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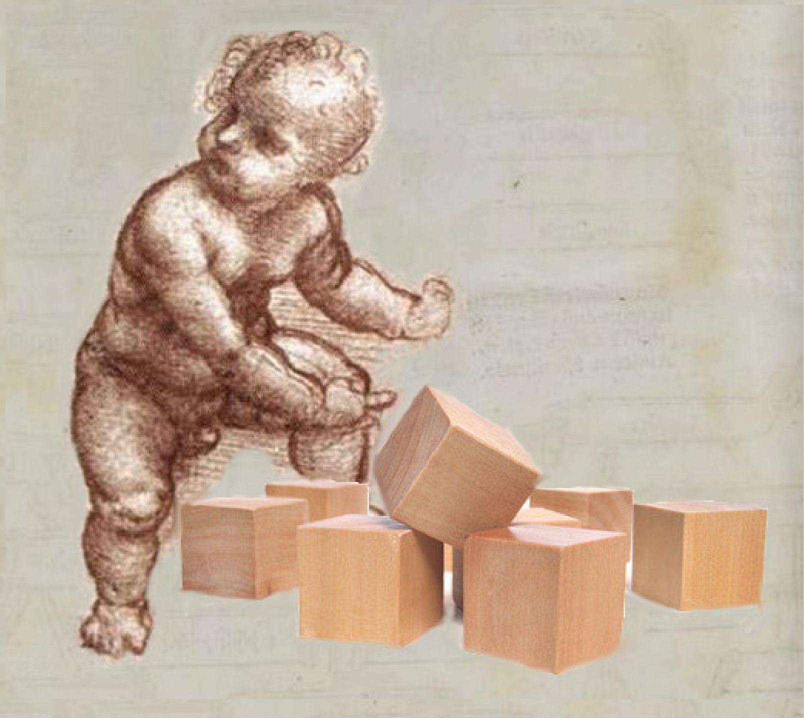
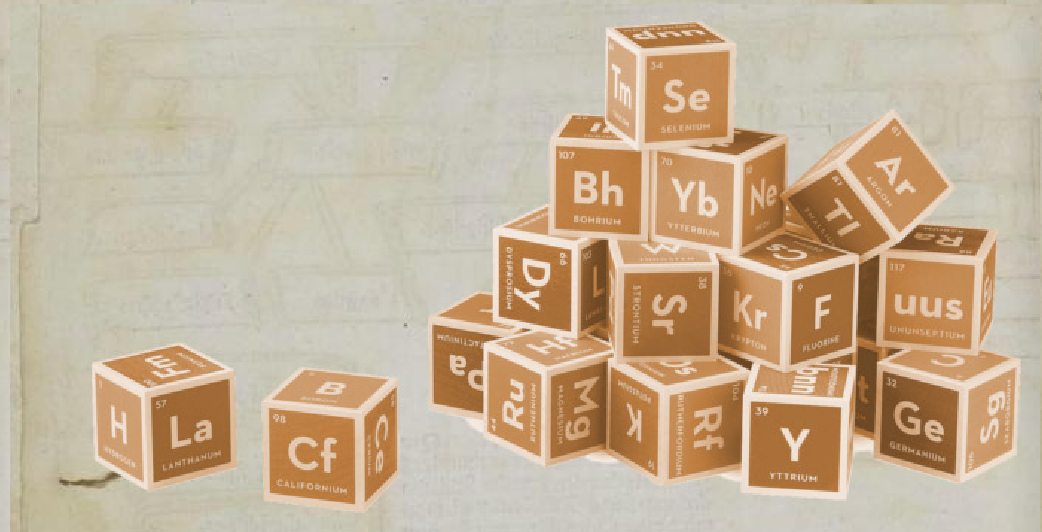
Lezione Leonardesche

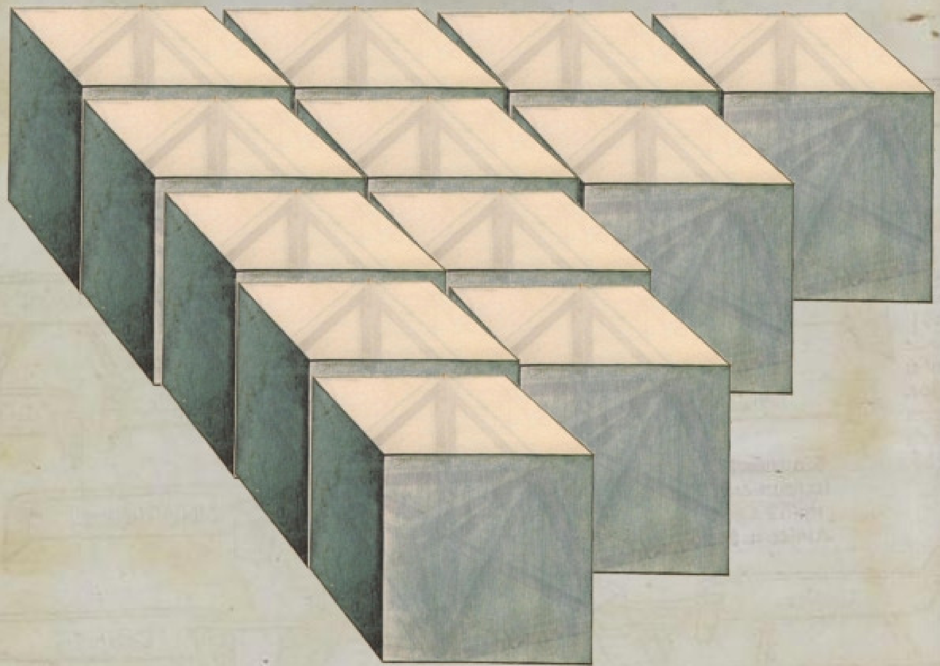
Other than the honor to speak in a long established lecture series, the only meaningful connection between boxcounting and Leonardo may be the following:

I believe Leonardo's mind was very geometric and visual, and he felt he understood something if he could draw it properly. Similarly, boxcounting attaches an image to otherwise quite abstract notions of modern algebraic geometry, something that can serve as an illustration in a modern equivalent of *De divina proportione*



Some mathematical physicists continue playing with blocks well into adulthood in hope of relating them to the building blocks of nature





2D Partition

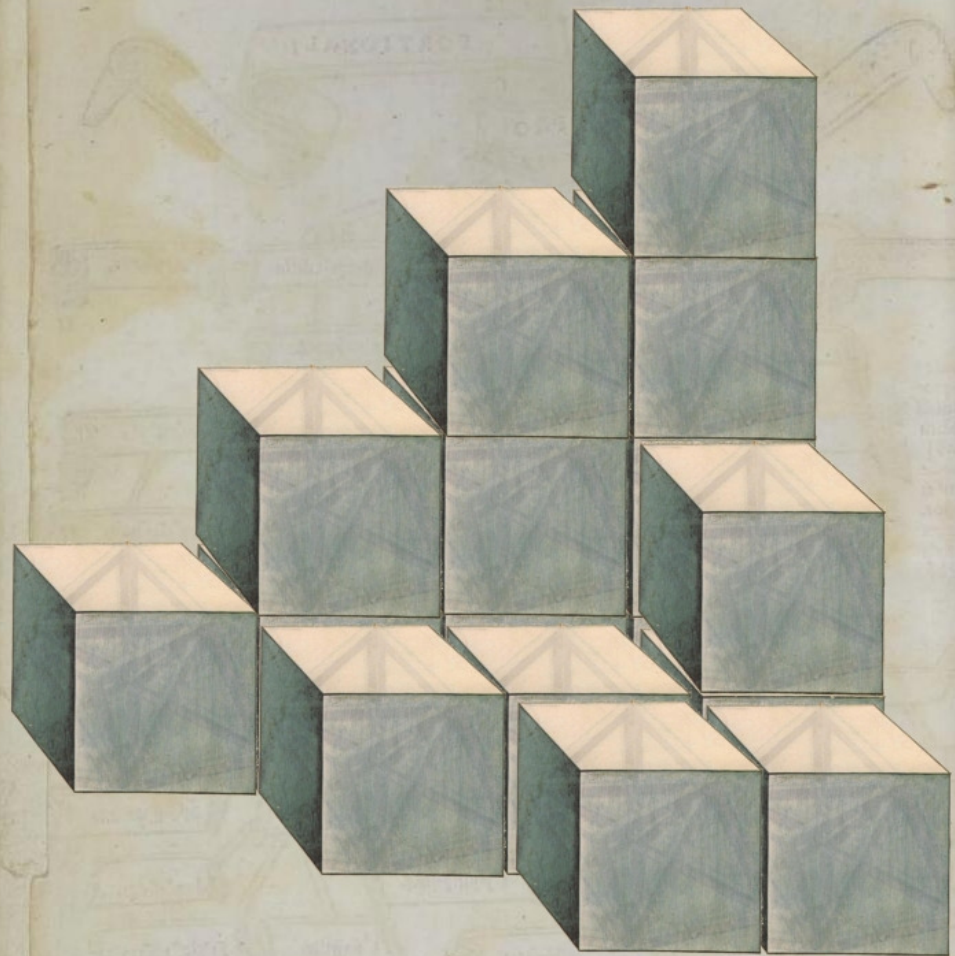
We think about 2-dimensional partitions ...

$\lambda=(4,3,2,2,1)$

$|\lambda|=12$

3D Partition

... and also about 3-dimensional ones ...



$|\pi|=16$

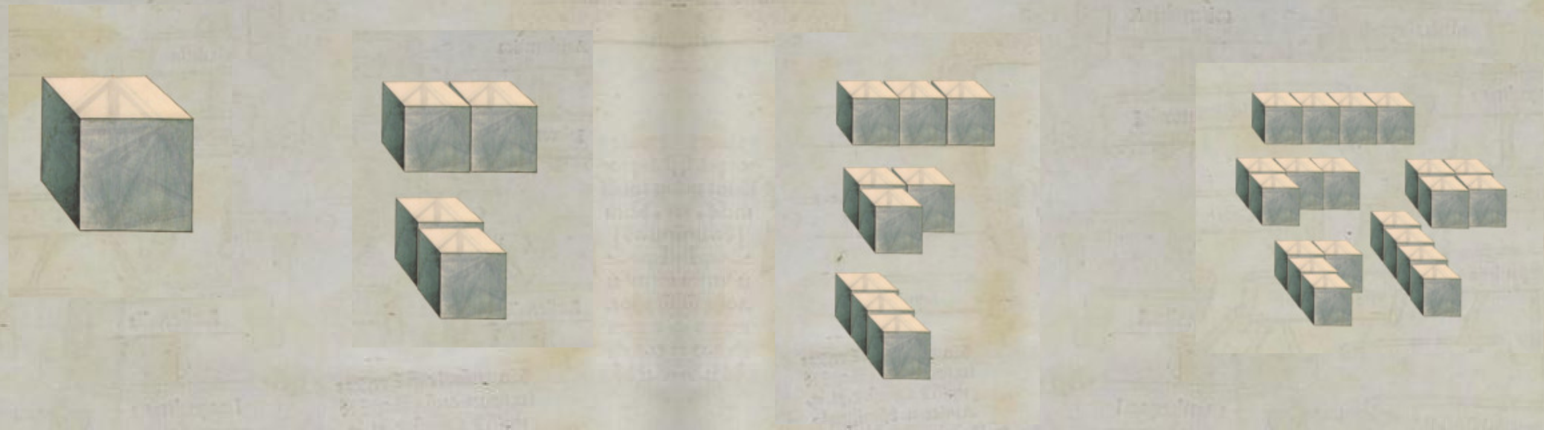
Often, our ideas say something interesting about very combinatorial sums over partitions, not unlike the generating function for 2D partitions

$$Z = \sum_{\lambda} q^{|\lambda|}$$

\# of boxes

which goes back to at least Euler in XVIII century.
Euler observed that ...

$$Z = 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots$$



$$= \prod_{n>0} (1-q^n)^{-1}$$

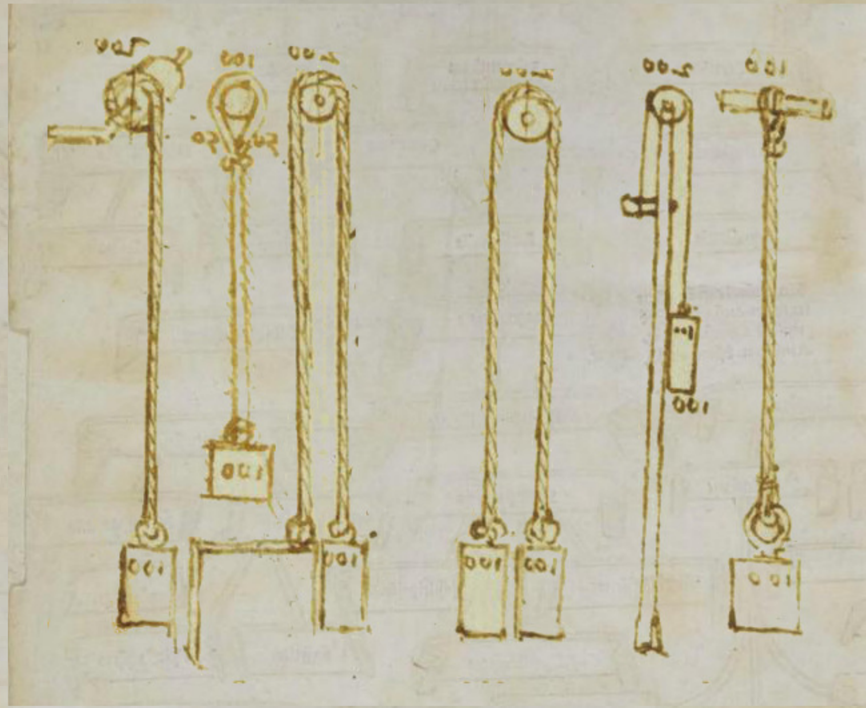
and discovered that

$$Z^{-1} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - q^{40} + \ldots.$$

In XIX century, Euler's formula for Z^{-1} became a special of case of the Jacobi triple product identity

Formulas for other powers of the function Z appeared in the XX century in the work of Dyson, Macdonald, Kac, among many others. We will see examples below

We will look at series of form



$$Z = \sum_{\lambda} q^{|\lambda|} \, \mathrm{weight}(\lambda)$$

for certain weight functions which are important from the combinatorial, representation-theoretic, probabilistic, and geometric points of view

Our weights will, in a sense, interpolate between the weight=1 case and the

Plancherel measure

which views partitions as irreducible representations of symmetric groups S(n)

and weights them by

$$\frac{(\dim \lambda)^2}{n!}$$

which may be interpreted as the percentage that the representation λ takes up in the regular representation of the group S(n)

there is a combinatorial formula

$$n!/\dim \lambda = \prod \text{hooks}$$

where the product is over all boxes in the partition and

$$\text{hook} = 1+\text{arm}+\text{leg}$$

We introduce the set

$$A\&L = \{(arm+1, -leg), (-arm, 1+leg)\}$$

and define two flavors of partition weights, the rational one and the exponential one by

$$w_{rat} = \prod_{a \in A\&L} \frac{m + (t,a)}{(t,a)}$$

$$w_{exp} = \prod_{a} \frac{1 - MT^{a}}{1 - T^{a}}$$

where

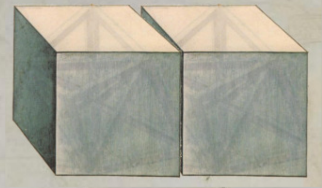
$$(t,a) = t_1 a_1 + t_2 a_2$$

$$T^{a} = T_1^{a_1} T_2^{a_2}$$

the exponential weight limits to the rational as

$$(T,M) \approx (1+t, 1+m)$$

for example, for



A&L={(1,0),(0,1)}

$$w_{rat} = \frac{(m+t_1)(m+t_2)}{t_1 t_2}$$

A&L={(2,0),(-1,1),(1,0),(0,1)}

$$w_{exp} = \frac{1 - MT_1^2}{1 - T_1^2} \times \text{3 more terms}$$

if m=0 or M=1 we get the uniform weight, while

$m \to \infty$, $q \to 0$, $t_1+t_2=0$ gives the Plancherel weight

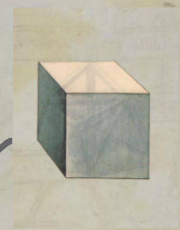
The evaluation of the corresponding weighted sums

$$Z_w = \sum_{\lambda} q^{|\lambda|} w(\lambda)$$

may be obtained as a corollary of a certain general theory deveoped by Carlsson, Nekrasov, and AO, or done directly, borrowing ideas from the 3-dimensional case.

The result is the following ...

In the rational case, we have



$$Z_{rat} = \sum_{\lambda} q^{|\lambda|} w_{rat}(\lambda) = \left(\prod_{n} (1-q^{n})^{-1} \right)^{\frac{(m+t_1)(m+t_2)}{t_1 t_2}}$$

which, for $t_1=-t_2=1$ becomes Macdonald identity for $sl(m)$, and gives a continuum of other formulas for the powers of Euler's partition function Z

In the exponential case, we need the following operation of **symmetric power** to state the result:

$$V \mapsto S^\bullet V = \mathbb{C} \oplus V \oplus S^2 V \oplus \ldots$$

field

It takes group representations to group representations and the character of $S^\bullet V$ may be computed using

$$S^\bullet(V_1 \oplus V_2) = S^\bullet V_1 \otimes S^\bullet V_2 \qquad S^\bullet t = \frac{1}{1-t}$$

monomial, trace of a 1-dimensional representation

For example,

$$S^{\bullet}\frac{t}{1-q} = S^{\bullet}\left(t + qt + q^2t + \dots\right)$$

$$= \prod_{n \geqslant 0} \frac{1}{1-q^n t}$$

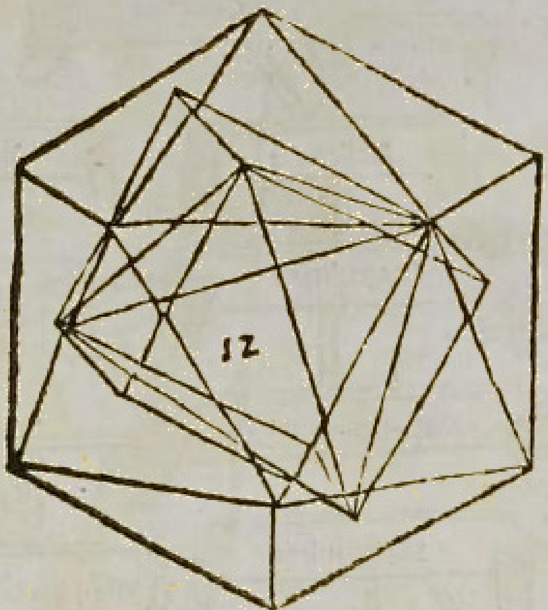
t=q gives Euler

a compact encoding of an infinite product. Similarly,

$$S^{\bullet}\frac{a-b}{1-q} = \prod_{n \geqslant 0} \frac{1-bq^n}{1-aq^n}$$

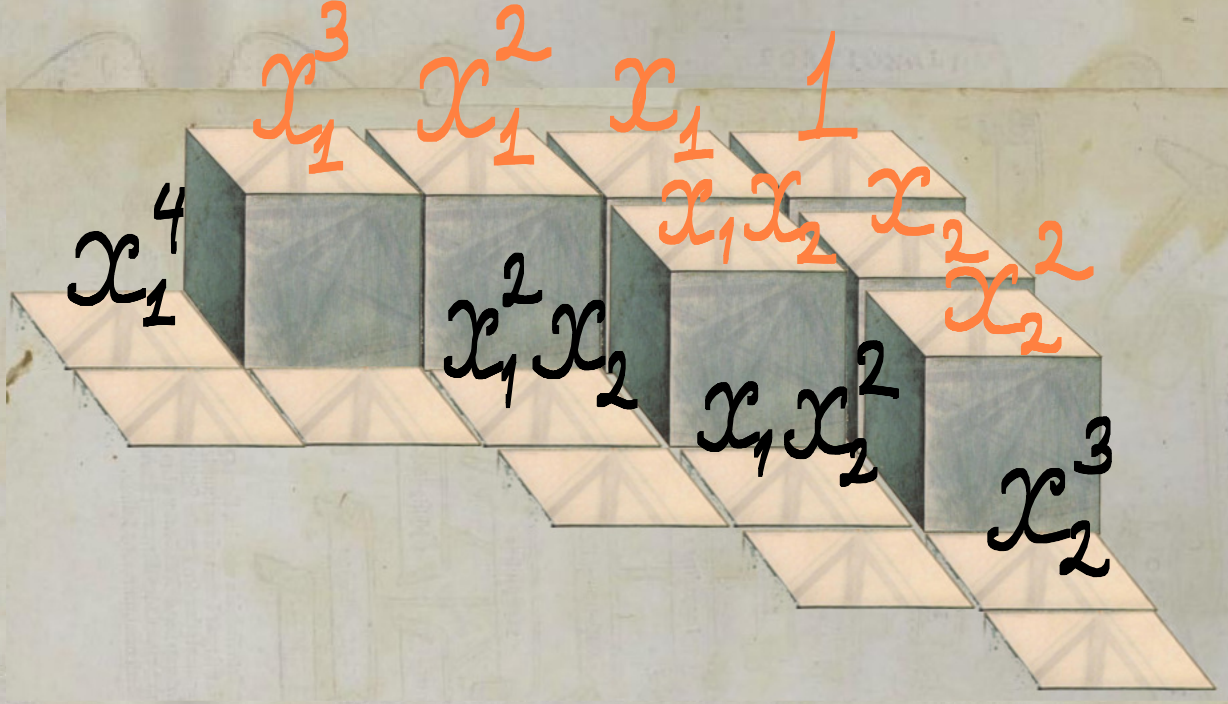
It is a theorem that

$$Z_{exp} = S^{\bullet} \frac{q(1-MT_1)(1-MT_2)}{(1-Mq)(1-T_1)(1-T_2)}$$



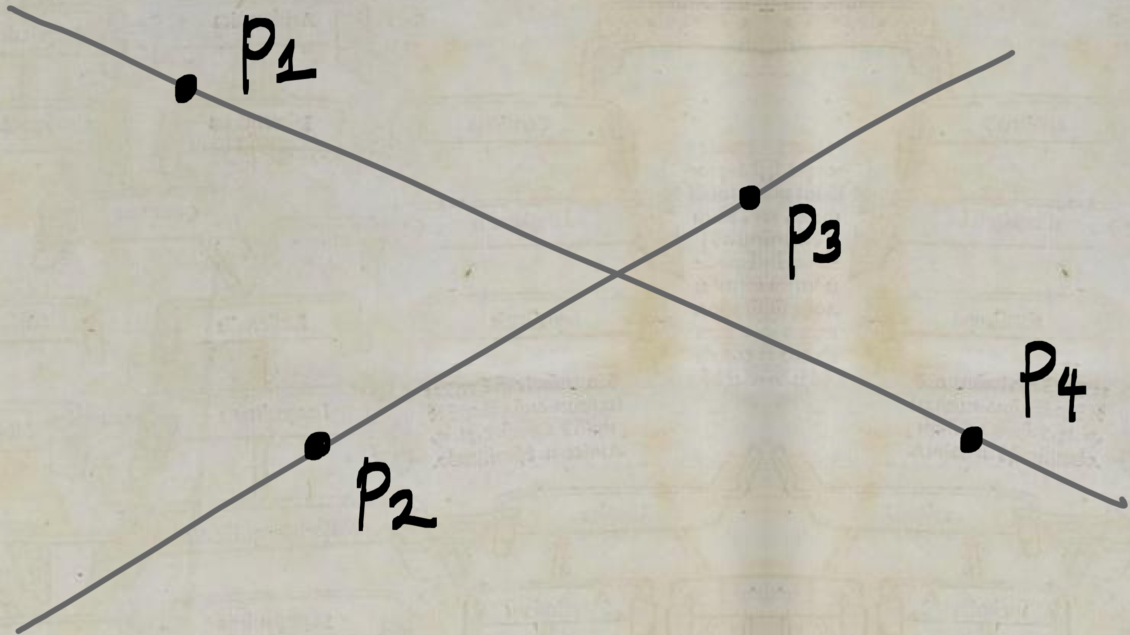
but what is the **geometric meaning** of this formula ?

a 2D partition is the same as a monomial <u>ideal</u> in the ring of polynomials in x_1 and x_2



most ideals in this ring are not monomial. Ideals of codimension **n** are parametrized by a smooth algebraic variety **Hilb(n)** of dimension **2n**

A generic point of **Hilb(n)** corresponds to polynomials **f** (x_1,x_2) vanishing at **n** distinct points



Monomial ideals are the opposite of generic. They correspond to ideals fixed under the action of the group **T** of diagonal matrices in **GL(2)**

The set **A&L** of a partition **λ** are the weights, that is, the joint eigenvalues, of the **T** action in the tangent space to **Hilb(n)** at the corresponding fixed point.

This connects Z_{exp} with the character of the action of **T** on <u>differential forms</u> on the Hilbert scheme, as follows

We have

$$\frac{1-Mx}{1-x} = \text{character } \mathbb{C}[x] - M \text{ character } \mathbb{C}[x]dx$$

and a k-fold product of such fractions gives

exterior forms

$$\text{character } \bigoplus_{i=0}^{k} (-M)^i \, \Omega^i_{\mathbb{C}^k}$$

For example, Hilb(1)=$\mathbb{C}^2$ and

$w_{exp}\Big($  $\Big) = \dfrac{(1-MT_1)(1-MT_2)}{(1-T_1)(1-T_2)}$

$$= \text{character}\left(\mathcal{O}_{\mathbb{C}^2} - M\,\Omega^1_{\mathbb{C}^2} + M^2\,\Omega^2_{\mathbb{C}^2}\right)$$

The Hilbert scheme can be covered by charts centered at monomial ideals and the Čech complex for this covering shows

$$Z_{exp} = \sum_{n,i \geq 0} (-M)^i q^n \operatorname{tr}_T \chi(\mathrm{Hilb}(n), \Omega^i)$$

where $T=\mathrm{diag}(T_1,T_2)^{-1}$ and Ω^i denotes the sheaf of exterior i-forms on **Hilb(n)**

If **Hilb(n)** were compact, this wouldn't depend on T, by Hodge theory

It can be argued geometrically that

$$Z_{exp} = S^{\bullet}\,\frac{}{(1-T_1)(1-T_2)}$$

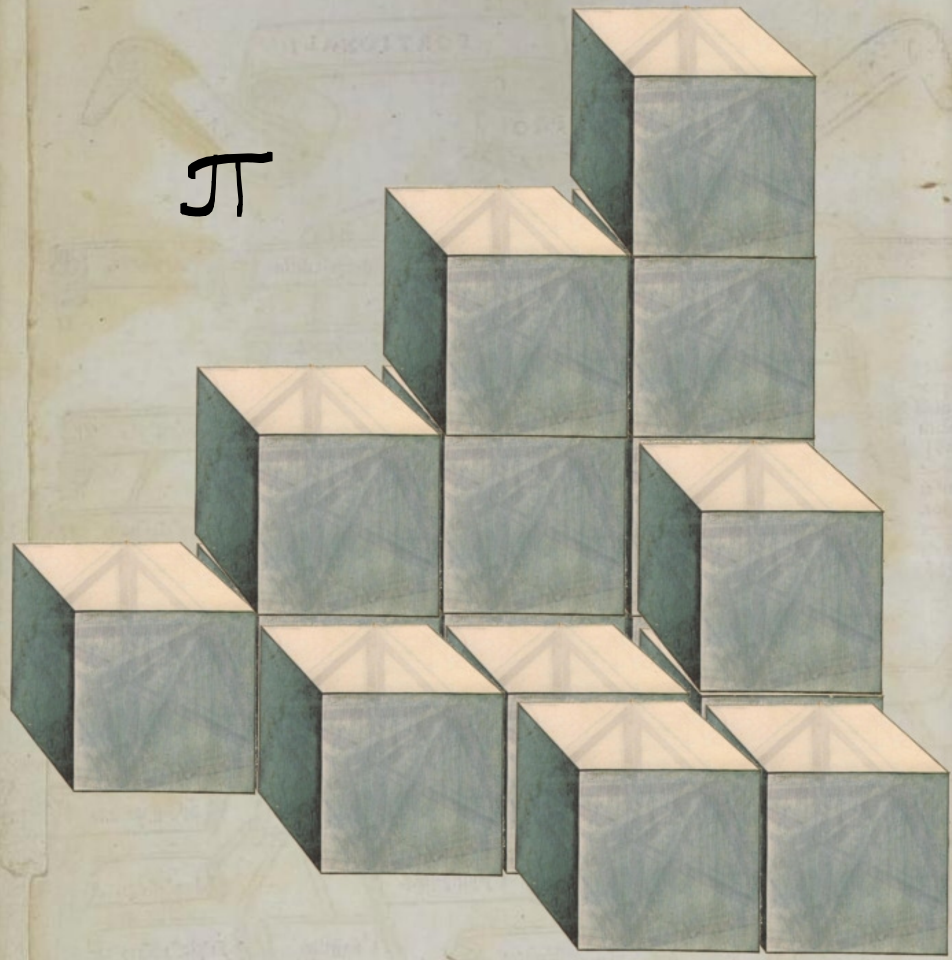
where the unknown part is, a priori, a Laurent polynomial in T_1, T_2. An easy combinatorial argument then shows

$$= q\,\frac{(1-MT_1)(1-MT_2)}{1-qM}$$

Q.E.D.

Now we progress to 3-dimensional partitions where there will be a generalization of McMahon's identity

π



$$\sum_{\pi} q^{|\pi|} = \prod_{n>0} (1-q^n)^{-n}$$

The relation between partitions and monomial ideals is the same in any dimension

but the geometry of the Hilbert scheme becomes more and more complex as the dimension grows. In 3D, it is already a quite singular scheme of, in fact, unknown dimension.

For a 3D partition π, one can define similarly the set **ALT (π)** of "arms, legs, and tails", formed by the weights of T^3 action in T_π**Hilb**. The beauty of certain virtual structures on Hilb in 3D is such that the weight

$$w_{3D}(\pi) = \prod_{b \in ALT} \frac{\hat{a}\left(T^{(1,1,1)-b}\right)}{\hat{a}\left(T^{b}\right)}$$

where $\hat{a}(x)=x^{1/2}-x^{-1/2}$, can still be computed from π in a simple combinatorial fashion for example

$w_{3D}\Bigg($  $\Bigg) = \dfrac{\prod_{i<j\leq 3}\left(T_i^{1/2}T_j^{1/2} - T_i^{-1/2}T_j^{-1/2}\right)}{\prod_{i\leq 3}\left(T_i^{1/2} - T_i^{-1/2}\right)}$

There is a rational version, for which [MNOP] proved long time ago that

$$\sum_{\pi} (-q)^{|\pi|} w_{3D,rat}(\pi) = \left(\prod_n (1-q^n)^n \right)^{\frac{(t_1+t_2)(t_1+t_3)(t_2+t_3)}{t_1 t_2 t_3}}$$

which for $t_1+t_2+t_3=0$ becomes McMahon's identity

The full exponential version was conjectured soon after by Nekrasov, but became a theorem only recently

To state it, we'll need symmetric powers $S^\bullet$ and variables

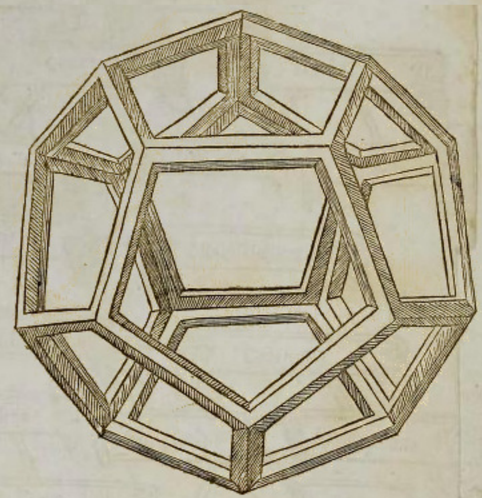
$$T_4 = q \Big/ \sqrt{T_1 T_2 T_3} \qquad T_5 = 1 \Big/ q\sqrt{T_1 T_2 T_3}$$

Theorem, conjectured by Nekrasov

$$\text{prefactor} \sum_{\pi} (-q)^{|\pi|} w_{3D}(\pi) =$$

$$= S^\bullet \frac{\sum_1^5 \left(T_i^{-1} - T_i\right)}{\prod_1^5 \hat{a}(T_i)}$$

What does this mean and why did
Nekrasov conjecture this ?



It has to do with the mysterious

M THEORY

which is supposed to unify supergravity with all other forces and matter fields in a unique 11-dimensional space-time harmony

M-theory has fields, namely the metric (graviton), its superpartner gravitino, and a 3-form analogous to the 4-vector potential in electromagnetism. They form a **single** representation of the sypersymmetry algebra.

The RHS of Nekrasov's conjecture is the supertrace of the action of

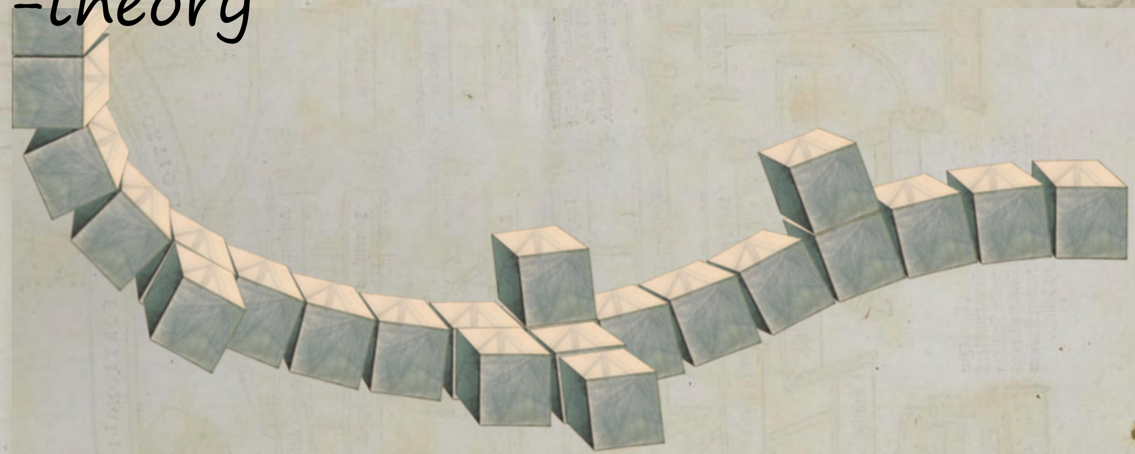
$$\mathrm{diag}(T_1,T_2,T_3,T_4,T_5) \in \mathbf{SU(5)}$$

on the fields of M-theory on the flat Euclidean space $\mathbf{C}^5=\mathbf{R}^{10}$

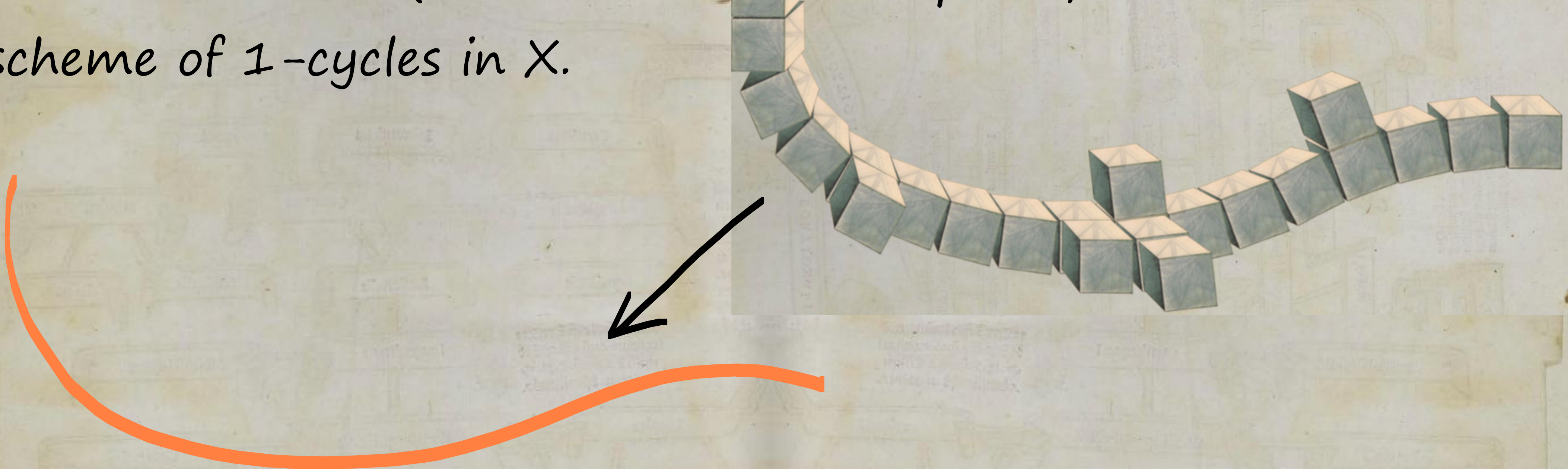
Thus, M-theory effectively sums up 3D boxcounting sums

M-theory also has extended objects, namely **membranes** (charged under 3-form) and their magnetic duals.

With Nekrasov, we conjectured a more general formula for summing up boxes "along a curve" in an algebraic 3-fold X in terms of membranes of M-theory



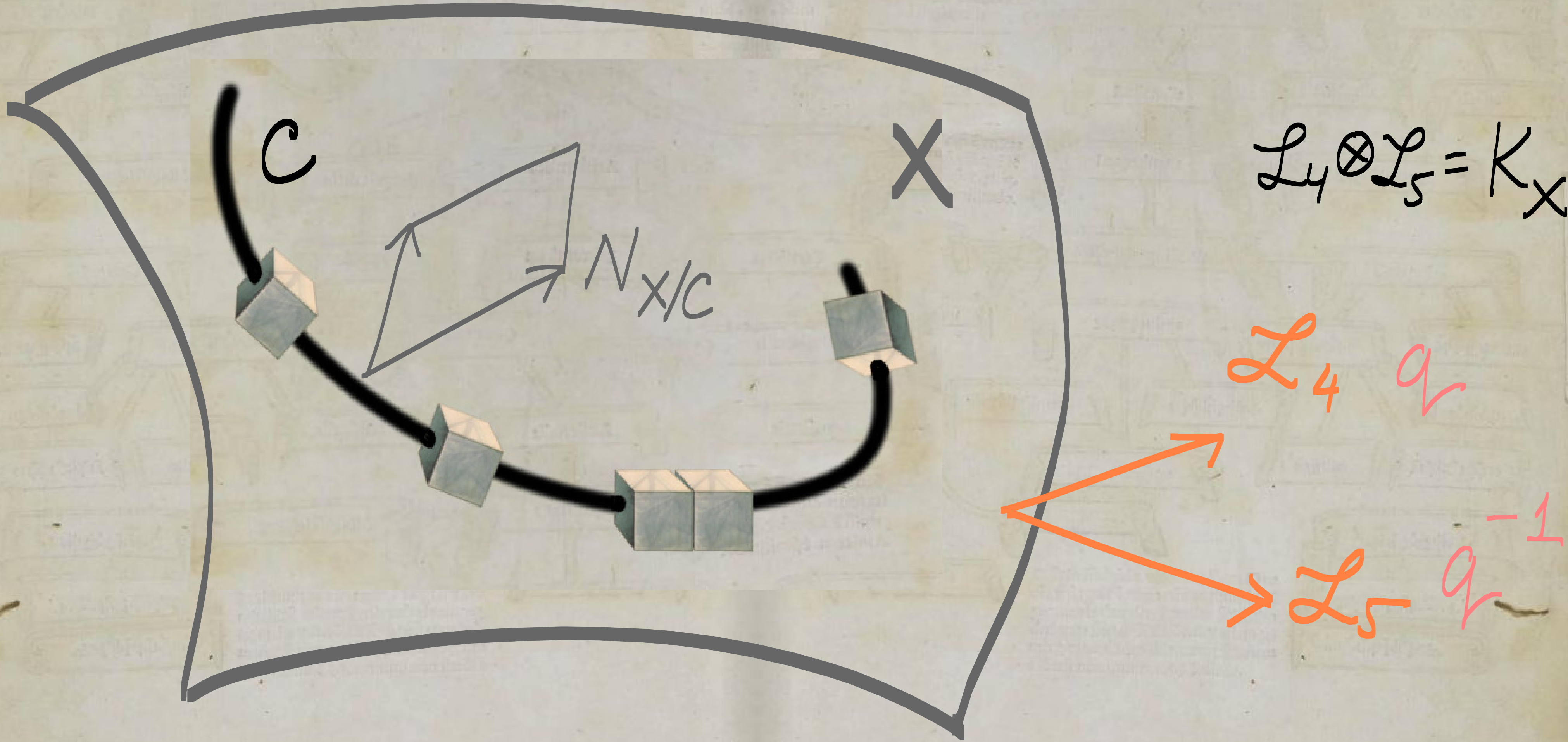
Allegories aside, this has to do with the map from the Hilbert scheme of curves (or other DT moduli spaces) in X to the Chow scheme of 1-cycles in X.



The push-forward of natural sheaves under this map with the weight $(-q)^{\chi}$ specializes to the previous boxcounting for $X=\mathbf{C}^3$ and the empty 1-cycle.

The way this works is best illustrated by the example of Pandharipande-Thomas spaces for a smooth curve C in X

moduli space = divisors D in C of degree n=0,1,2,3,...



virtual structures:

$$\mathrm{Obstruction} = H^{\bullet}(\mathcal{O}_D \otimes \Omega^1{}_C \otimes L_4{}^{-1} \otimes L_5{}^{-1})$$

$$K_{vir} = \det(\mathrm{Obs} - \mathrm{Def})$$

$$\tilde{\mathcal{O}}_{vir} = (-q)^n\, \mathcal{O}_{vir} \otimes \Big(K_{vir} \otimes \det H^{\bullet}(\mathcal{O}_C(D) \otimes (L_4 - L_5))\Big)^{1/2}$$

where L_4 and L_5 are line bundles analogous to the T_4 and T_5 variables from before.

we get

$$\chi\left(\mathrm{PT}, \tilde{\mathcal{O}}_{\mathrm{vir}}\right) \propto \tilde{S}^{\bullet} H^{\bullet}\left(C, q\mathcal{L}_4 \oplus q^{-1}\mathcal{L}_5\right)^{\vee}$$

corresponds to C moving in 4th and 5th dimensions

where

$$\tilde{S}^{\bullet} V = \det{}^{1/2} V \cdot S^{\bullet} V$$

Curiously, this is a 2-parameter (L_4 and L_5) generalization of another formula of Macdonald, the one for the Hodge numbers of a symmetric power of a curve.

Little is proven beyond this, but there is a strategy how to attack the general conjectures.

K-theoreric DT theory of 3-folds is, I believe, currently being solved along the lines similar to what [...OP,MO,MOOP,PP,...] did in cohomology. If one can replicate the same reductions in M-theory, that would be the proof.

in conclusion, one can see
quite far through here