RANDOM SURFACES

[P. Di Francesco, E. Guitter, J. Bouttier, R. Kedem]

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1. 2D Quantum Gravity & Matrix models

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1. 2D Quantum Gravity & Matrix models

maps / integrability

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1. 2D Quantum Gravity & Matrix models

   maps / integrability

2. Geodesic distance in planar maps

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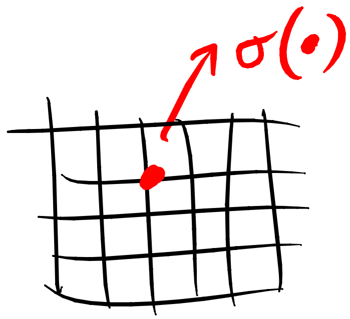
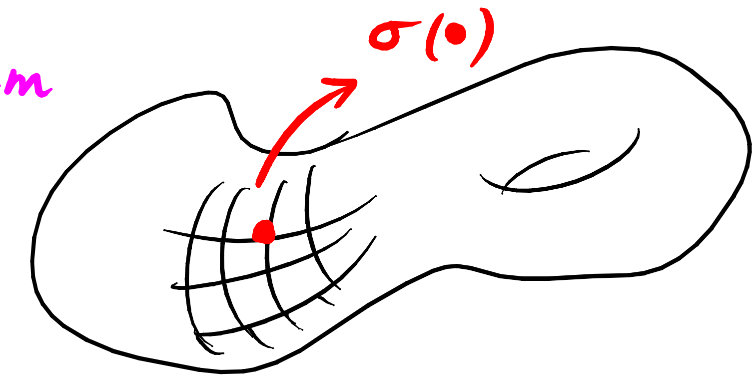
1. 2D Quantum Gravity & Matrix models
   maps / integrability
2. Geodesic distance in planar maps
   discrete integrability

PLANAR MAPS & GEODESICS



2D quantum Gravity $\longrightarrow$ (Euclidian) [90's]



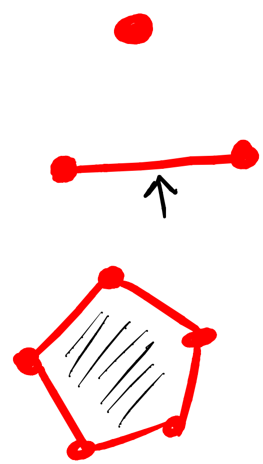
2D lattice model

2D random tessellation

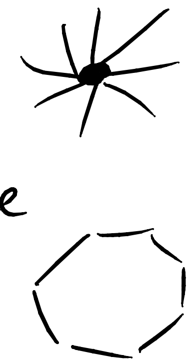
- Matrix models, exact solutions
- Integrability (KP / KdV tau-functions)
- Correlations at fixed geodesic distance?

MAPS: 2D embedding of a connected graph with notions of:

- vertex
- edge
- face

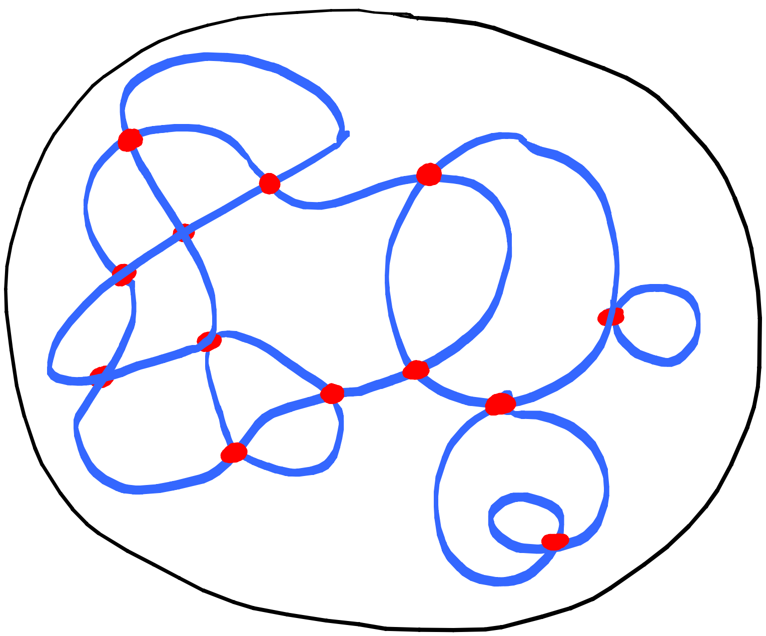


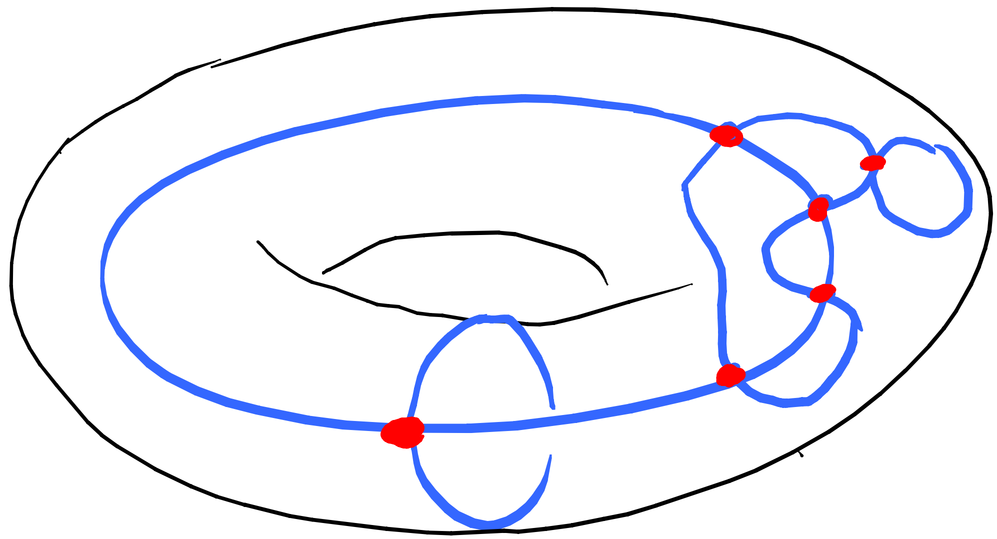
- degree
  =
  valence



usually, rooting at an edge or a vertex

Examples of 4-valent maps





GENUS 0

GENUS 1

MATRIX MODELS

- statistical ensemble $\{M,$ $N\times N$ Hermitian matrices$\}$ equipped w/ measure:

$$d\mu(M) = \frac{1}{Z_0} \underbrace{dM}_{\text{Haar measure } \prod dM_{ii} \prod_{i<j} d(\mathrm{Re}\, M_{ij})\, d\,\mathrm{Im}(M_{ij})} e^{-N\,\mathrm{Tr}\left(\frac{M^2}{2} - \sum_{k\geq 1} g_{2k} \frac{M^{2k}}{2k}\right)}$$

- Normalization / Partition function

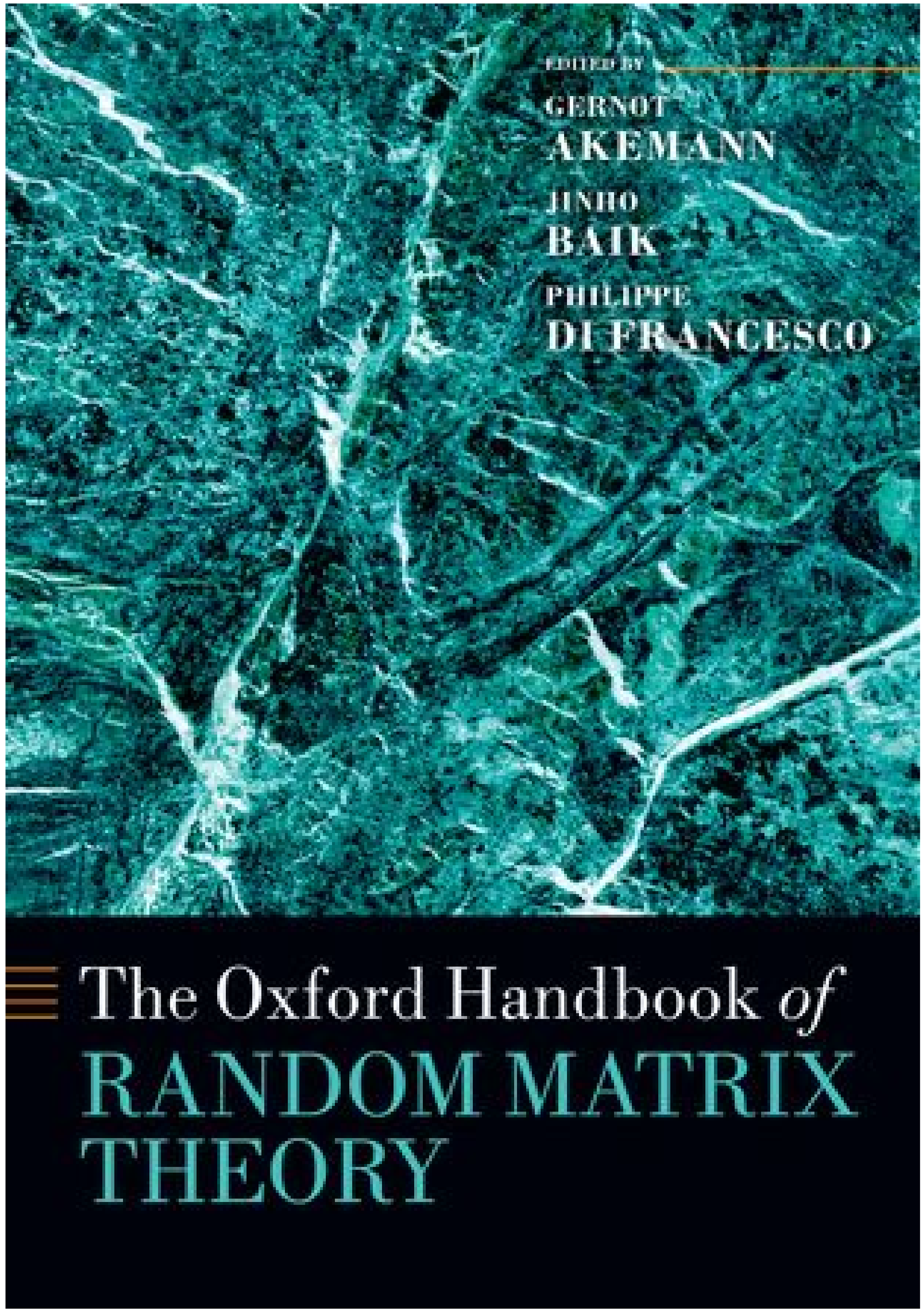
$$Z_0 = \int dM\, e^{-N\,\mathrm{Tr}\left(\frac{M^2}{2}\right)} \qquad \text{(Gaussian)}$$

$$Z(N;\{g_{2k}\}) = \int d\mu(M)$$

THM $\log Z(N;\{g_{2k}\})$ = generating function for connected maps $\mathcal{M}$ of even valences with weight g_{2k} per $2k$-vertex and $N^{\chi(\mathcal{M})}$ where $\chi(\mathcal{M})$ = Euler characteristic of the map, namely $2 - 2 \times$ genus of the underlying Riemann surface

$$\log Z(N;\{g_{2k}\}) = \sum_{h\geq 0} N^{2-2h} \sum_{\substack{\text{connected} \\ \text{maps } \mathcal{M} \\ \text{genus } h}} \prod_k g_{2k}^{\,V_{2k}(\mathcal{M})} \Big/ |\mathrm{Aut}\,\mathcal{M}|$$

$V_{2k}(\mathcal{M})$ = # vertices of degree $2k$ in $\mathcal{M}$



42 Chapters

919 pages

or

....

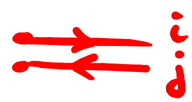
THM $\log Z(N;\{g_{2k}\})$ = generating function for connected maps m of even valences with weight g_{2k} per $2k$-vertex and $N^{\chi(m)}$ where $\chi(m)$ = Euler characteristic of the map, namely $2 - 2\times$genus of the underlying Riemann surface

Proof = "combinatorial lego"

1. expand $Z = \left\langle \sum_{n_k} \prod_k \frac{1}{n_k!}\left(N g_{2k} \frac{\mathrm{Tr} M^{2k}}{2k}\right)^{n_k} \right\rangle$ ← Gaussian average

$d\mu_0(M) = dM\, e^{-N\mathrm{Tr}\left(\frac{M^2}{2}\right)}$

2. Represent:

M_{ij} 

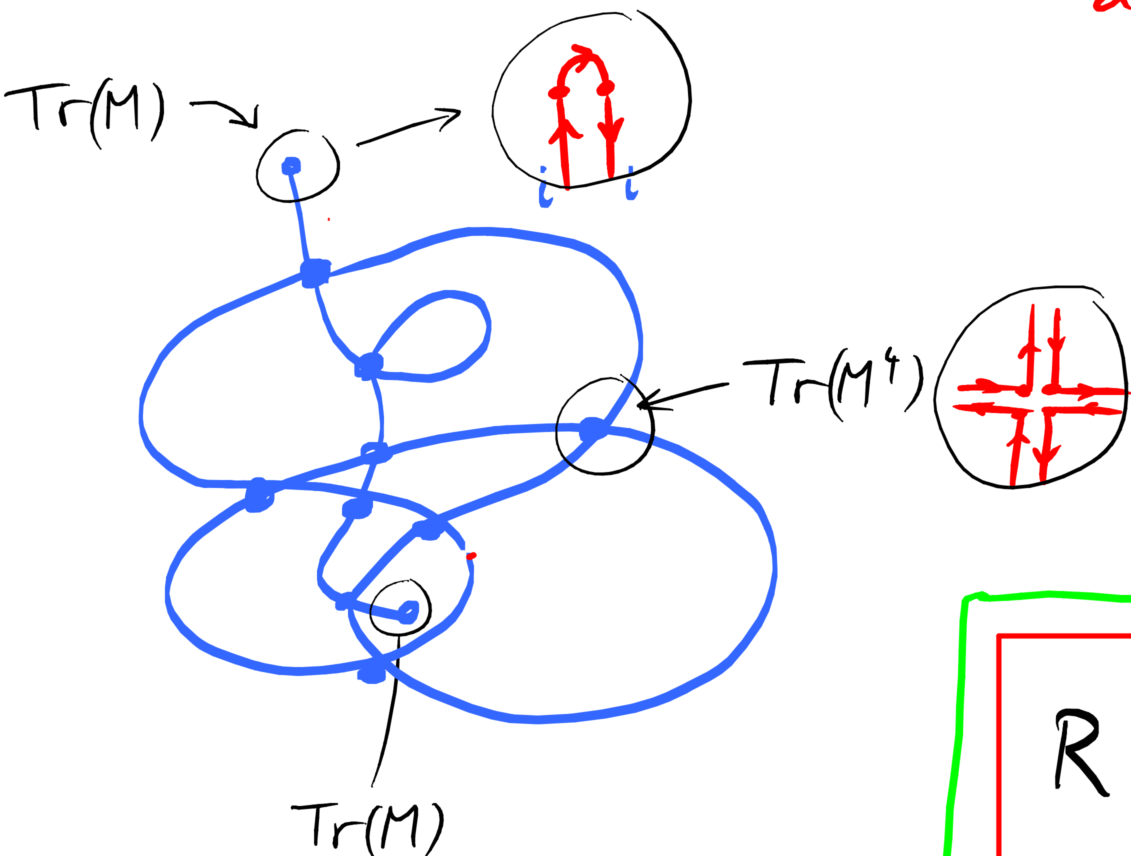
vertex $\mathrm{Tr}(M^{2k})$ 

3. Wick THM pairings of M_{ij}'s via propagators $\langle M_{ij} M_{kl} \rangle = \frac{\delta_{il}\delta_{jk}}{N}$ interpreted as edges M_{ij}  W

Ex: the 2-leg planar (genus 0) map gen. func.

$$R(\{g_{2k}\}) = \lim_{N\to\infty} \int [\mathrm{Tr}(M)]^2 \, d\mu(M)$$

$$d\mu(M) = d\mu_0(M)\, e^{+N\,\mathrm{Tr}\left(\sum_k g_{2k} \frac{M^{2k}}{2k}\right)}$$



THM R is the unique solution with formal power series expⁿ in the g's to the algebraic eqn:

$$\boxed{R = 1 + \sum_{k\geq 1} \binom{2k-1}{k} g_{2k} R^k}$$

Results :

- compute matrix integrals : orthogonal polynomial technique
- Integrability : Z = tau-function of the KdV hierarchy (Miwa vars $= g_{2k}/2k$) $\longrightarrow$ genus expansion

Missing :

- an explanation for too simple answers
- intrinsic geometry

COMPUTING THE MATRIX INTEGRAL

0. Problem = compute $Z = \int dM\, e^{-N \operatorname{Tr} V(M)}$

1. Go to eigenvalues $\quad M = U m U^{\dagger}; \; m = \operatorname{diag}(m_i)$

$U \in U(N)/U(1)^N \; ; \; m \in \mathbb{R}^N$.

$$dM = dU\, \Delta(m)^2 \quad ; \quad \Delta(m) = \prod_{i<j} (m_i - m_j) \text{ Vandermonde}$$

$$Z = \int d^N m\, \Delta(m)^2 e^{-N \operatorname{Tr} V(m)}$$

2. Use orthogonal polynomials $\quad d\mu(x) = dx\, e^{-NV(x)}$ on $\mathbb{R}$

- $\Delta(m) = \det_{1 \le i,j \le N} \left(p_{i-1}(m_j) \right)$

p_m = any monic polynomial of degree m.

COMPUTING THE MATRIX INTEGRAL

- Pick p_m orthogonal wrt: $d\mu(x) = dx\, e^{-NV(x)}$ on $\mathbb{R}$

namely $(p_m, p_n) = \int_{\mathbb{R}} d\mu(x)\, p_m(x)\, p_n(x) = h_n\, \delta_{m,n}$

$\uparrow$ norms

- Then: $Z = \sum_{\sigma,\tau \in \mathfrak{S}_N} \mathrm{sgn}(\sigma\tau) \int d\mu(m_1)\cdots d\mu(m_N) \prod_{i=1}^{N} p_{\sigma(i)-1}(m_i)\, p_{\tau(i)-1}(m_i)$

$$= \sum_{\sigma,\tau \in \mathfrak{S}_N} \mathrm{sgn}(\sigma\tau) \prod_{1}^{N} \left(p_{\sigma(i)-1}, p_{\tau(i)-1}\right)$$

$\sigma = \tau$ by orthogonality

$$= N! \prod_{i=1}^{N} h_{i-1}$$

COMPUTING THE MATRIX INTEGRAL

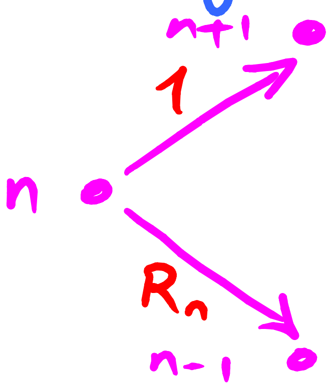
3. Recursion relations for h_i:

Q: operator x. on $p_n(x)$: $\quad x p_n(x) = (Q p_n)(x)$

P: operator $\frac{d}{dx}$ on $p_n(x)$: $\quad \frac{d}{dx} p_n(x) = (P p_n)(x)$

- Then: $\quad x p_n(x) = p_{n+1}(x) + R_n\, p_{n-1}(x) \qquad R_n = \dfrac{h_n}{h_{n-1}}$

(use self duality of Q: $\quad (Qp_n, p_m) = (p_n, Q p_m)$)



$\Rightarrow$ powers of Q described by paths.

COMPUTING THE MATRIX INTEGRAL

- $$(P p_n, p_{n-1}) = n h_{n-1} = \int p_n' \, p_{n-1} e^{-NV(x)} dx$$

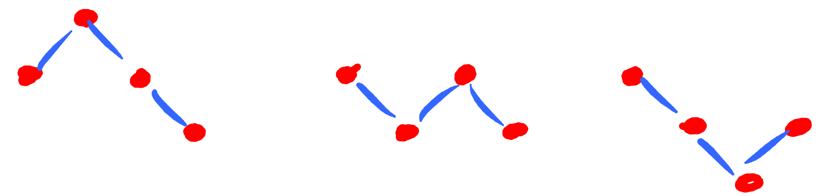
$$\underset{\text{by parts}}{=} \underbrace{\cancel{- \int p_n p_{n-1}' e^{-NV(x)}}}_{\text{by orthoganality}} + N \int V'(x) p_n p_{n-1} e^{-NV(x)} dx$$

$$= N(V'(Q) p_n, p_{n-1})$$

$$\Longleftrightarrow \quad \frac{n}{N} = \frac{(V'(Q) p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} = R_n - \sum_{k \geq 2} g_{2k} \underbrace{\frac{(Q^{2k-1} p_n, p_{n-1})}{(p_{n-1}, p_{n-1})}}_{\substack{\text{Polynomial of} \\ R_n,\, R_{n\pm1}, \ldots, R_{n\pm(k-1)} \\ \text{of degree } k.}}$$

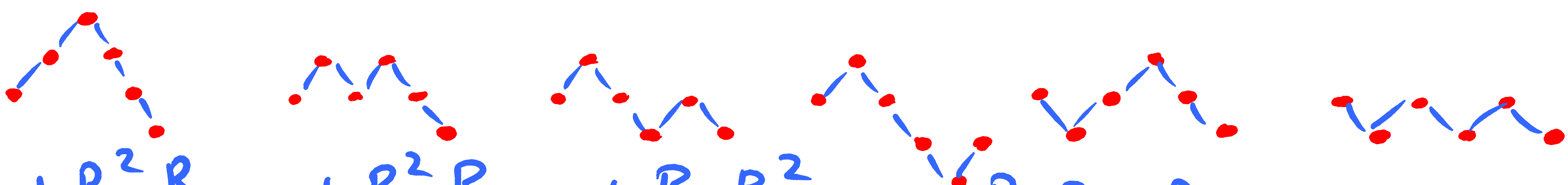
Examples

$k=2$ (4 valent) : $\dfrac{(Q^3 p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} = R_{n+1}R_n + R_n^2 + R_n R_{n-1}$

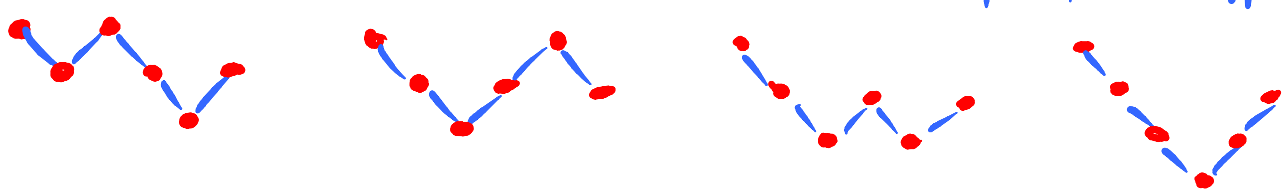


$k=3$ (6 valent) : $\dfrac{(Q^5 p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} =$

$R_{n+2}R_{n+1}R_n + R_n^2 R_{n-1} + R_n R_{n-1}^2 + R_{n+1}R_n R_{n-1} + R_n^2 R_{n+1} + R_n^3$



$+ R_n^2 R_{n-1} + R_n^2 R_{n-1} + R_n R_{n-1}^2 + R_n R_{n-1} R_{n-2}$



$\binom{5}{3} = 10$ terms

COMPUTING R

$$R = \int (\operatorname{Tr} M)^2 \, d\mu(M) = \frac{1}{Z}\int d^N m \, \Delta(m)^2 \Big(\underbrace{\sum_1^N m_i^2}_{N \text{ terms}} + \underbrace{\sum_{i \neq j} m_i m_j}_{N(N-1) \text{ terms}} \Big) e^{-N \operatorname{Tr} V(m)}$$

- $$\int m_1^2 \, d\mu(M) = \frac{1}{Z} \sum_{\sigma,\tau \in S_N} \operatorname{sgn}(\sigma\tau) \left(m_1^2 P_{\sigma(1)-1}(m_1), P_{\tau(1)-1}(m_1)\right) \prod_2^N \left(P_{\sigma(i)-1}, P_{\tau(i)-1}\right)$$

$\sigma = \tau$ by orthogonality

$$= \frac{1}{Z} \sum_{i=0}^{N-1} \frac{(Q^2 P_i, P_i)}{(P_i, P_i)} = \sum_0^{N-1} (R_{i+1} + R_i)$$

- $$\int m_1 m_2 \, d\mu(M) = \frac{1}{Z} \sum_{\sigma,\tau} \underset{= -1}{\operatorname{sgn}(\sigma\tau)} \Big(m_1 P_{\sigma(1)-1}, \underset{=}{P_{\tau(1)-1}}\Big)\Big(m_2 P_{\sigma(2)-1}, \underset{= \sigma(1)}{P_{\tau(2)-1}}\Big) \prod_3^N \left(P_{\sigma(i)-1}, P_{\tau(i)-1}\right)$$

$\Rightarrow \sigma(2) = \sigma(1) \pm 1$

$\sigma = \tau$ on $[3, N]$

$$= -2 \sum_0^{N-1} R_i$$

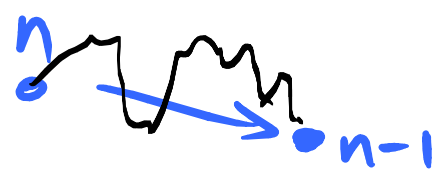
<u>Total</u> $= R_N \xrightarrow[N \to \infty]{} R$ and theorem follows.

COMPUTING R

$$R_N \to R = \underset{\left(\frac{n}{N} \text{ at } n=N\right)}{1} + \sum_{k\geq 2} g_{2k} \, R^k \binom{2k-1}{k}$$

$$\downarrow$$

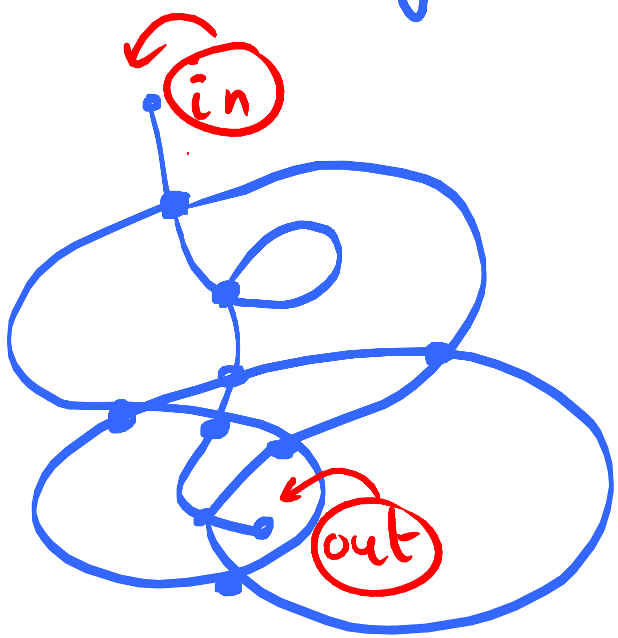
(# paths of $2k-1$ steps from

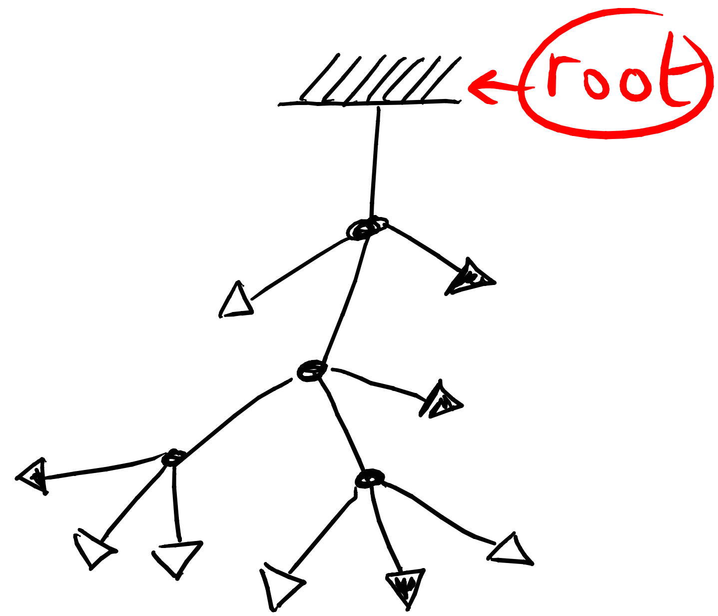


w/ weight R / descent.)

A COMBINATORIAL APPROACH

{tetravalent planar maps with 2 legs} $\longleftrightarrow$ {rooted 4-valent blossom trees with 1 $\downarrow$ at each vertex}





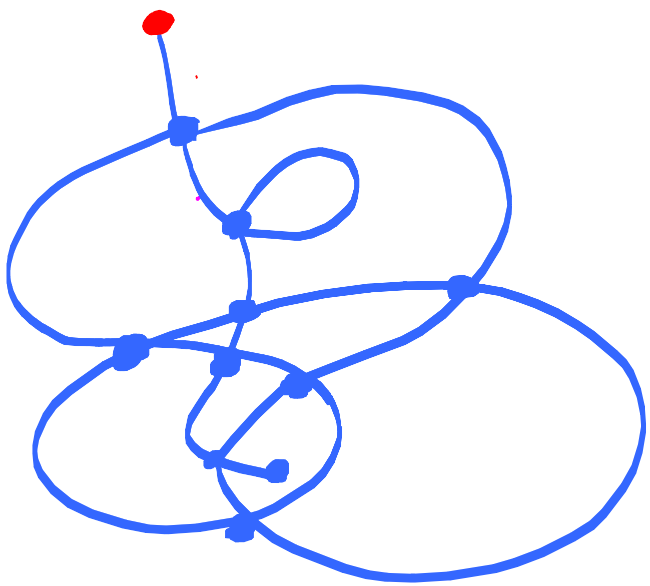
- Explains algebraic eqn (TREES).
- Allows to track distance between legs

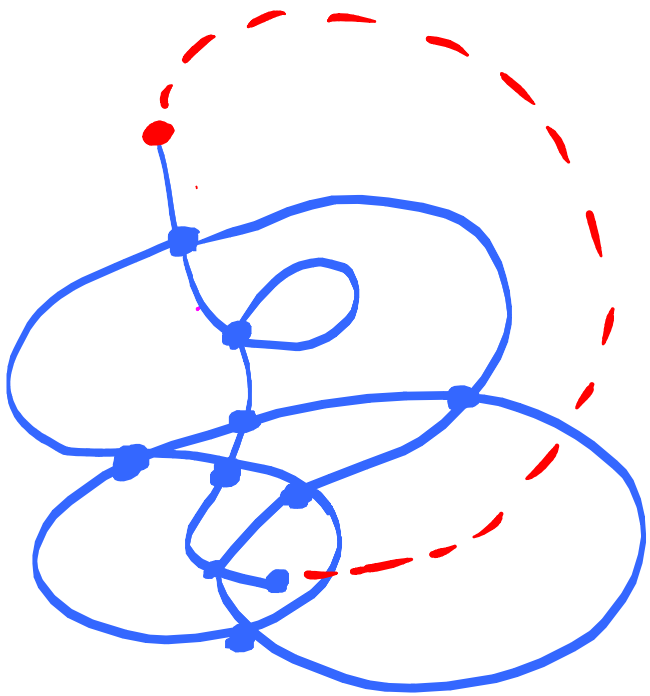
- Model for discrete random surfaces

$\downarrow$

Planar map

- tetravalent
- 2 legs



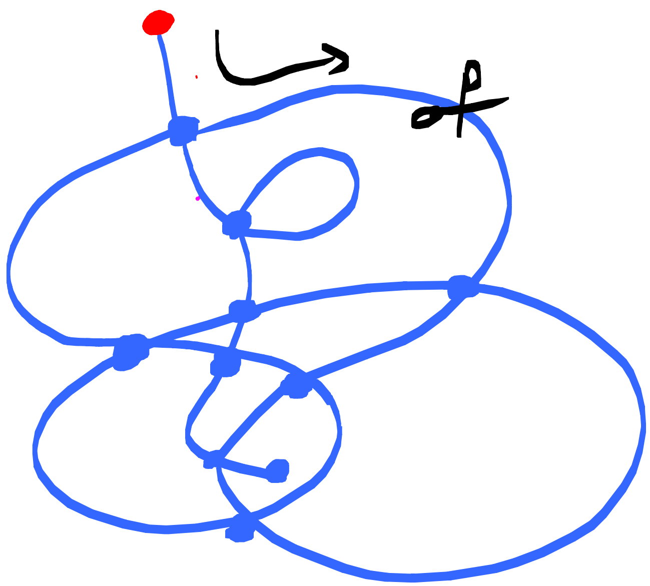


- intrinsic geometry

Planar map

- tetravalent
- 2 legs
- geodesic distance between the legs = 2 here

MAP-TREE BIJECTION



- cutting algorithm

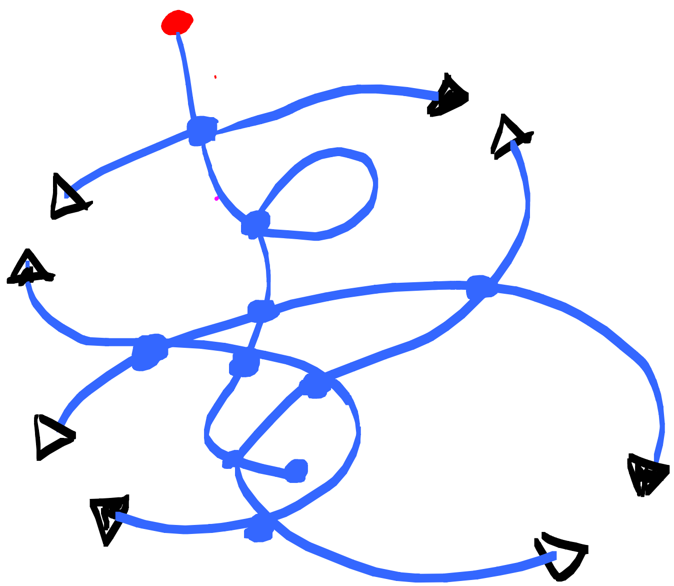
(A) 1. walk along border of external face

2. for each edge, cut if it doesn't disconnect the graph

3. replace

   with

MAP-TREE BIJECTION



- cutting algorithm

A 1. walk along border of external face

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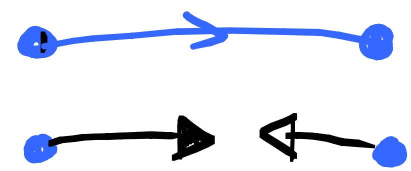
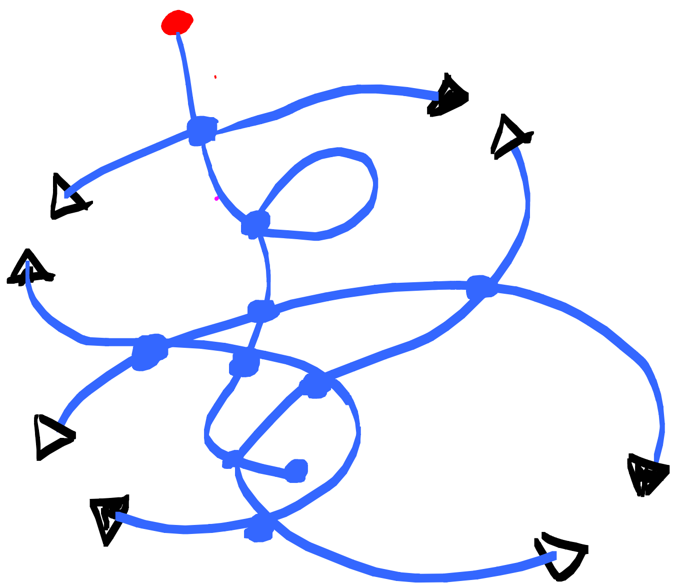


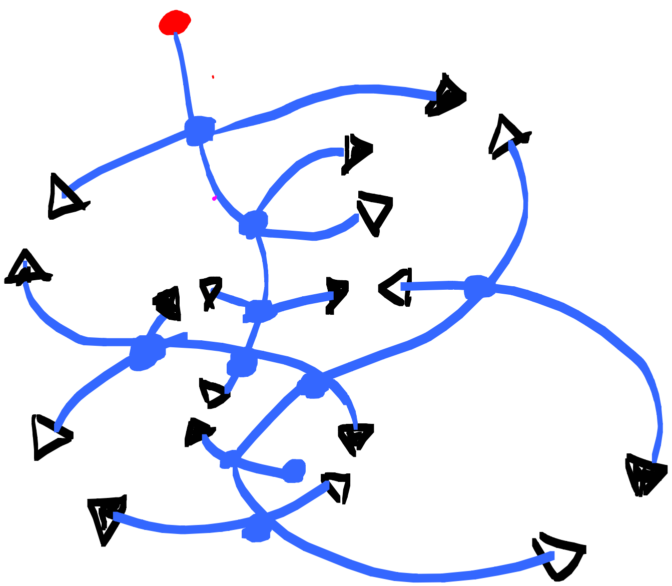
MAP-TREE BIJECTION



- cutting algorithm

(B) repeat (A) with this new planar map

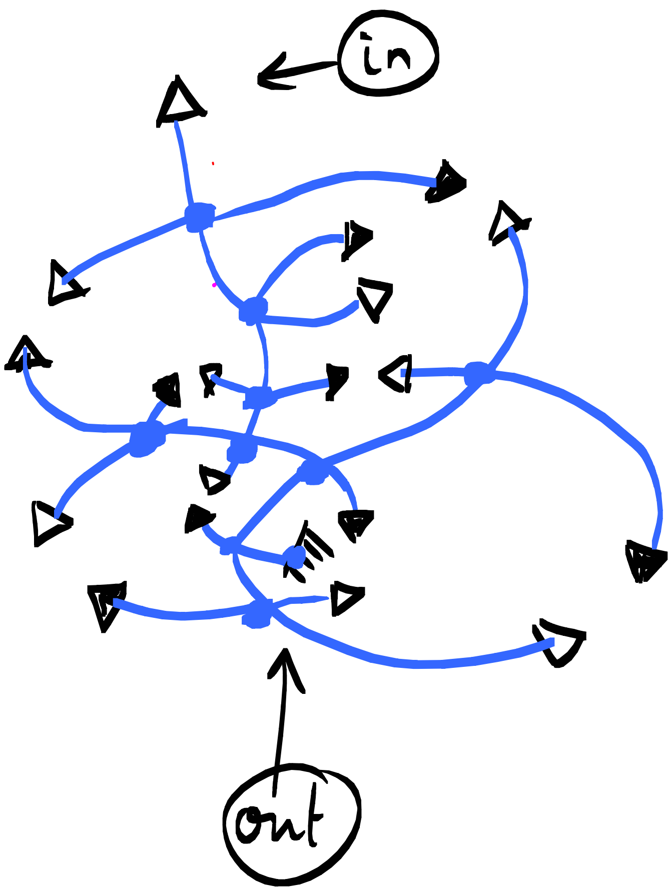
MAP-TREE BIJECTION



- cutting algorithm

(B) repeat (A) with this new planar map etc. until we have a tree (only 1 face).

MAP-TREE BIJECTION



Change the out vertex into a root

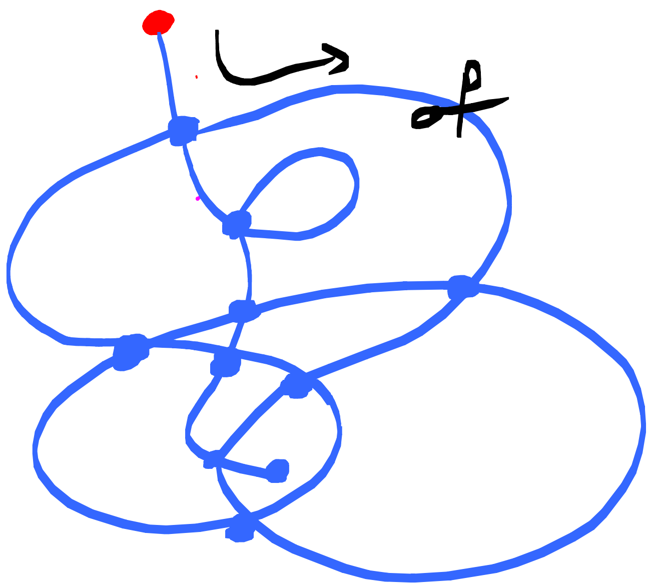
in vertex into a ↓

→ Blossom-tree = tree with ↓ and ↓ leaves and

(i) #(↓) = #(↓) + 1

(ii) exactly 1 ↓ at each inner vertex

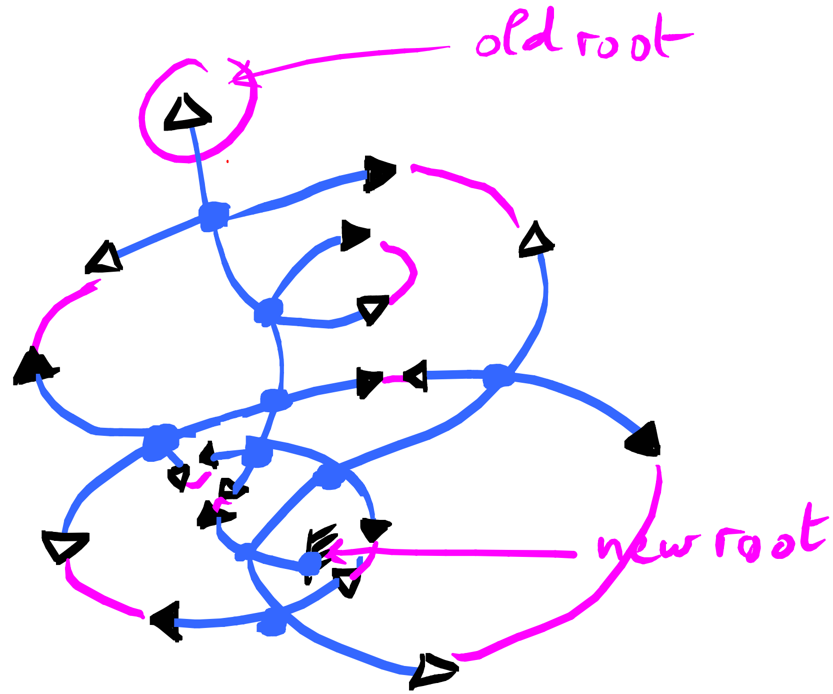
MAP-TREE BIJECTION



bijection

$\longleftrightarrow$

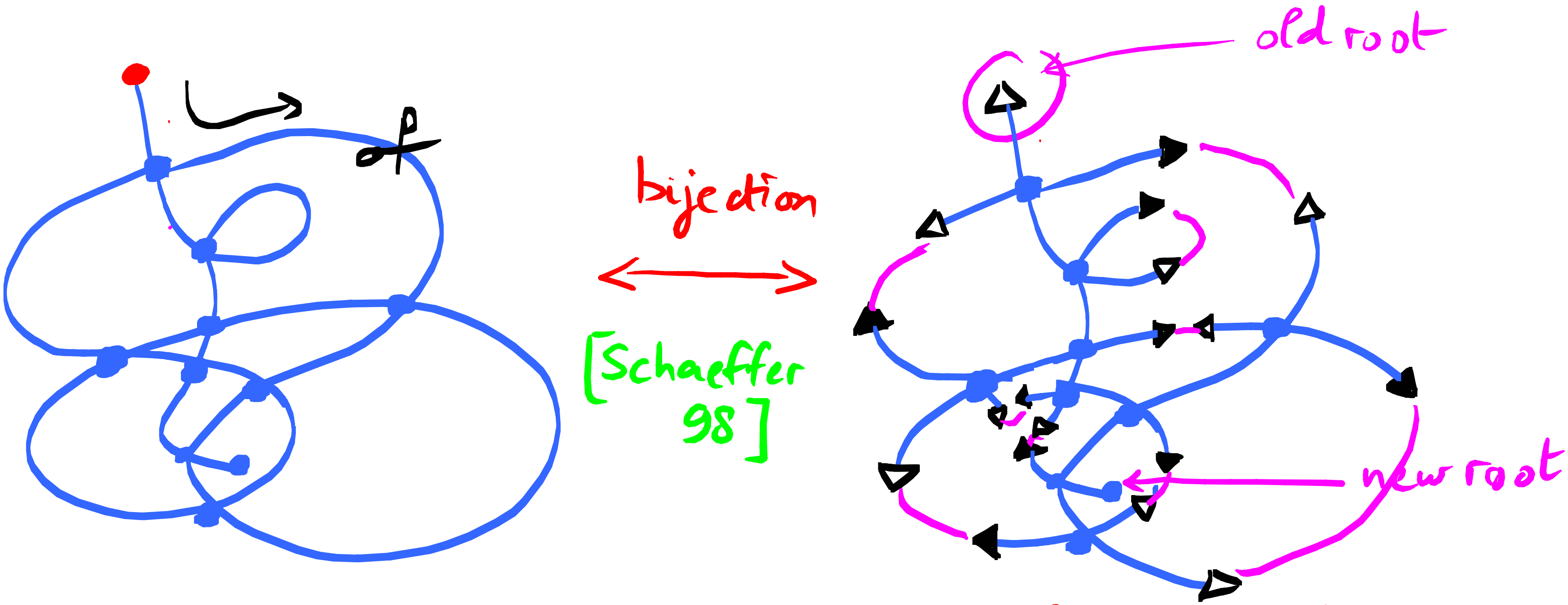
[Schaeffer 98]



"change" $= \# \downarrow_{\triangledown} - \# \downarrow_{\blacktriangledown} = +1$

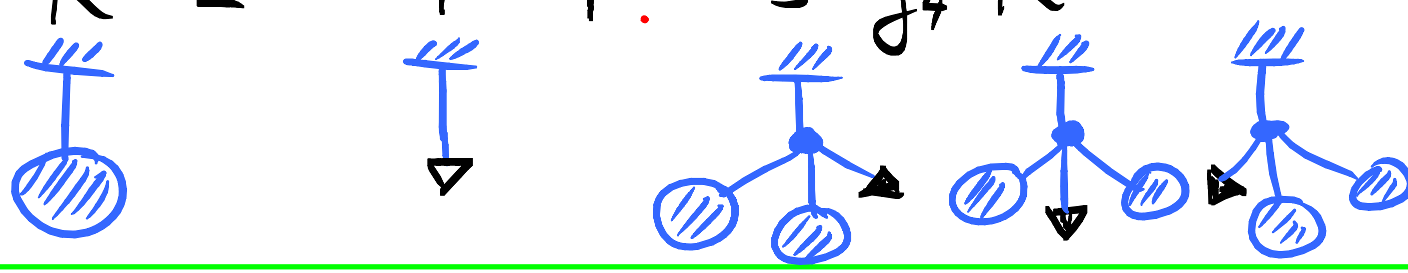
Prop change of any descendant subtree is $+1$

MAP-TREE BIJECTION



R = generating function for rooted blossom trees

$$R = 1 + 3 g_4 R^2$$

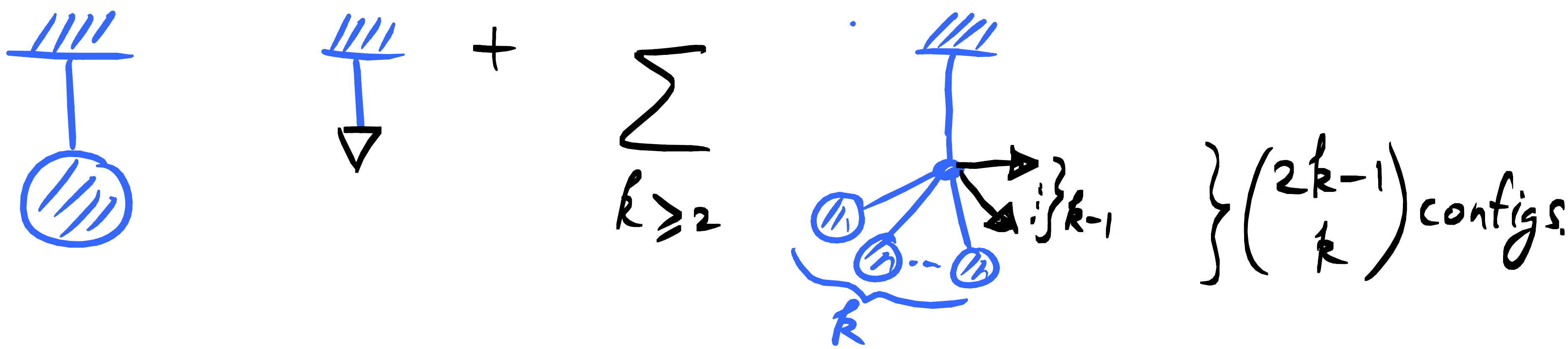


Generalization to arbitrary even valences $g_2, g_4, \ldots$

- same cutting algorithm

$\{$two-leg planar maps$\}$ $\longleftrightarrow$ $\left\{\begin{array}{l}\text{rooted planar blossom} \\ \text{trees with exactly} \\ k-1 \downarrow \text{ at each} \\ 2k\text{-valent vertex}\end{array}\right\}$

$$R = 1 + \sum_{k\geq 2} g_{2k} \binom{2k-1}{k} R^k$$



CRITICAL POINTS

Pure gravity:

$$R = 1 + 3g R^2 \Rightarrow R = \frac{1-\sqrt{1-12g}}{6g}$$

$g_c = \frac{1}{12}$ = critical point

$$\#\text{ maps with } n \text{ vertices} = \frac{3^n}{n+1}\binom{2n}{n}$$

$$\sim \frac{1}{\sqrt{\pi}} \frac{12^n}{n^{3/2}}$$

$$\#\text{ unmarked maps} \sim \frac{12^n}{n^{7/2}} = \frac{\mu^n}{n^{3-\gamma_{st}}}$$

$\gamma_{st} = -\frac{1}{2}$ string susceptibility. $R_c - R \sim (g_c - g)^{-\gamma_{st}}$

CRITICAL POINTS

General maps with even valence:

$$g_4 = g \;;\; g_{2k} = \gamma_{2k}\, g^{k-1} \;;\; \rho = g R$$

$$\Rightarrow \quad g = \varphi(\rho) = \rho - \sum_2^d \gamma_{2k}\, \rho^k$$

- Critical point: if γ_{2k} generic g_c such that $\begin{cases} \varphi'(\rho_c) = 0 \\ g_c = \varphi(\rho_c) \end{cases}$

  $\gamma_{str} = -\frac{1}{2}$ as $g_c - g \sim \frac{(\rho_c - \rho)^2}{2} \varphi''(\rho_c)$

- Multicritical point: γ_{str} fine-tuned $g_c = \varphi(\rho_c)$

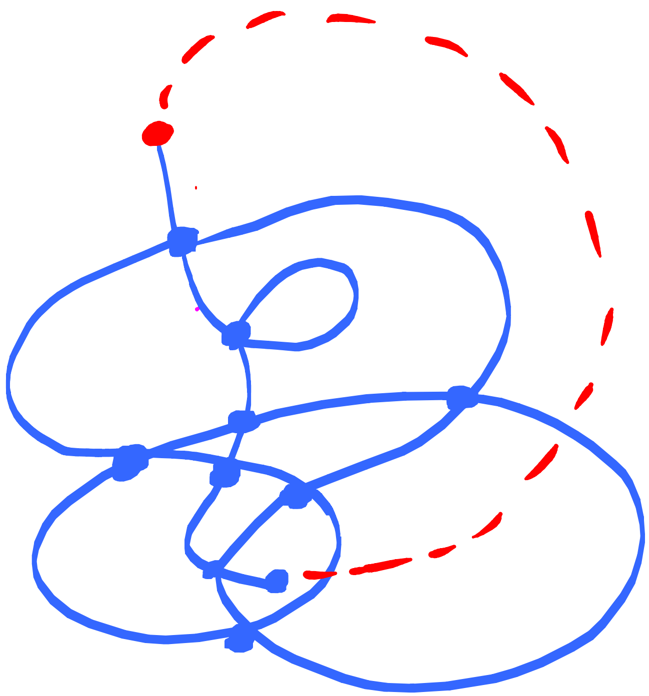
  $\varphi'(\rho_c) = \varphi''(\rho_c) = \dots = \varphi^{(m)}(\rho_c) = 0$

  then $\gamma_{str} = -\frac{1}{m+1}$ as $g_c - g \sim \frac{(\rho_c - \rho)^{m+1}}{(m+1)!} \varphi^{(m+1)}(\rho_c)$

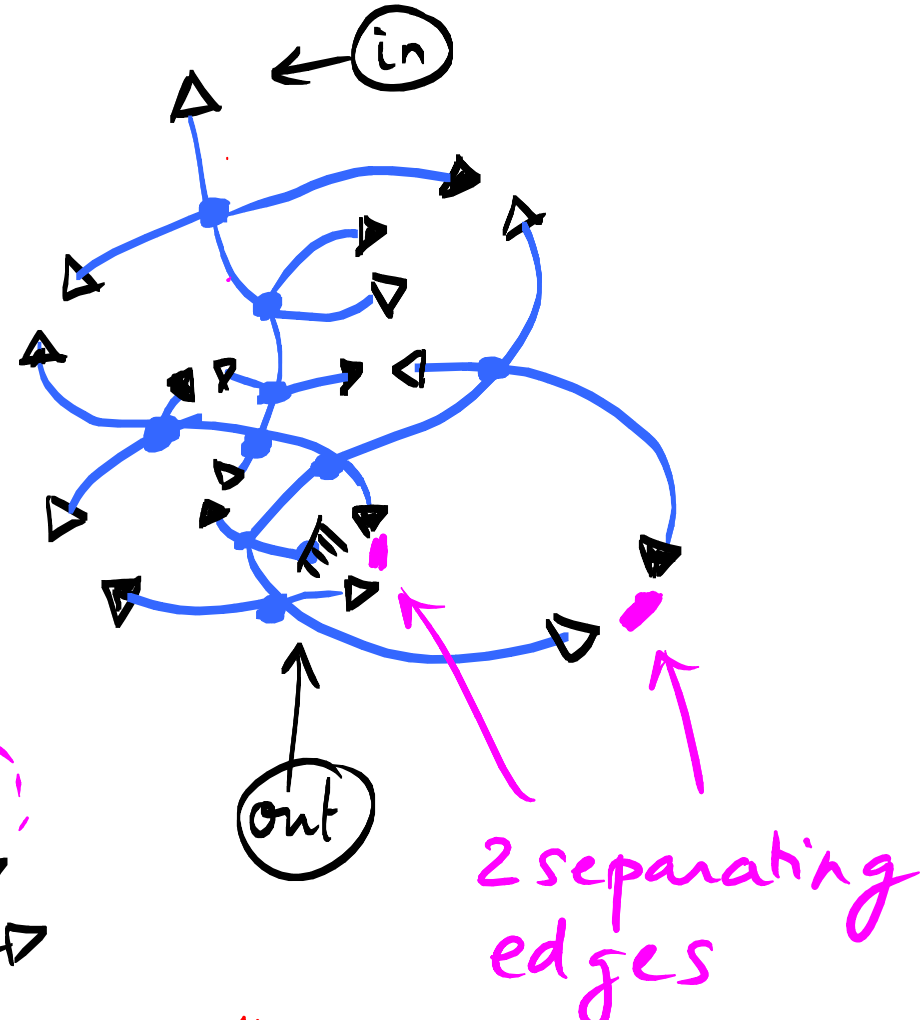
$$\boxed{\#\,\text{maps}_{n\text{ vertices}} \sim g_c^{-n} / n^{3-\gamma_{str}}}$$

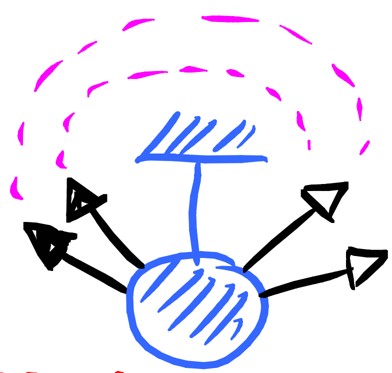
another universality class!

GEODESIC DISTANCE:



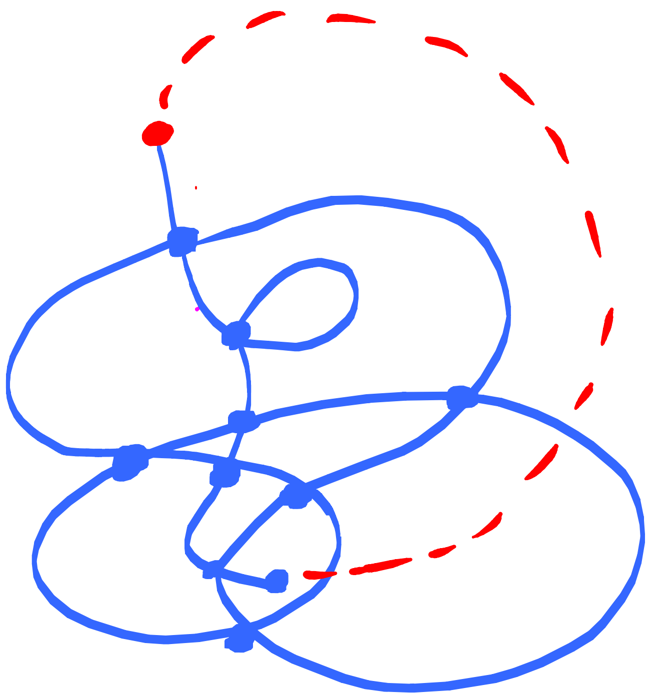
$d = 2$



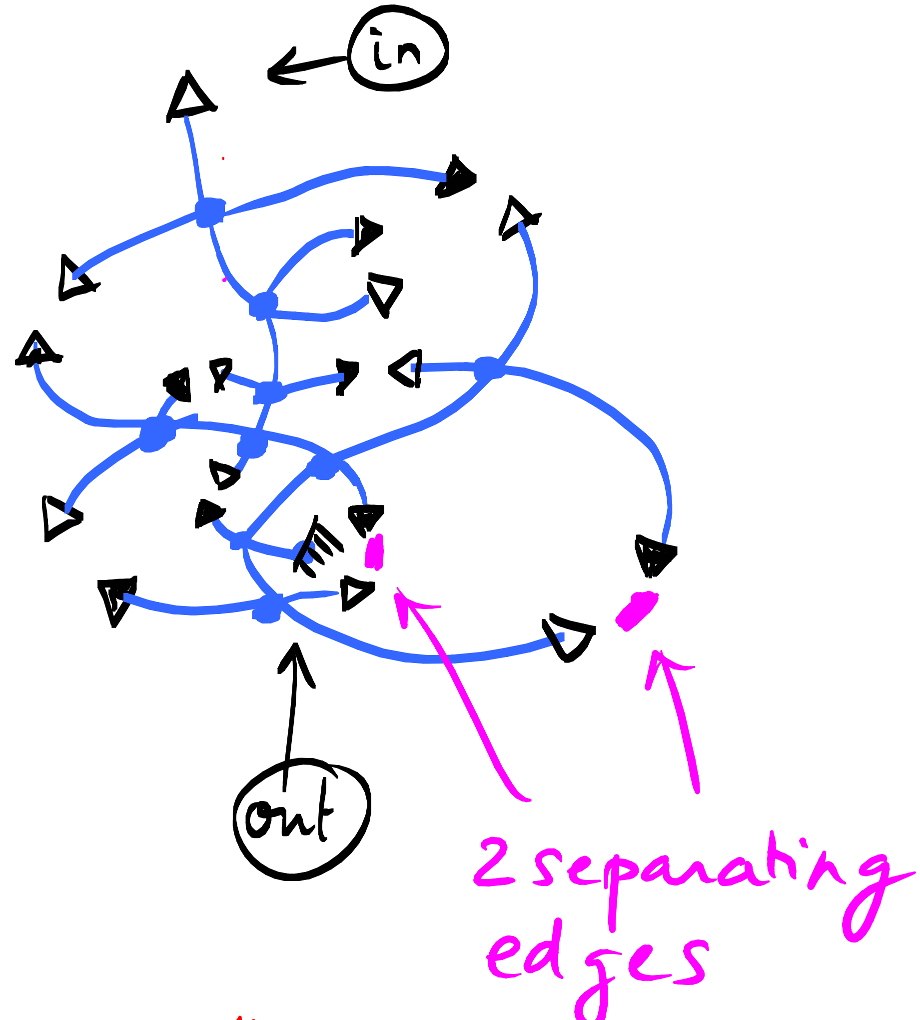


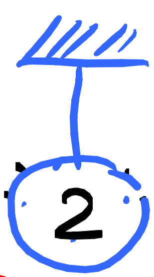
TWO "EXCESS BLACK LEAVES" ON THE LEFT

GEODESIC DISTANCE:



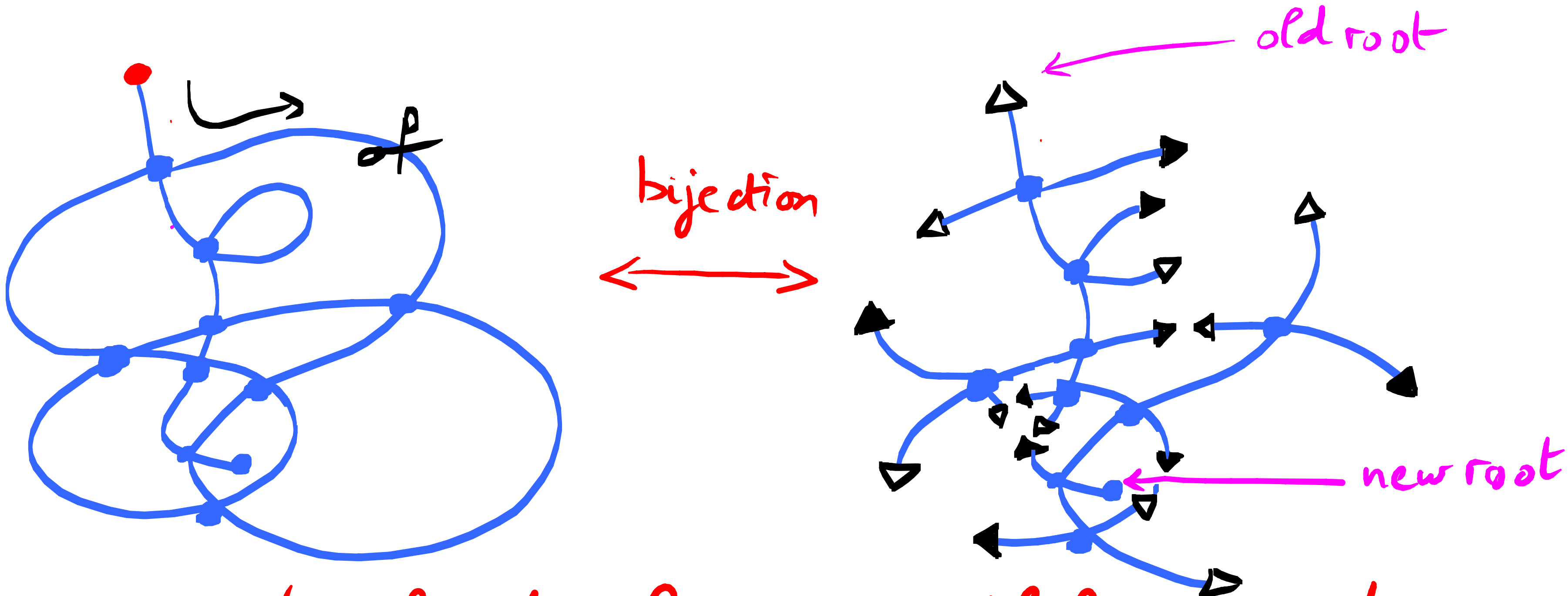
$d = 2$





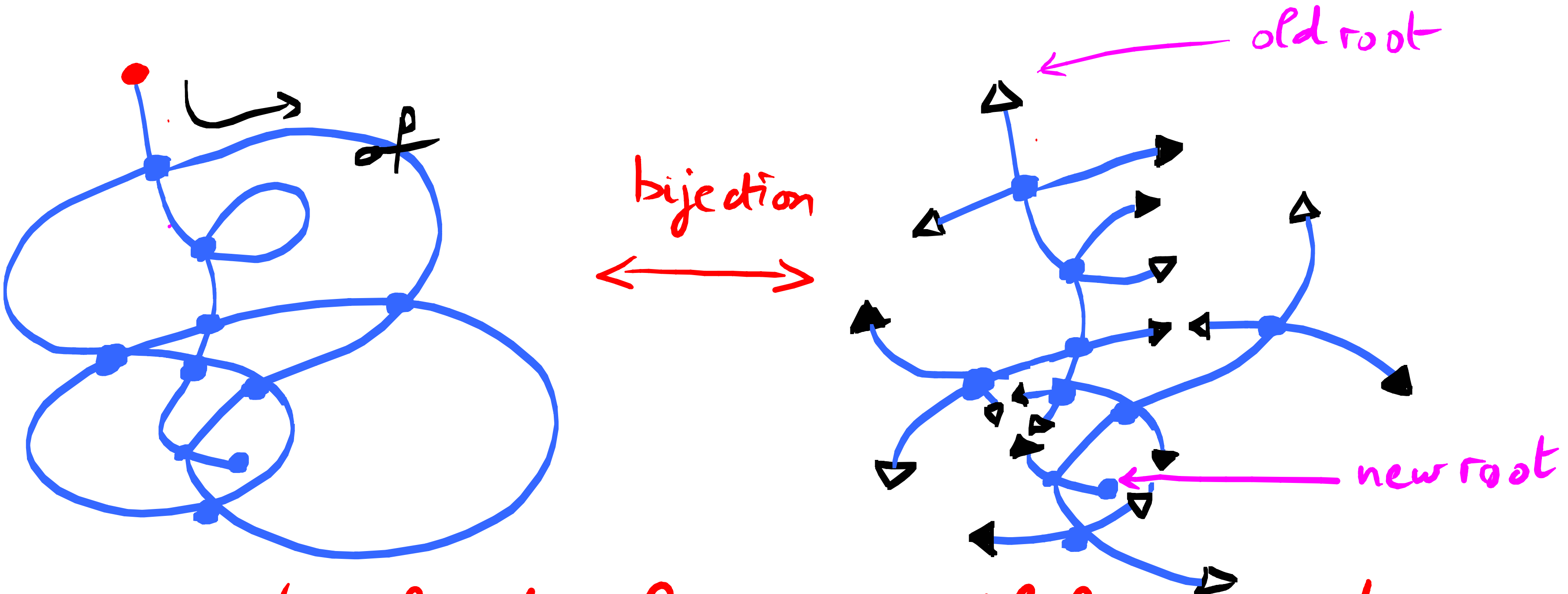
TWO "EXCESS BLACK LEAVES" ON THE LEFT

MAP - TREE BIJECTION



R_n = generating function for maps with legs at distance $\leq n$

MAP-TREE BIJECTION (4-valent case)



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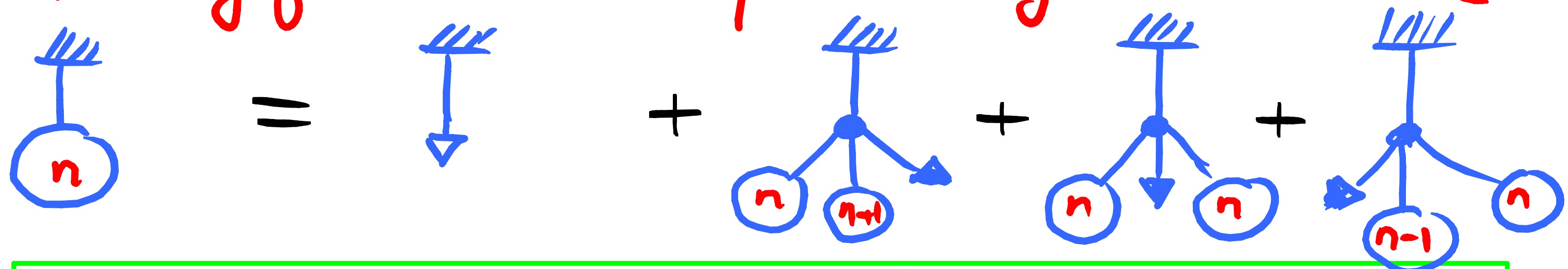

$$\boxed{R_n = 1 + g R_n \left(R_{n+1} + R_n + R_{n-1} \right)}$$

INTEGRABILITY

$$0 = R_n - 1 - gR_n(R_{n+1} + R_n + R_{n-1}) \quad (*) \qquad (R_{-1} = 0,\ R_\infty = R)$$

- This is a discrete integrable equation (n = time)

$$\boxed{\begin{array}{l} \exists\ \varphi(x,y) = xy(1-g(x+y)) - x - y \ \text{ such that} \\ \varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1}) \times (*) \end{array}}$$

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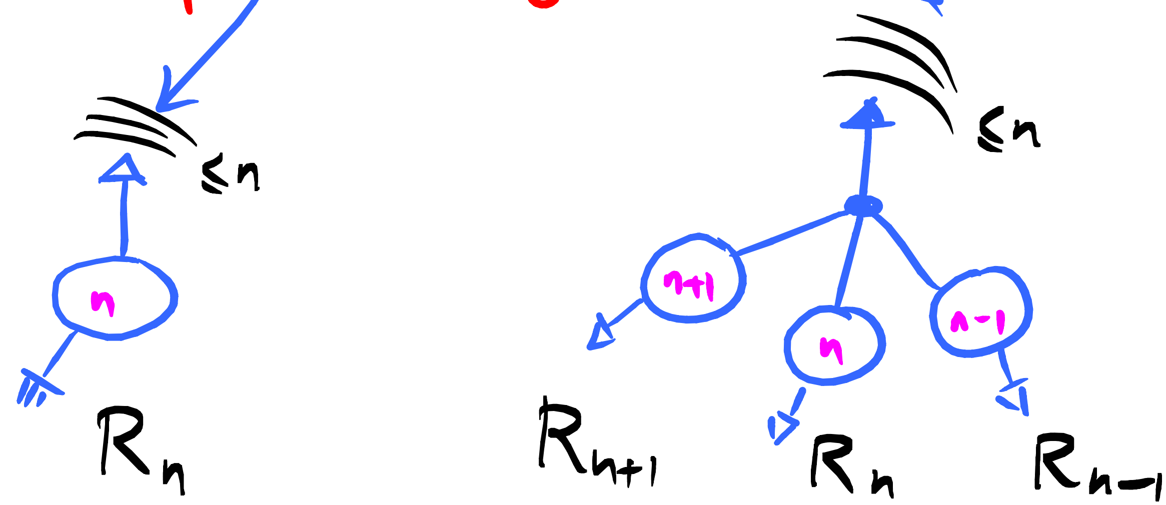
Discrete 1st integral of motion

CONSERVATION LAW

Depth of a leaf = distance to external face

Conservation law : $\#(\triangledown \ \text{depth} \le n) - \#(\blacktriangledown \ \text{depth} \le n) = 1$

is independent of n!



$$\boxed{R_n - g R_{n+1} R_n R_{n-1} = R_0}$$ (indep of n).

equivalent to $\varphi(R_n, R_{n+1}) = $ const modulo $(*)$

SOLUTION

use $R_\infty = R$ and expand $R_n = R(1-\rho_n)$

ρ_n "small" $\to 0$ as $n\to\infty$. Linearize eqn arnd R:

$$R(1-\rho_n) - 1 - g\,R^2(1-\rho_n)(3-\rho_{n+1}-\rho_n-\rho_{n-1}) = 0$$

$$-R\rho_n + gR^2(4\rho_n+\rho_{n+1}+\rho_{n-1}) = 0$$

look for $\rho_n \sim x^n$ then $\boxed{x+\frac{1}{x}+4=\frac{1}{gR}}$

- write higher orders:

$$R_n = R(1-\rho_n^{(1)} - \rho_n^{(2)} \cdots)$$

$d_1 x^n$ $\quad d_2 x^{2n}$ ... recursion relations

$\to$ The series sums nicely...

INTEGRABILITY & SOLUTION

$$0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) \quad (*) \qquad (R_{-1}=0,\ R_\infty = R)$$

- Exact soliton solution [DF, Bouttier, Guitter 05-07]

$$\boxed{\begin{aligned} R_n &= R\, \frac{1-x^{n+1}}{1-x^{n+2}}\, \frac{1-x^{n+4}}{1-x^{n+3}} \\ R &= 1 + 3gR^2 \\ x + \frac{1}{x} + 4 &= 1/gR, \quad |x|<1. \end{aligned}}$$

- $d_F = 4$
- scaling functions
- continuum limit [Le Gall, Miermont, Bouttier, Guitter, ...]

- other boundary conditions

u_n = elliptic function of n $\qquad$ x = angle variable

FRACTAL DIMENSION

$$\frac{R_n|_{g^N}}{R_0|_{g^N}} = \#\{\text{vertices in a ball of radius } n \text{ in maps w/ } N \text{ vertices}\}$$

$$\sim \frac{3}{56} n^4 \quad (N \to \infty)$$

$$d_F = 4$$

Proof: $R_n = R \dfrac{1-x^{n+1}}{1-x^{n+2}} \dfrac{1-x^{n+4}}{1-x^{n+3}}$

set $v = gR \Rightarrow R = \dfrac{1}{1-3v}$; $x = \dfrac{1-4v-\sqrt{(1-2v)(1-6v)}}{2v}$.

$$R_n|_{g^N} = \oint \frac{dg}{2i\pi g^{N+1}} R_n(g)$$

use variable v, do saddle pt around critical value $v_c = \frac{1}{6}$.

$v = v_c\left(1 + i\,\xi/\sqrt{N}\right)$

SCALING

near critical maps (4valent case) $g = g_c(1-\varepsilon^4)$ $g_c = \frac{1}{12}$

$$\Rightarrow \begin{cases} R = 2(1-\varepsilon^2) \\ x = 1-\sqrt{6}\,\varepsilon \end{cases} \qquad x^n \sim e^{-n\sqrt{6}\,\varepsilon}$$

has a nice limit if $r = n\varepsilon$ finite (= renormalized geodesic distance). $(d_F = 4)$

$\mathcal{F}(r)$ = partition function for continuous random surface with 2 marked pts at distance $\geq r$

$$= \lim_{\varepsilon \to 0} \frac{R - R_n}{\varepsilon^2 R} = \frac{3}{\sinh^2\left(\sqrt{\frac{3}{2}}\, r\right)}$$

$\mathcal{G}(r)$ = idem at distance $r = -\frac{\partial \mathcal{F}}{\partial r} = 3\sqrt{6}\,\frac{\cosh(\sqrt{3/2}\, r)}{\sinh(\sqrt{3/2}\, r)^3}$

- Alternatively, start from: $R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$

and insert $R_n = R\left(1 - \varepsilon^2 \mathcal{F}(r = n\varepsilon)\right)$, expand in ε.

$(= R_c(1 - \varepsilon^2(1+\mathcal{F})))$

order 4 in ε:
$$\begin{cases} \mathcal{F}''(r) - 3\mathcal{F}(r)^2 - 6\mathcal{F}(r) = 0 \\ \lim_{r\to 0} \mathcal{F}(r) = \infty \qquad \lim_{r\to\infty} \mathcal{F}(r) = 0 \end{cases}$$

- set $u = 1 + \mathcal{F}$ then $\boxed{u^2 - \frac{u''}{3} = 1}$

(Painlevé I)

SCALING

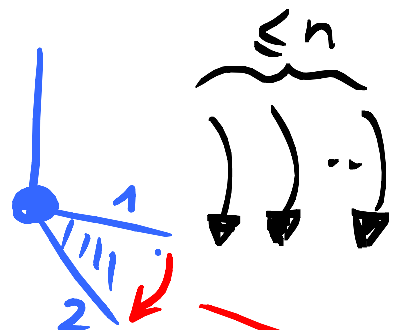
- $$R_n / g^N \sim \frac{12^N}{N^{3/2}} \int_0^{+\infty} d\xi \, \xi^2 e^{-\xi^2} \mathcal{U}\left(r\sqrt{-i\xi}\right)$$

$$\Rightarrow \boxed{P(r) = \int_0^{+\infty} d\xi \, \xi^2 e^{-\xi^2} \mathcal{U}\left(r\sqrt{-i\xi}\right)}$$

is the probability that dist $\geq r$ in random surfaces with 2 marked points.

Generalization to arbitrary even valences

Describe the local environment of a vertex in a blossom-tree via :

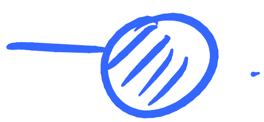


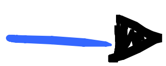
corner transfer operator

→ state $|n\rangle$

→ operator action $Q|n\rangle = R_n\,|n-1\rangle + |n+1\rangle$

config 1 is

either 

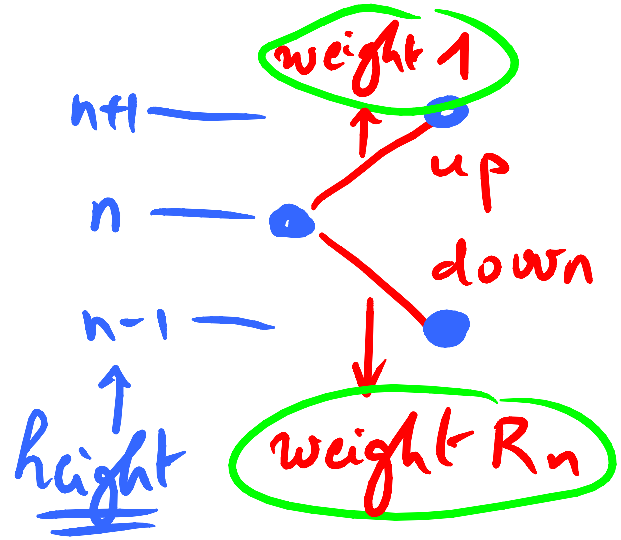
or 

$$\boxed{\begin{array}{l}\text{Recursion relation:}\\ R_n = 1 + \sum_{k\geq 1} g_{2k}\langle n-1|Q^{2k-1}|n\rangle\end{array}}$$

PATH INTERPRETATION OF Q :

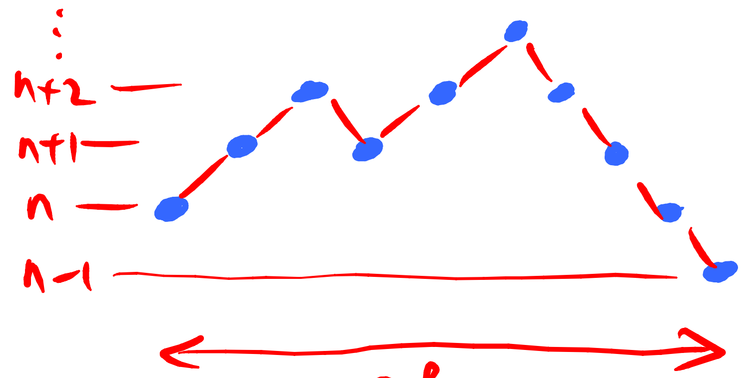
$$Q|n\rangle = |n+1\rangle + R_n|n-1\rangle$$

$\langle n|$ dual basis: $\langle m|p\rangle = \delta_{m,p}$



$$\langle n-1|Q^{2k-1}|n\rangle$$

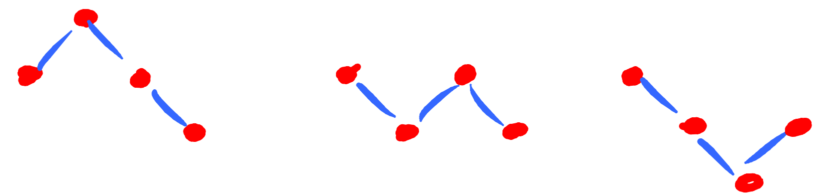
$$= \sum_{\substack{\text{paths}\\ n\to n-1}}$$



$$= \mathrm{Pol}(R_n, R_{n\pm1}, R_{n\pm2} \ldots)$$

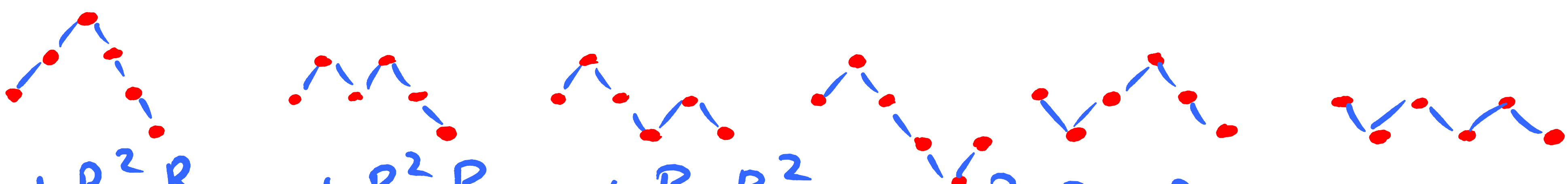
Examples

$k=2$ (4 valent) : $\langle n-1|Q^3|n\rangle = R_{n+1}R_n + R_n^2 + R_nR_{n-1}$

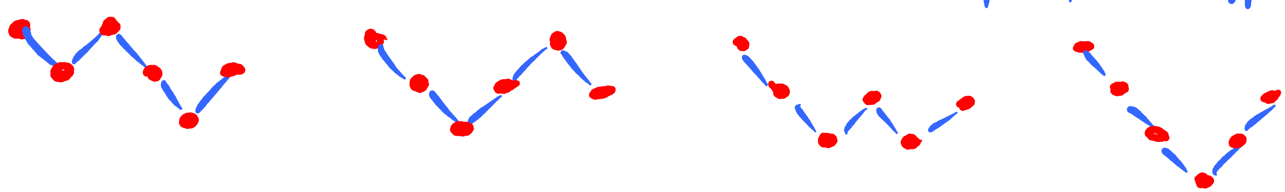


$k=3$ (6 valent) : $\langle n-1|Q^5|n\rangle =$

$$R_{n+2}R_{n+1}R_n + R_n^2R_{n-1} + R_nR_{n-1}^2 + R_{n+1}R_nR_{n-1} + R_n^2R_{n+1} + R_n^3$$



$$+ R_n^2R_{n-1} + R_n^2R_{n-1} + R_nR_{n-1}^2 + R_nR_{n-1}R_{n-2}$$



$\binom{5}{3} = 10$ terms

$$\boxed{R_n = 1 + \sum_{m \geq k \geq 2} g_{2k} \langle n-1 | Q^{2k-1} | n \rangle}$$

$$Q|n\rangle = |n+1\rangle + R_n |n-1\rangle$$
$$\langle m | n \rangle = \delta_{m,n}$$

THM This is integrable

THM This has the following exact solution ((m)-soliton solution)

$$R_n = R \frac{u_n \, u_{n+3}}{u_{n+1} \, u_{n+2}}$$

$$u_n = \det_{1 \leq i,j \leq m} \left(\delta_{ij} - \lambda_i x_j^{n+2i-2} \right)$$

[m discrete soliton τ-function]

$|x_i|<1$; $\left(x_1, x_2, \ldots, x_m \;,\; \frac{1}{x_1}, \frac{1}{x_2}, \ldots \frac{1}{x_m}\right)$

= roots of the linearized characteristic equation (obtained by expanding the equation for $R_n = R(1-\alpha x^n)$ at first order in α).

- Access to fractal properties by taking a critical continuum limit
- Scaling 2pt function obeys Painlevé eqn.

MULTICRITICAL 2pt FUNCTION

- fine-tune γ_{2k} to get to $\gamma_{str} = -\frac{1}{m+1}$
- m of the x_i tend to 1 in the multicritical limit

$$\mathcal{F}(r) = -2\frac{\partial^2}{\partial r^2}\log W\left(\{\sinh(a_i r)\}_{1\le i\le m}\right)$$

W: Wronskian; a_i: determined by γ_{2k}.

$$\mathcal{G}(r) = -\mathcal{F}'(r)$$

(2pt function of multicritical random surface with 2 marked pts at dist r)

obeys higher diff^{al} eqn of the KdV hierarchy

• Differential equation for $\mathcal{U} = 1 + \mathcal{F}$

$$R_{m+1}[\mathcal{U}] = R_{m+1}[1] \quad \text{where}$$

$$\mathrm{Res}\left(\left(d^2 - \mathcal{U}\right)^{m+\frac{1}{2}}\right) = R_{m+1}[\mathcal{U}].$$

Ex: • $\mathcal{U}'' - 3\mathcal{U}^2 = -3 \quad (m=1)$

• $\mathcal{U}^{(4)} - 10\,\mathcal{U}\,\mathcal{U}'' - 5(\mathcal{U}')^2 + 10\,\mathcal{U}^3 = 10 \quad (m=2)$

Obtained by writing $[P, Q] = 0$

$$Q = d^2 - \mathcal{U}\,; \quad P = d^{2m+1} + a_1 d^{2m-1} + \cdots + a_{2m}$$

INTEGRABLE BRANCHING PROCESSES

Re-consider the eqn : $R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$

and write it as $$R_n = \frac{1}{1 - g(R_{n+1} + R_n + R_{n-1})}$$

- R_n = g.f of vertex-labeled trees with root label n and the "sicilian mama" rule = children stay close enough
  $\text{child}(n) \in \{n, n \pm 1\}$. n = position/embedding
- $R_{-1} = 0$; $\lim_{n \to \infty} R_n = R$

Branching process : $\left\{ \begin{array}{l} (1) \text{ arbitrary planar tree} \\ (2) \text{ embedding in } \mathbb{Z}. \end{array} \right.$

- genealogical tree
- proba(k children) $= (1-p)\,p^k \qquad k=0,1,2\ldots$
- diffusion of position: proba $\frac{1}{3}$ of $(n+1, n$ or $n-1)$ if parent at n
- $E_n(t)$ = Proba of extinction of a family in time $\leq t$

$$E_n(t) = \frac{1-p}{1-\frac{p}{3}\left(E_{n+1}(t-1)+E_n(t-1)+E_{n-1}(t-1)\right)} \; ; \; E_n(0)=0$$

$$R_n = \frac{E_n(\infty)}{1-p}$$

$\rightarrow$ proba of extinction of a family sitting at n at the first generation, and staying at $n \geq 0$ (presence of a wall).

$$g = \frac{p(1-p)}{3}$$

- Note: $S_n = 1-(1-p)R_n$ = probability of survival

- g_c is attained at $p = p_c = \frac{1}{2}$ (GW tree).

- 1 wall $n \geq 0 \Rightarrow$ $\boxed{\begin{cases} S_n = 1 - \dfrac{1-|2p-1|}{2p} \dfrac{1-x^{n+1}}{1-x^{n+2}} \dfrac{1-x^{n+4}}{1-x^{n+3}} \\ x = \dfrac{1-\sqrt{|2p-1|}}{1+\sqrt{|2p-1|}} \end{cases}}$

(EXACT SOLUTION)

no wall:

$n \to \infty$ $\quad S_n \to S(p) = \dfrac{1-2p-|1-2p|}{2p} = 0$ if $p < \frac{1}{2}$ or average # of children $= 1 = p/1-p$

(GW tree almost surely finite).

ESCAPE FROM A SEGMENT

- Solutions of $R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$
  with $R_{-1} = 0$ & $R_{L+1} = 0$; $S_n = 1 - (1-p) R_n$.

Elliptic exact solution:

$$\boxed{R_n = R \frac{u_{n+1}}{u_{n+2}} \frac{u_{n+4}}{u_{n+3}}}$$

$$u_n = \Theta_1(n\alpha) \quad \text{(Jacobi theta fctn)}$$

$$x = e^{2i\pi\alpha} \quad \text{and} \quad \begin{cases} R = 4 \dfrac{\Theta_1(\alpha)\Theta_1(2\alpha)}{\Theta_1'(0)\Theta_1(3\alpha)} \left(\dfrac{\Theta_1'}{\Theta_1}(\alpha) - \dfrac{1}{2}\dfrac{\Theta_1'}{\Theta_1}(2\alpha)\right) \\ g = \dfrac{\Theta_1'(0)\,\Theta_1(3\alpha)}{16\Theta_1(\alpha)^2\Theta_1(2\alpha)\left(\dfrac{\Theta_1'}{\Theta_1}(\alpha) - \dfrac{1}{2}\dfrac{\Theta_1'}{\Theta_1}(2\alpha)\right)^2} \end{cases}$$

$$\alpha = \frac{1}{L+5}$$

$$R_n \sim 2\left(1 - \varepsilon^2\, \mathcal{U}(r)\right) \qquad \mathcal{U}(r) = 2\,\wp\left(\sqrt{\tfrac{3}{2}}\, r\right) \ \text{(Weierstraß)}.$$

BOUNDED MAPS ...

CONCLUSION

2D Quantum Gravity

(Decorated random surfaces)

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Exact solution
(matrix model)

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geodesic distance

Labeled Trees

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INTEGRABILITY

Question

1. Relation between geodesic distance and genus expansion via orthogonal polynomials

almost same equation

Trees $[P, Q] = 0$ $\qquad Q = d^2 - \mathcal{U}(r)$

geodesic distance

Matrix Model $[P, Q] = 1$ $\qquad Q = d^2 - u(x)$

topological exp^n parameter