{vertices} \\ (i,j)}} \text{weights}(z_i, w_j)$$

$$\prod_{i,j} c(z_i, w_j)$$

THM

$$Z_n = \frac{\prod_{i,j} a(z_i,w_j)\, b(z_i,w_j)}{\prod_{i<j} (z_i - z_j)(w_i - w_j)} \det\left\{ \frac{1}{a(z_i,w_j)\, b(z_i,w_j)} \right\}_{1 \le i,j \le n}$$

(Korepin – Izergin)

recursion relation + symmetries (from commutation of Transfer matrices).





Homogeneous limit: $\begin{cases} z_i \longrightarrow r & \forall i \\ w_j \longrightarrow r^{-1} & \forall j \end{cases}$

$$a(z_i, w_j) \longrightarrow qr - q^{-1}r^{-1} = a(r, r^{-1})$$
$$b(z_i, w_j) \longrightarrow q^{-1}r - qr^{-1} = b(r, r^{-1})$$
$$c(z_i, w_j) \longrightarrow q^2 - q^{-2} = c(r, r^{-1})$$

$$Z_n(q,r) = \frac{(ab)^{n^2}}{c^n} \det_{0 \le i,j \le n-1} \left(\left(\frac{d}{du}\right)^i \left(\frac{d}{dv}\right)^j \left[\frac{c(u,v)}{a(u,v)\, b(u,v)} \right] \Big|_{\substack{u=r \\ v=r^{-1}}} \right)$$

↑ by Taylor expansion around the limit

• Note: $$\frac{c(u^{-1},v)}{a(u^{-1},v)\,b(u^{-1},v)} = \frac{1}{uv-q^{-2}} - \frac{1}{uv-q^{2}}$$

Taylor-expand:

• Define: $(A_+)_{i,j} = \left(\frac{d}{du}\right)^i \frac{(dv)^j}{j!} \frac{1}{uv-q^2}\Big|_{\substack{u=r^{-1}\\ v=r^{-1}}}$ $(A_-)_{i,j} = \text{idem}\ (q\to q^{-1})$

• Introduce upper triangular matrix $U(\alpha,\beta)$

$$U(\alpha,\beta)_{i,j} = \begin{cases} \binom{j}{i}\alpha^i\beta^j & \text{if } i\le j \\ 0 & \text{if } i>j \end{cases}$$

$$\begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 & \cdots \\ 0 & \alpha\beta & 2\alpha\beta^2 & 3\alpha\beta^3 & \cdots \\ 0 & 0 & \alpha^2\beta^2 & 3\alpha^2\beta^3 & \\ 0 & 0 & 0 & \alpha^3\beta^3 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

$i,j \in \mathbb{Z}_+ \qquad \alpha,\beta \in \mathbb{C}^*$

THM

$$A_+ = -\frac{1}{r^{-2}-q^2}\left[U^t(\alpha,\beta)\right]^{-1} U(\alpha',\beta')$$

with: $\alpha = \frac{1-q^2r^2}{r}$ $\beta = \frac{q^2-q^{-2}}{r^2-q^2}$ $\alpha' = -q^2r^2\beta$ $\beta' = -\frac{1}{\alpha}$

Proof:

1. generating fun. $(U(\alpha\beta)) = \dfrac{1}{1-\beta w(1+\alpha z)} = Y$

2. $U^t(\alpha,\beta)^{-1} = U(-1/\beta, -1/\alpha)$; gf $= \dfrac{1}{1+\frac{1}{\alpha}z(1-\frac{1}{\beta}w)} = X$

3. generating function $(A_+) = \dfrac{1}{(r^{-1}+z)(r^{-1}+w)-q^2} = X*Y$

THM

$$A_+ = -\frac{1}{r^{-2}-q^2}\left[U^t(\alpha,\beta)\right]^{-1} U(\alpha',\beta')$$

with: $\alpha = \frac{1-q^2r^2}{r}$ $\beta = \frac{q^2-q^{-2}}{r^2-q^2}$ $\alpha' = -q^2r^2\beta$ $\beta' = -\frac{1}{\alpha}$

Proof:

1. generating fun. $_t$ ($U(\alpha\beta)$) = $\dfrac{1}{1-\beta w(1+\alpha z)} = Y$

2. $U^t(\alpha,\beta)^{-1} = U(-1/\beta, -1/\alpha)$; gf = $\dfrac{1}{1+\frac{1}{\alpha}z(1-\frac{1}{\beta}w)} = X$

3. generating function (A_+) = $\dfrac{1}{(r^{-1}+z)(r^{-1}+w)-q^2} = X*Y$

Holds true for any finite truncation to $i,j \in [0, n-1]$

set $V = U^t(\alpha, \beta)$ $\quad \bar{U} = U(\alpha', \beta')$

$\bar{V} = V(q \to q^{-1})$ $\quad \bar{U} = U(q \to q^{-1})$

then:

$$A_+ = \frac{1}{q^2 - r^{-2}} V^{-1} U$$

$$A_- = \frac{1}{q^{-2} - r^{-2}} \bar{V}^{-1} \bar{U}$$

$$\det(A_- - A_+) = \det(A_-) \det\left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} \bar{U}^{-1} \bar{V} V^{-1} U \right]$$

$$\propto \det\left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} (\bar{V} V^{-1})(U \bar{U}^{-1}) \right]$$

$U \bar{U}^{-1} = U(-1, 1)$

$\bar{V} V^{-1} = U^t(-y, x)$

(Proof by convolution of q-f.)

Collecting all prefactors, we get:

$$Z^{(n)}_{ASM}(x,y) = \det\left((1-\nu)I + \nu G\right)$$

$$\nu = \frac{r^{-2}-q^{-2}}{q^2-q^{-2}} \qquad G = U^t(-y,x)\, U(-1,1)$$

$$G_{ij} = \sum_{k\geq 0} \binom{i}{k} y^k \binom{j}{k} x^{i-k}$$

$\Rightarrow$ generating function of $(1-\nu)I + \nu G = f_{ASM}(z,w)$

THM

$$f_{ASM}(z,w) = \frac{1-\nu}{1-zw} + \frac{\nu}{1-zx-w-(y-x)zw}$$

Final identity:

$$Z^{(n)}_{DPP} = \det(I + M) = \det((1-\nu)I + \nu G) = Z^{(n)}_{ASM}$$

Proof:

note that

$$(1-z)(1-(1-\nu)w)\, f_{DPP}(z,w) - \left(1-\frac{z}{1-\nu}\right)(1-w)\, f_{ASM}(z,w)$$

$$= \Big(\underbrace{x\nu(1-\nu) - y(1-\nu) - \nu}_{=0}\Big) \times \text{rational fraction}(z,w)$$

$$\left(x = \left(\frac{q^2-q^{-2}}{q^{-1}r-qr^{-1}}\right)^2,\ y = \left(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}}\right)^2,\ \nu = \left(\frac{r^{-2}-q^{-2}}{q^2-q^{-2}}\right),\ 1-\nu = \left(\frac{q^2-r^{-2}}{q^2-q^{-2}}\right) \right)$$

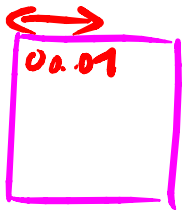
+ **Remark**: Let $A = (A_{ij})_{i,j \geq 0}$ $\quad F = \sum_{i,j \geq 0} A_{ij} z^i w^j$

then $\boxed{(1-\lambda z)(1-\mu w) F(z,w)}$ is the generating function for $(I - \lambda S) A (I - \mu S^t)$ where

$S_{i,j} = \delta_{i,j+1}$ "shift" matrix strictly lower triangular $\Rightarrow$ the determinant is unchanged.

Proof completed for $z = 1$

REFINEMENTS

- ASM 

$z^{\#0\text{'s left of 1 in 1st row (ASM)}}$ ← $\#a_1$'s
- DPP 

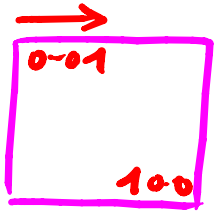
$z^{\#\text{parts} = n}$

modifies the last column of the matrix only
→ change the g.f accordingly, same idea of proof

$$Z_{ASM}(x, y, z) = Z_{DPP}(x, y, z)$$ (MRR)

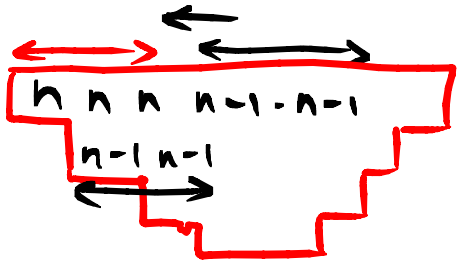
DOUBLE REFINEMENT

- ASM



$z^{\#0\text{'s left of top 1}}\, w^{\#0\text{'s right of bottom 1}}$

- DPP

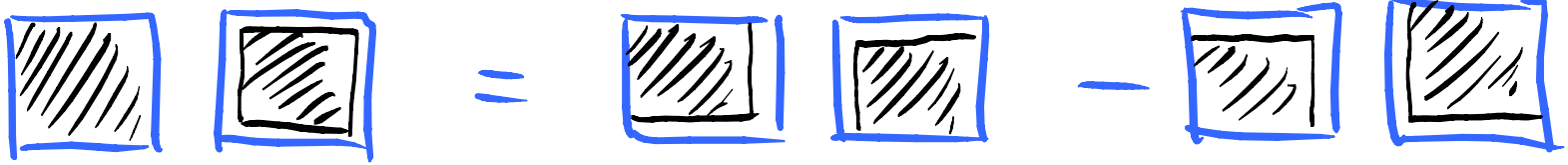


$z^{\#(\text{parts}=n)}\, w^{\#(\text{parts}=n-1)}$

$\times\, w^{\#(\text{rows of length } n-1)}$

THM $Z_{ASM}(x, y, z, w) = Z_{DPP}(x, y, z, w)$

Proof: use Desnanot-Jacobi Lewis-Carroll to relate $Z(z,w)$ to $Z(z,1)$ $Z(w,1)$ and $Z(1,1)$



For both $Z = Z_{ASM}$ and $Z = Z_{DPP}$:

$$(z-w) Z_N(z,w)\ Z_{N-1}(1,1)$$
$$= :(z-1)w\ Z_N(z,1)\ Z_{N-1}(1,w)$$
$$- (w-1)z\ Z_N(1,w)\ Z_{N-1}(z,1)$$

(z,w affects only 2 columns of the matrix).

CONCLUSION

MRR PROVED by method of generating fctns for the matrix of which we take the det

something special about the class of gen.-fctns

→ Bijection ASM – DPP ? (– TSSCPP ?)
fermions? (– O(n) ?)

→ More refinements ?
other spectral parameters ?

→ Generalizations : DPP with symmetries
↔ ASM with symmetries [in progress]

→ q-deformation : $|D| = \sum a_{ij}$ for a DPP

$$\sum_{D \in DPP(n)} q^{|D|} = \prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}$$

$$[j]_q = \frac{1-q^j}{1-q}$$

$$j!_q = [1]_q [2]_q \cdots [j]_q$$

pb = q-enumeration of ASMs?

→ Razumov-Stroganov for DPP ?

REMARK : integrability of 1+1D Lorentzian gravity v/s integrability of the 6V model

Critical varieties

- $\varphi(g,a) = \dfrac{1-g^2(1-a^2)}{ga}$
- $2\Delta(a,b,c) = \dfrac{a^2+b^2-c^2}{ab} = \dfrac{1+y-x}{\sqrt{x}}$

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not same!

$x = g^2 \qquad y = g^2a^2$

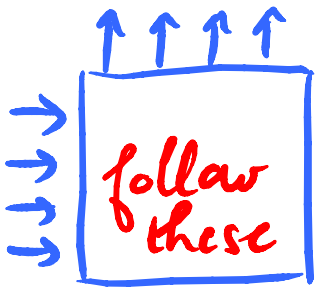
$\begin{cases} x \leftrightarrow y \ ? \\ a \hookrightarrow c \ ? \end{cases}$

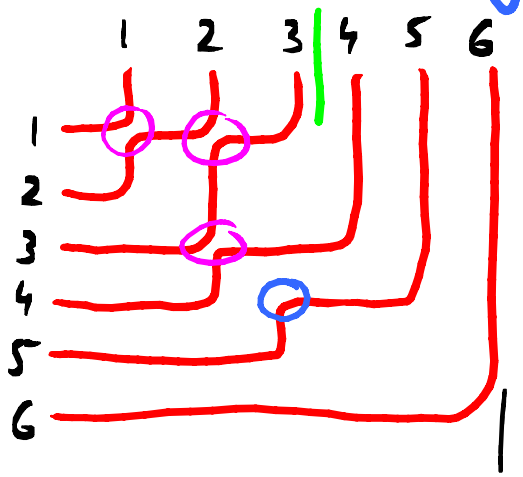
IT's A SMALL WORLD!

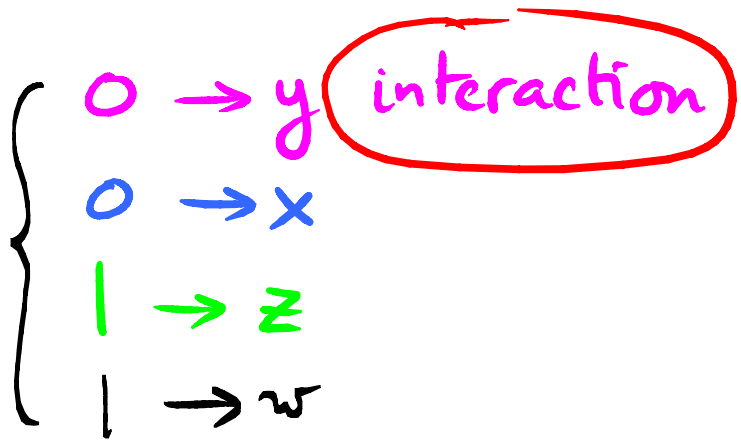
FREE V/S NON-FREE FERMIONS

- DPP, TSSCPP → non-intersecting paths
  = free fermians

- 6V, ASM → osculating paths







The determinant identity disentangles the interaction → make it more precise?