Characters and difference equations

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Characters and difference equations

A. Okounkov

based on joint works with Mina Aganagic

and with Roman Bezrukavnikov

Consider $\mathfrak{sl}_2 = \{ 2\times 2$ matrices with trace $= 0 \}$

$$= \text{span of } \underset{e}{\underset{\parallel}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}},\ \underset{h}{\underset{\parallel}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}},\ \underset{f}{\underset{\parallel}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}}$$

with commutation relations $[h,e] = 2e$, $[h,f] = -2f$

$[e,f] = h$

the simplest simple Lie algebra provided char $\neq 2$

Its representation theory is a always a nice appetizer bite for a proper course in Lie algebras and representations.

Here is an easy module to construct, known as the Verma module

$$M = \text{generated by a vector } |\lambda\rangle \text{ s.t. } \quad \begin{aligned} h\,|\lambda\rangle &= \lambda\,|\lambda\rangle \\ f\,|\lambda\rangle &= 0 \end{aligned}$$

λ — ground field

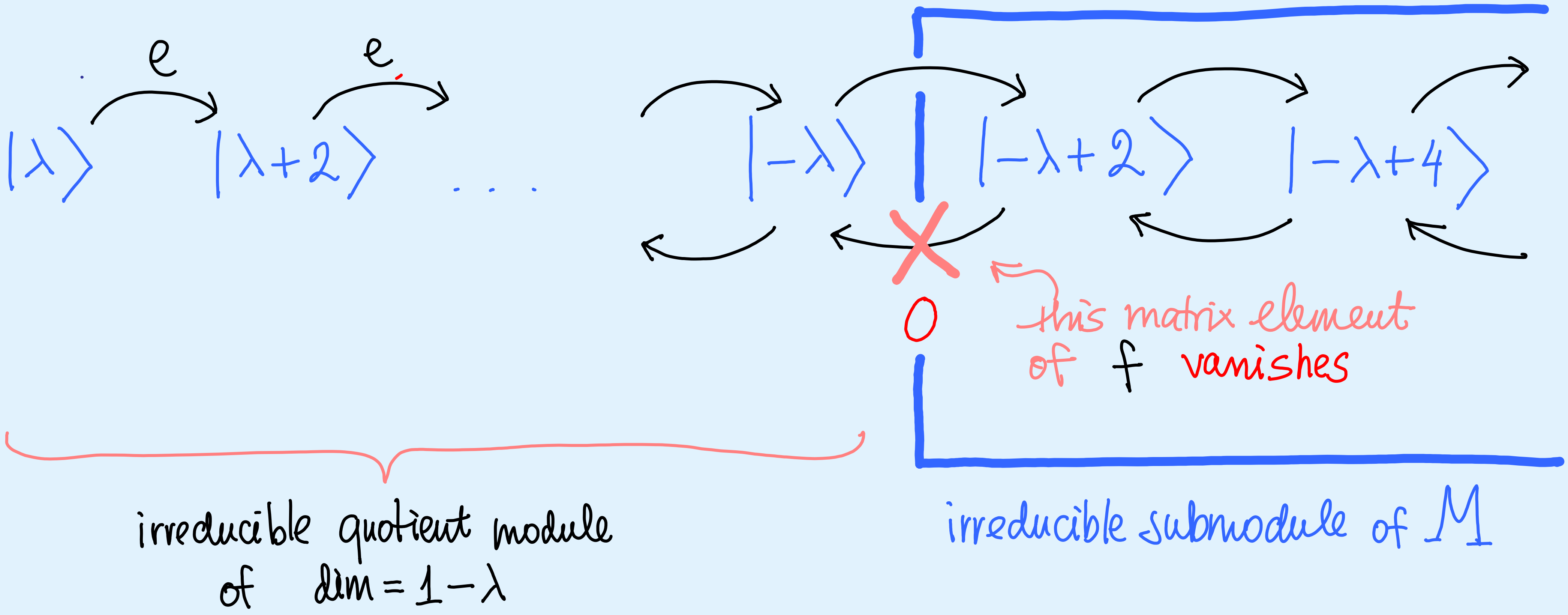
from the commutation relations, we have

$$\begin{tikzcd}[column sep=large]
|\lambda\rangle \arrow[r, bend left, "e"] & e\,|\lambda\rangle = |\lambda+2\rangle \arrow[l, bend left, "f", "-\lambda"'] \arrow[r, bend left, "e"] & |\lambda+4\rangle \arrow[l, bend left, "f", "-2\lambda-2"'] \arrow[r, bend left, "e"] & \cdots\cdots \arrow[r, bend left, "e"] & |\lambda+2k\rangle \arrow[l, bend left, "f", "\underline{\underline{-k(\lambda+k-1)}}"'] \arrow[r, bend left, "e"] & {} \arrow[l, bend left, "f"]
\end{tikzcd}$$

$-\lambda$ with coefficient

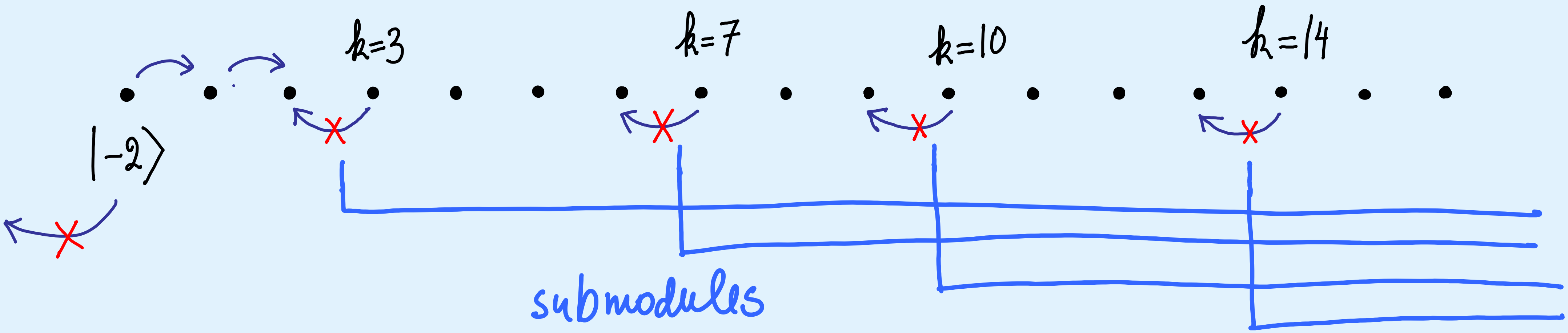
E.g. if $\mathrm{char} = 0$ and $\lambda \neq 0, -1, -2, \ldots$ this is <u>irreducible</u>

If char $= 0$ and $\lambda \in \{0, -1, -2, \dots\}$ then



In general, the decomposition of Verma modules into irreducibles in char $=0$ is one of the central problems solved by the Kazdan-Lusztig theory in the hands of Beilinson, Bernstein, Brylinski, Kashiwara, Ginzburg, Soergel etc. etc.

If char $= p$ and $\lambda \in \mathbb{Z}/p$ then



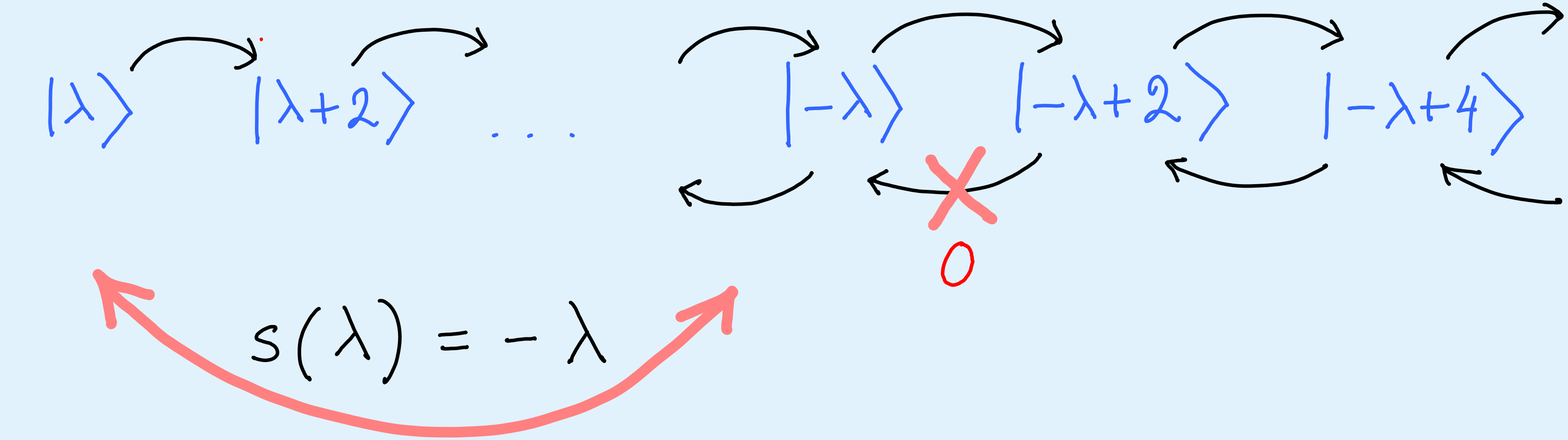
We have

$$f\, e^{k} |\lambda\rangle = 0 \quad \text{when} \quad k = 0,\, 1-\lambda \mod p$$

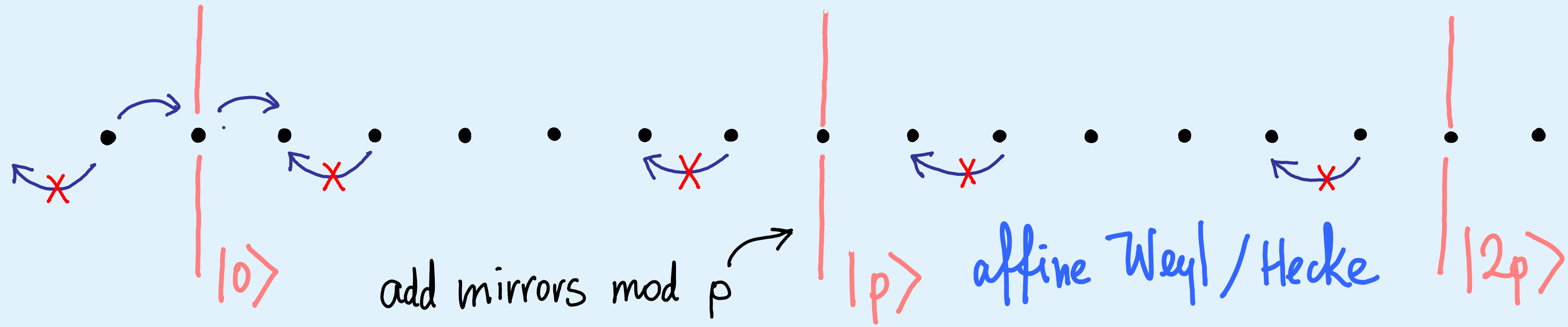
and so we get an infinite chain of submodules with irreducible finite-dimensional quotients that repeat periodically there exists a char=p version of the KL theory, which is an amazing piece of mathematics created by Bezrukavnikov, Williamson, their collaborators, and many other people...

It contains at least 2 kinds of phenomena: those that stabilize for $p \gg 0$ and the more elusive transient phenomena for p of moderate size. Williamson's deep insights into the latter were recognized by the 2016 CMI prize. Today, we will talk about the former...

Obviously, this picture has something to do with reflections



and with the Weyl group W that they generate for general $\mathfrak{g}$. The Hecke algebra, in which $s^2 = 1$ is deformed is, in fact, fundamental for KL theory. Similarly, for char $= p$



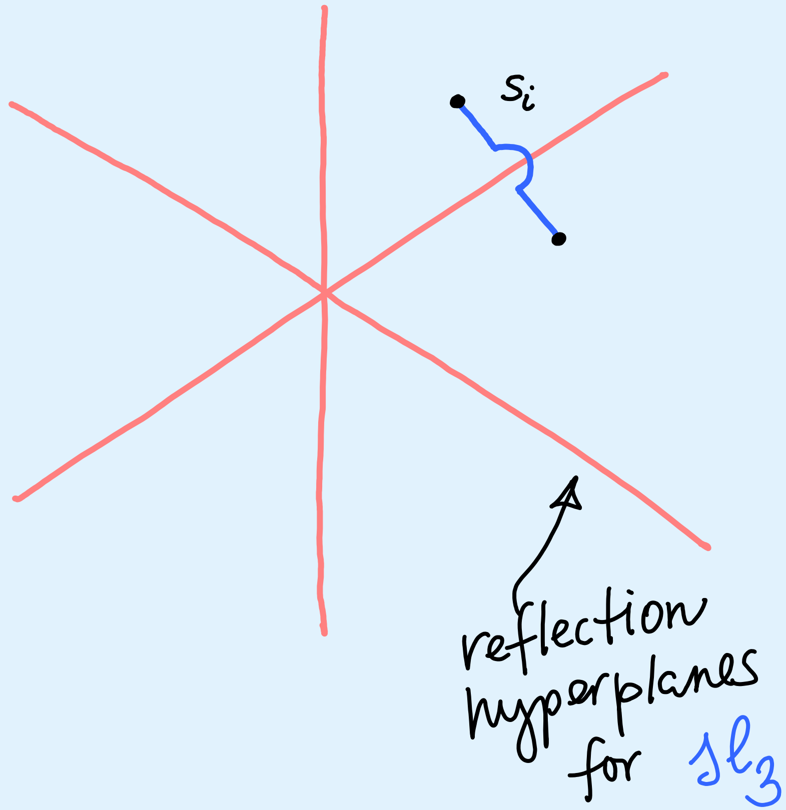
In general

$$\text{Hecke algebra} = \text{Braid group } \mathcal{B} \Big/ s_i^2 = \ldots$$

where

$$\mathcal{B} = \pi_1\left(\mathbb{C}^{rk} \setminus \text{refl hyperlanes} \,/\, W \right)$$

Weyl group

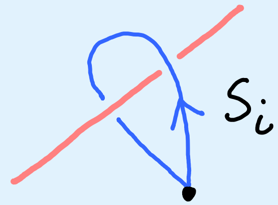


in the affine situation,

$$\mathbb{C}^{rk} \rightsquigarrow (\mathbb{C}^\times)^{rk}$$

a very classical source of algebras of the form

$$\pi_1\Big((\mathbb{C}^\times)^{rk} \setminus \text{hyperplanes} \Big) \Big/ \text{fix eigenvalues of}$$



is <u>monodromy</u> of flat connections, that is, differential equations with prescribed singularities

In fact, there is a very large supply of important algebras (e.g. Cherednik algebras) that are not exactly $\mathcal{U}(\mathfrak{g})$, e.g. they do not have a Weyl group, but which have a KL theory in char $= p \gg 0$ with a certain monodromy group in place of the Hecke algebra

[Bezrukavnikov – O.]

What I want to discuss today will, I believe, simplify many things in [BO]

For differential equations, we'll start with the hypergeometric function

Gauß

$$F\left[\begin{matrix} a, b \\ c \end{matrix} \middle| z\right] = \sum_{k \geq 0} z^k \frac{(a)_k (b)_k}{\underbrace{(1)_k}_{= k!} (c)_k}$$

where $(a)_k = a(a+1)(a+2)\dots(a+k-1) = \dfrac{\Gamma(a+k)}{\Gamma(a)}$

Riemann <u>characterized</u> it as the solution of a 2nd order ODE in z with 3 regular singular points $z = 0, 1, \infty \in \mathbb{P}^1$

Like $\mathfrak{sl}_2$, the hypergeometric function is a perfect appetizer bite for a course that is considerably harder to find in the course catalogs...

Nonetheless, very important.

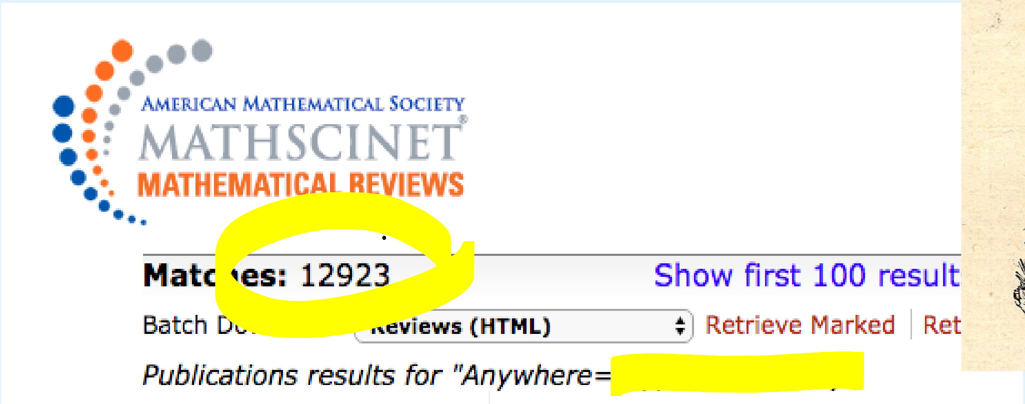
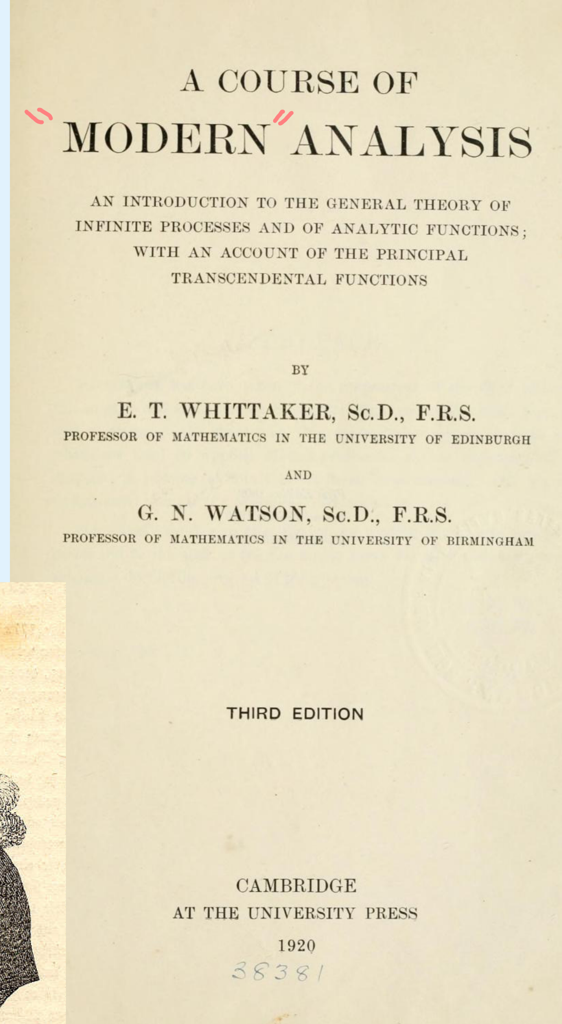



Fig. 1795. — Dronte.


A COURSE OF

"MODERN" ANALYSIS

AN INTRODUCTION TO THE GENERAL THEORY OF INFINITE PROCESSES AND OF ANALYTIC FUNCTIONS; WITH AN ACCOUNT OF THE PRINCIPAL TRANSCENDENTAL FUNCTIONS

BY

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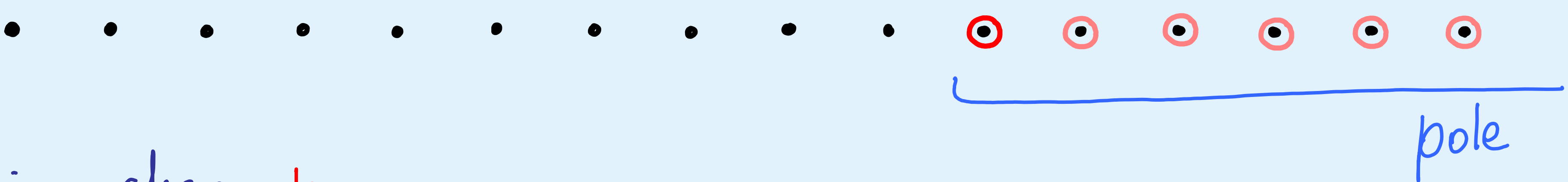
Observe that the series has the same structure of poles

$$F\left[\begin{matrix} \cdots \\ c \end{matrix} \,\middle|\, z\right] = \sum_{k \geq 0} z^k \frac{\cdots}{k!\,(c)_k}$$

as the structure of submodules in the $\mathfrak{sl}_2$ Verma modules

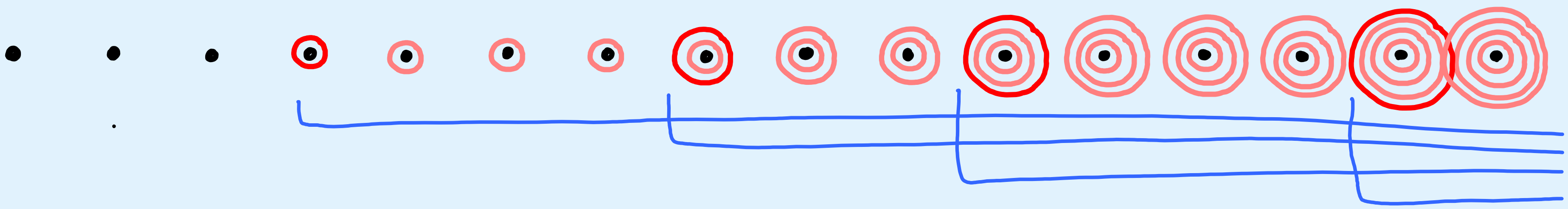
in char $= 0$

$k = 1 - c, \quad c = 0, -1, -2, \ldots$



in char $= p$

$k \equiv 0 \qquad k \equiv 1 - c$



This phenomenon can be seen more cleanly and adequately in the world of <u>q-analogs</u>, in which additive variables become multiplicative and both differential and difference equations become q-difference equations, where $q \in \mathbb{G}_m$

For instance,

$$\frac{1}{\Gamma_q(x)} \stackrel{\text{def}}{=} \prod_{i=0}^{\infty} (1 - q^i x)$$

converges for $|q|<1$
entire in x with
Simple zeros $x = q^0, q^{-1}, q^{-2}, \ldots$

instead of
$x = 0, -1, -2, \ldots$

solves

$$\Gamma_q(qx) = (1-x)\,\Gamma_q(x)$$

vanishes at $x = 1 \in \mathbb{G}_m$

deformation of $\Gamma(x+1) = x\,\Gamma(x)$

the symmetrized q-analogs

$$[n]_q = q^{n/2} + q^{n/2-1} + \dots + q^{-n/2}$$

$=$ centered Hodge polynomial for $\mathbb{P}^{n-1}$

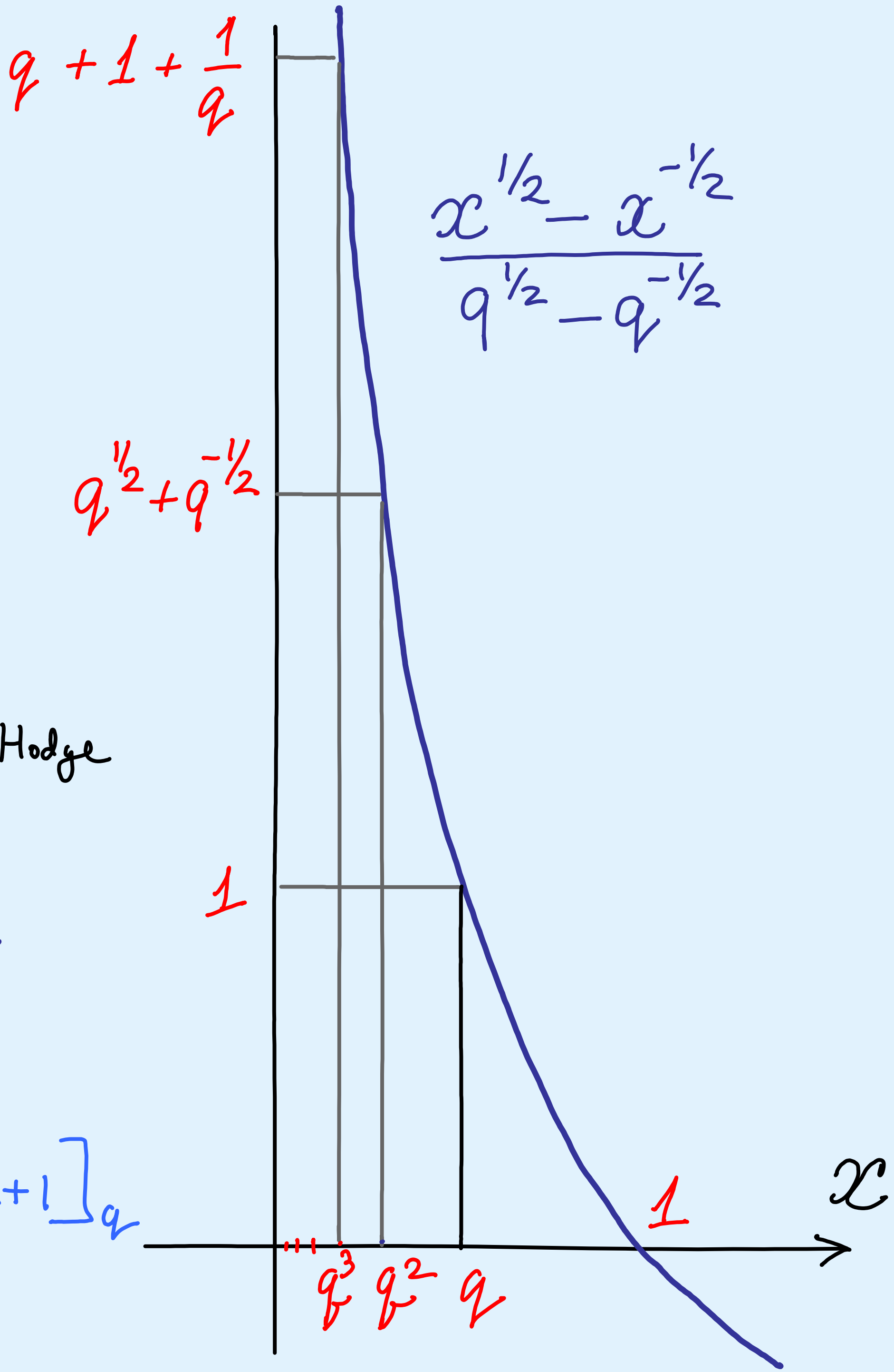
$$\operatorname{tr}\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \in SL(2)_{\text{Hodge}}$$

typically replace integers in the q-world.

Satisfy e.g.

$$[n+1]_q\,[m+1]_q - [n]_q\,[m]_q = [n+m+1]_q$$

also satisfied by motives of $\mathbb{P}^n$



Adopting the notation

$$(a)_k := \frac{\Gamma_q(q^k a)}{\Gamma_q(a)} = (1-a)(1-qa)\dots(1-q^{k-1}a)$$

we can use the same type of formula

$$\Phi\left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| \, z \right] = \sum_{k \geq 0} z^k \frac{(a)_k (b)_k}{(q)_k (c)_k}$$

for the q-hypergeometric function, which now solves a q-difference equation of order 2 in z, as well as in a, b, c

A much more symmetric object!

Evidently, the terms in

$$\phi\left[\begin{matrix}\cdots\\ c\end{matrix}\,\middle|\, z\right] = \sum_{k\geq 0} z^k \frac{\cdots}{(q)_k\,(c)_k}$$

have the following poles in q and c

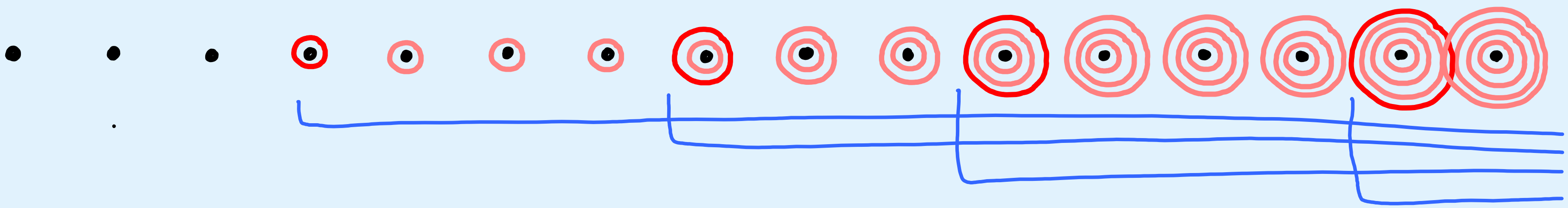
$c = q^\lambda$

if $q \notin \sqrt{1}$

$k = 1-\lambda \qquad \lambda = 0, -1, -2, \dots$

if $q^m = 1$

$k \equiv 0 \qquad k \equiv 1-\lambda \mod m$



Lie algebras like $\mathfrak{sl}_2$ also have q-analogs, known as quantum groups e.g.

$$[e, f] = \text{the } q\text{-analog of } h = [h]_q$$

$$\begin{tikzcd}
|\lambda\rangle \arrow[r, bend left, "e"] & |\lambda+2\rangle \arrow[l, bend left, "-[\lambda]_q"] \arrow[r, bend left, "e"] & \cdots \arrow[r, bend left, "e"] & |\lambda+2k\rangle \arrow[l, bend left, "-[k]_q[\lambda+k-1]_q"] \arrow[r, bend left, "e"] & {} \arrow[l, bend left]
\end{tikzcd}$$

Again $\quad f e^k |\lambda\rangle = 0 \iff k \equiv 0, 1-\lambda$ modulo m

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \;$ order(q)

In general, quantum groups at $q^m = 1$ behave in the same way as char $= m$ for $m \gg 0$

[Andersen–Jantzen–Soergel, ..., Bezrukavnikov, ...]

Main idea: Verma modules over quantum groups (and hence Verma modules in char $p \gg 0$) break up in exactly the same way as solutions of certain q-difference eq.

Should work for some much broader class of algebras than $U\mathfrak{g}$, see below...

Why? In the rest of the talk I hope to explain one answer

It involves certain ideas from mathematical physics and enumerative geometry

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- with SUSY, $H = (\text{Dirac})^2$ and it easie to ask about the index(Dirac) as a representation of symmetries. Invariant under deformations!

$$\text{Even} \underset{\not{D}}{\overset{\not{D}}{\rightleftarrows}} \text{Odd} \qquad \text{index} = \text{Ker}_{\text{even}} - \text{Ker}_{\text{odd}}$$

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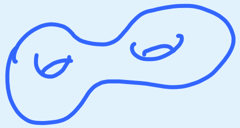
- QFT / random field are, in this respect, like QM with an infinite-dimensional configuration space at large distances / low energies we may describe states of a physical system as a modulated vacuum, that is a map

$(T(y), p(y), \dots) \in$

Parameter ("moduli") space X
vacuum states

Since indices are invariant under deformation, we may want to compute them in the theory of maps f to X

In dim $= 2+1$ and with the amount of SUSY that we want, these will be holo maps from $B =$  to a holo symplectic X

What we want to compute is the index (Dirac) for moduli spaces $\mathcal{M}$ of holo maps $f : B \to X$

While singular, with the proper setup, $\mathcal{M}$ has a <u>virtual</u> $\hat{A}$-genus

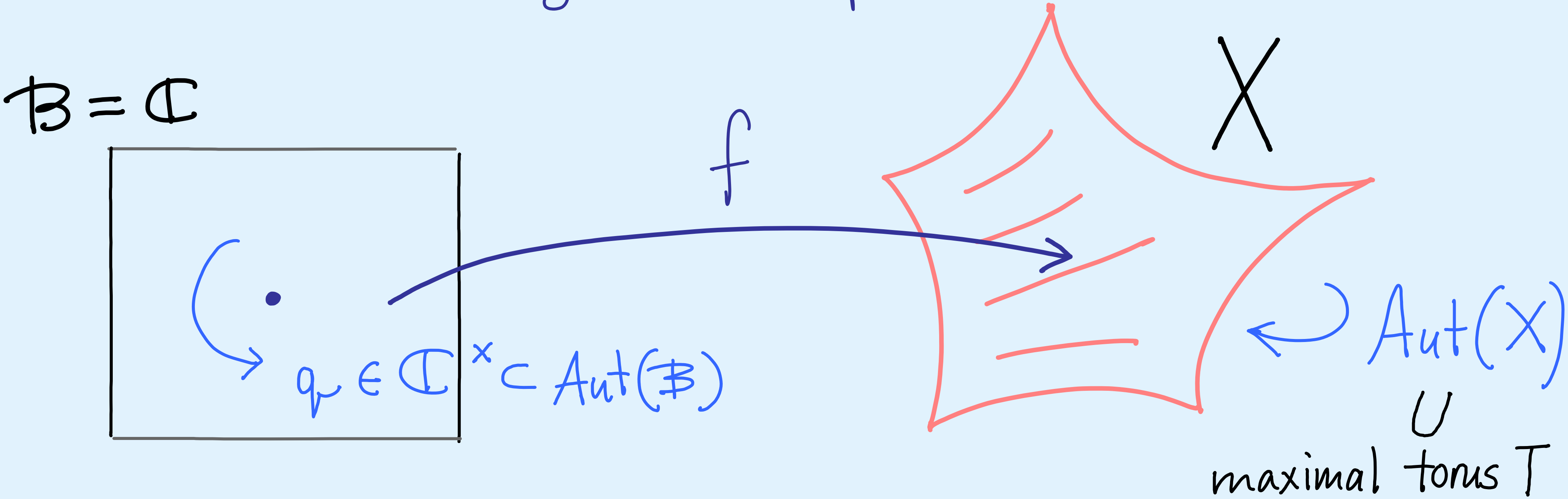
$$\hat{\mathcal{O}}_{\mathcal{M}} \in K_{\mathrm{Aut}}(\mathcal{M})$$

acts on B and X

and the index we want is defined as $\chi(\hat{\mathcal{O}}_{\mathcal{M}})$, graded by the action of Aut and also by $\deg f \in H_2(X)_{\mathrm{eff}}$

record by $z^{\deg}$
where $z \in H^2(X, \mathbb{Z}) \otimes \mathbb{C}^\times$

The central role belongs to the equivariant counts



which have the form

$$V = \sum_{d \in H_2(X,\mathbb{Z})_{\text{eff}}} z^d \, \chi\big(\mathcal{M}_d, \hat{\mathcal{O}}_\mu\big)$$

∞-dimensional space with finite-dimensional $q \times T$ eigenspaces

$$\cap$$

$K_{eq}(pt)_{\text{localized}}$ = rational functions of q and T

The poles in

$$V = \sum z^d \, \chi\left(\mathcal{M}_d, \hat{\mathcal{O}}_\mu\right)$$

in equivariant variables are due to the noncompactness of both source and target of $f: B \to X$ and are under excellent geometric control

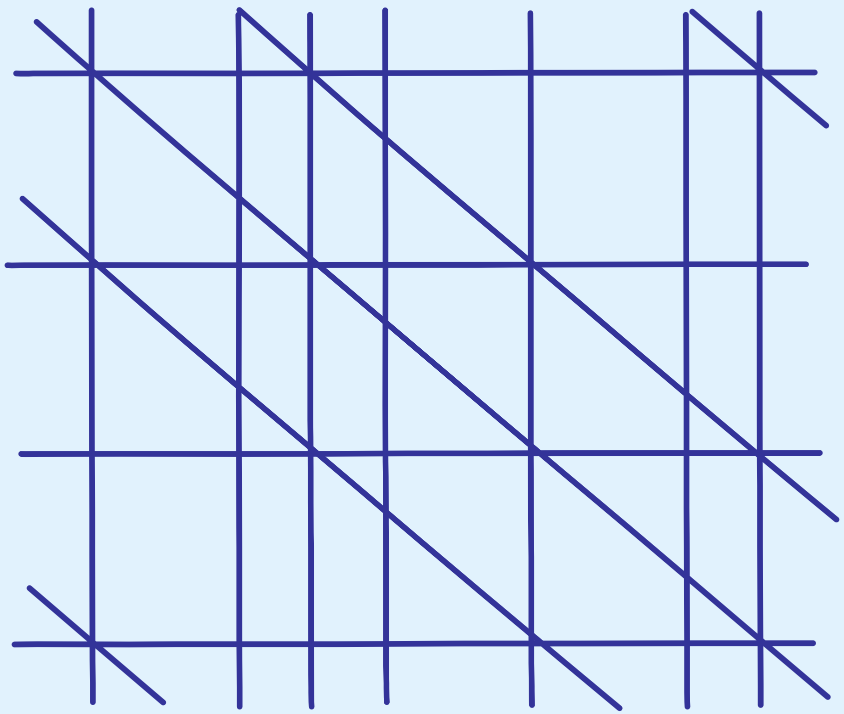
There has been a lot of work on these functions and they have been understood as the present day analogs of the hypergeometric functions, namely fundamental solutions of certain very special q-difference equations in both degree-counting (a.k.a Kähler) variables z, and the equivariant variables in T.

Classical hypergeometry is related to the simplest targets like $X = T^*\mathbb{P}^1$ or $X = T^*\mathbb{P}^n$.

Really new examples start with $X = \text{Hilbert scheme}(\mathbb{C}^2)$

weight or curve class

$$\langle \alpha, x \rangle \in \mathbb{Z}$$



$\mathrm{Lie}\, A$ or $H^2(X)$

$\cap$

$\mathrm{Aut}(X, \omega)$

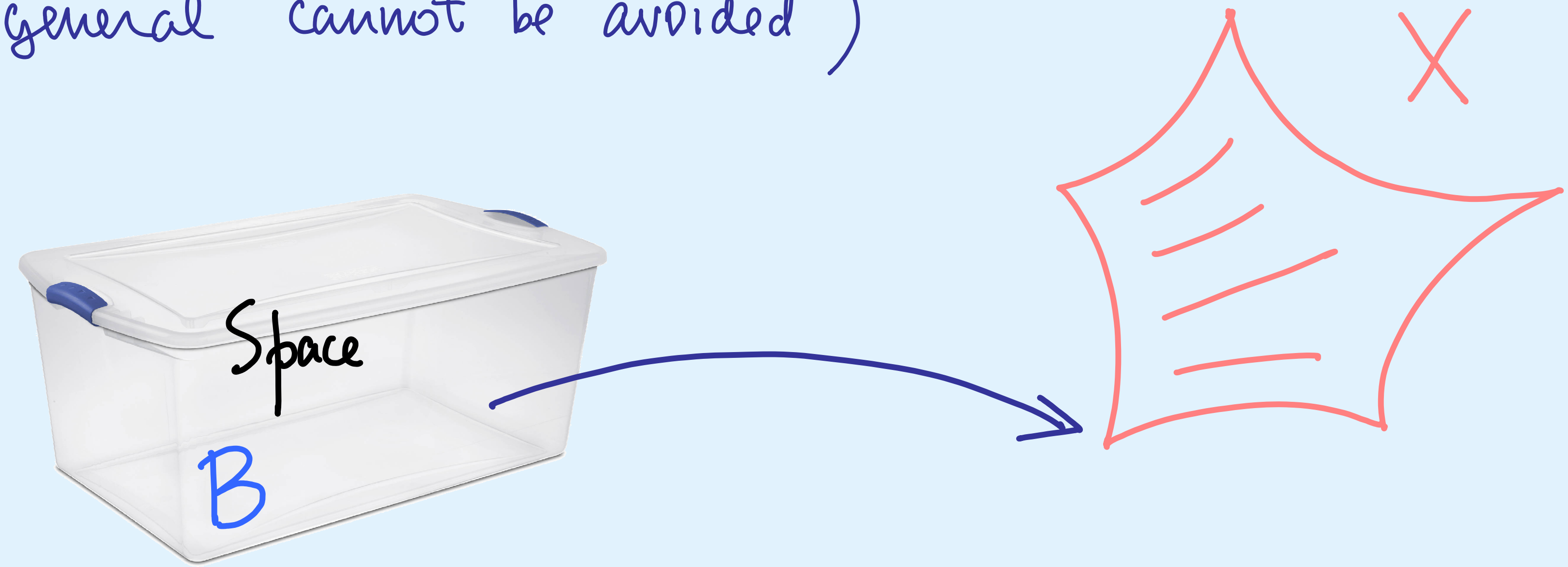
The difference equations in either set of variables come from a certain groupoid associated to a rational hyperplane arrangement.

An object more general than Hecke algebras in that it needs no Weyl group

Recall that X in this context is algebraic symplectic.
It is possible and useful to quantize it, that is, study associative noncommutative deformations of the sheaf $\mathcal{O}_X$ and the corr. noncommutative algebras of global sections.

However, the algebra whose Verma module will be breaking up, comes not from X but from the quantization of a dual symplectic singularity $X^\vee$
(same q-diff equations! with an exchange of variables)

A basic question in both physics and mathematics is: what to do when the map f hits a singularity (which, in general cannot be avoided)



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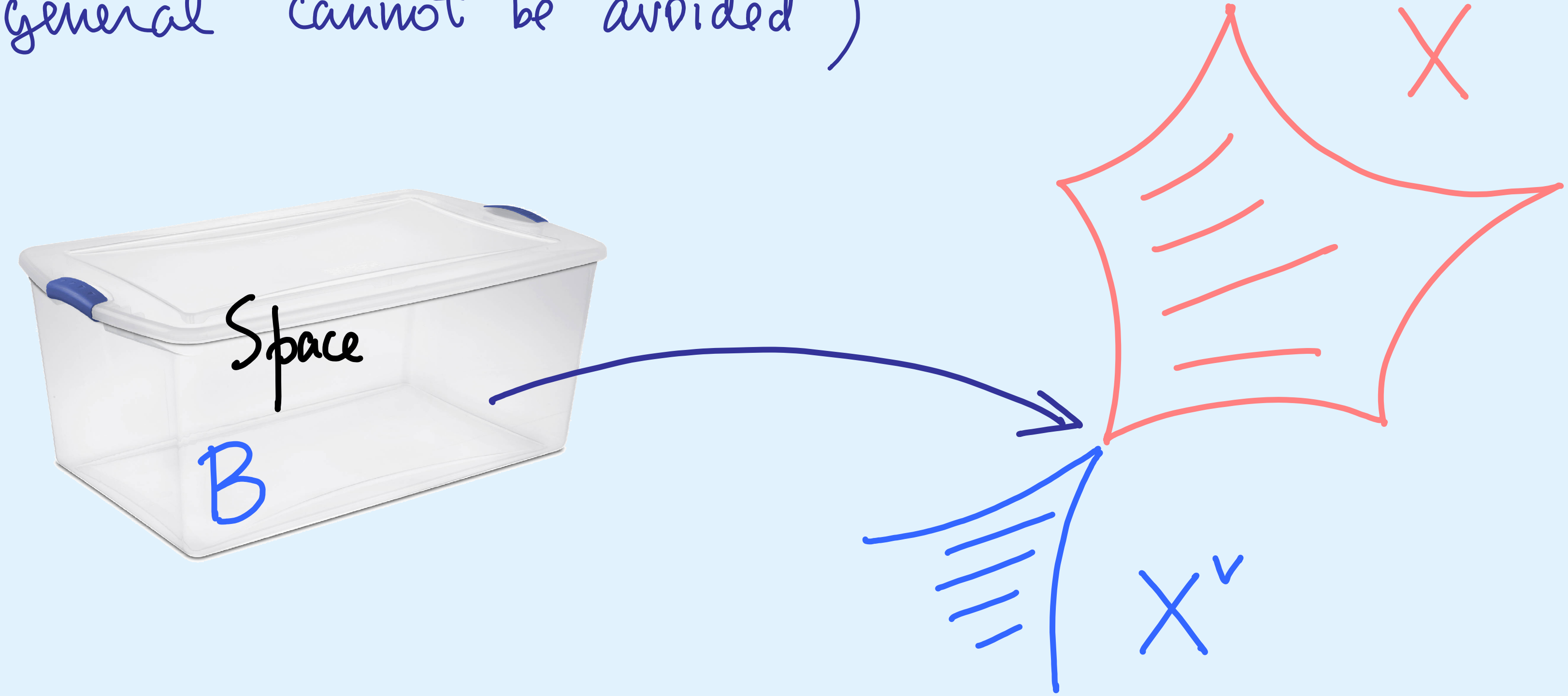


Some degrees of freedom that could be ignored before become important now. Measuring them acts as an operator in theory described by maps to X. A condensation of such insertions takes us to a new component $X^\vee$ of the moduli of vacua.

Distilling the mathematical essence of this, Nakajima defined a quantization of $X^\vee$ as a certain algebra acting on $K_{crit}(\mathcal{M})$.

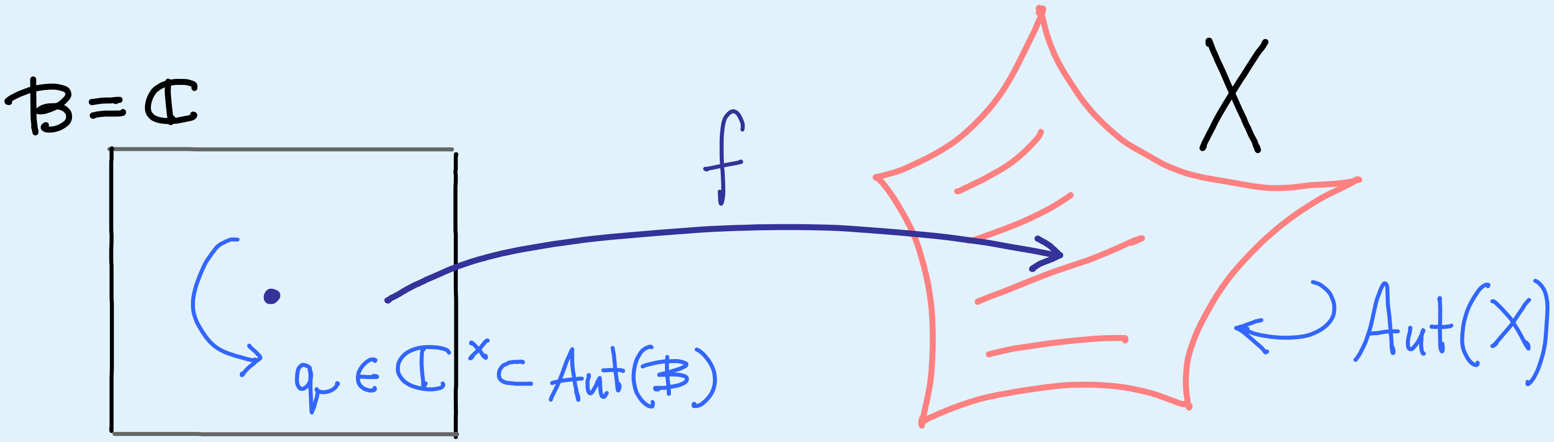
Also very important work by Braverman-Finkelberg-Nakajima, Gaiotto, Webster, and their collaborators.

Here $\mathcal{M}$ is the moduli of maps $f: B \to X$ and it has K_{crit} because it may be written as $\{dW = 0\}$. The function W is scaled nontrivially by Aut, so there is a loss of equivariance $\mathrm{Aut}_{CY} \subset \mathrm{Aut}$. Easy to see that

difference eq. become too simple, no poles

$$\star \quad \chi(\mathcal{M}, \hat{\mathcal{O}}_{\mathcal{M}})\Big|_{\mathrm{Aut}_{CY}} = \mathrm{rk}\, K_{crit}(\mathcal{M})$$

In particular, for $\mathbb{B} = \mathbb{C}$ and $\mathcal{M}$ = the moduli of



$K_{crit}(\mathcal{M})$ is the universal Verma module for the quantized $\mathcal{X}^\vee$ ($=$ quantum sl_2 for $X = X^\vee = T^*\mathbb{P}^1$)

It breaks up for exactly the same geometric reasons as those that create the poles in $\chi(\mathcal{M}, \hat{\mathcal{O}}_{\mathcal{M}})$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