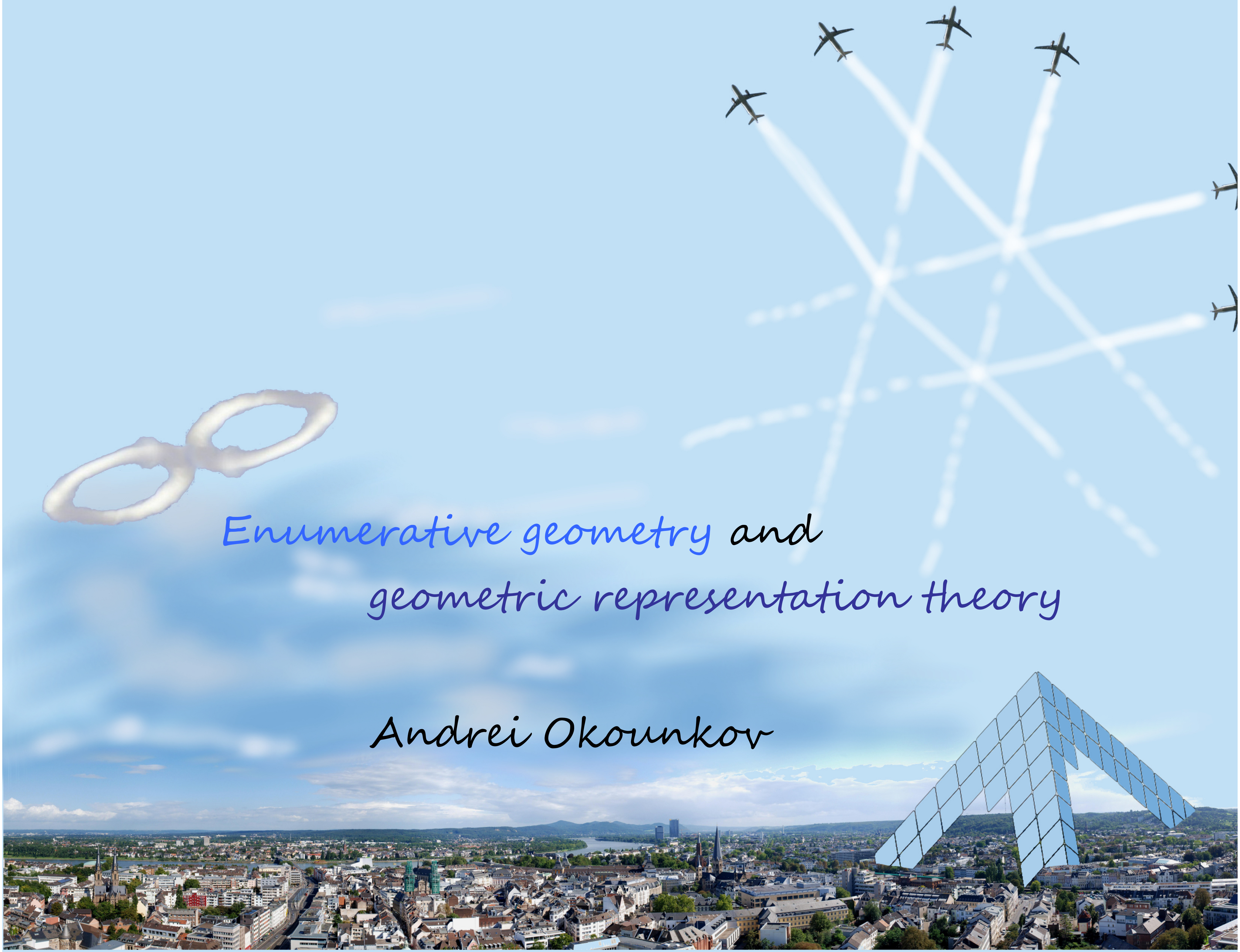
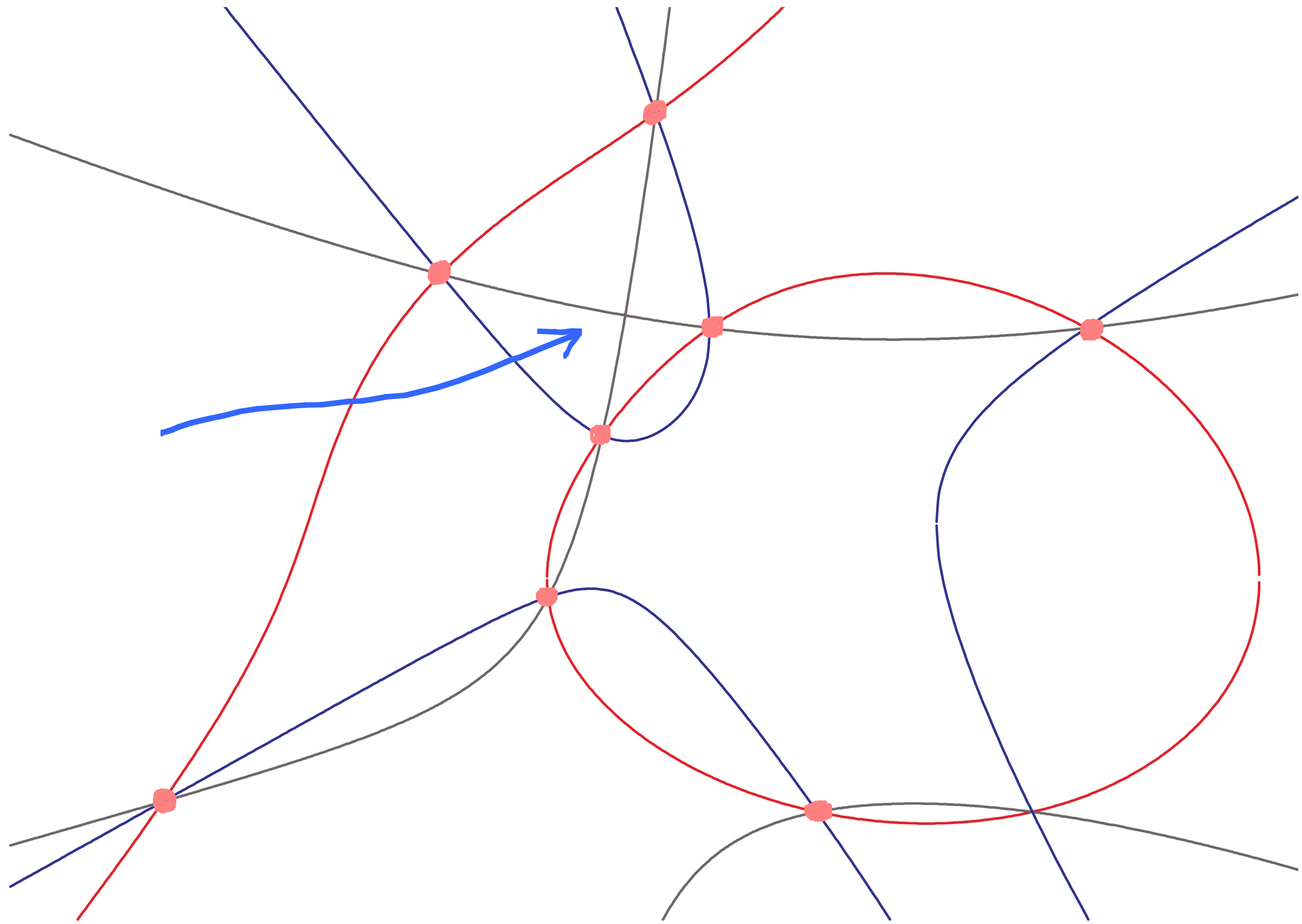


Enumerative geometry and geometric representation theory

Andrei Okounkov

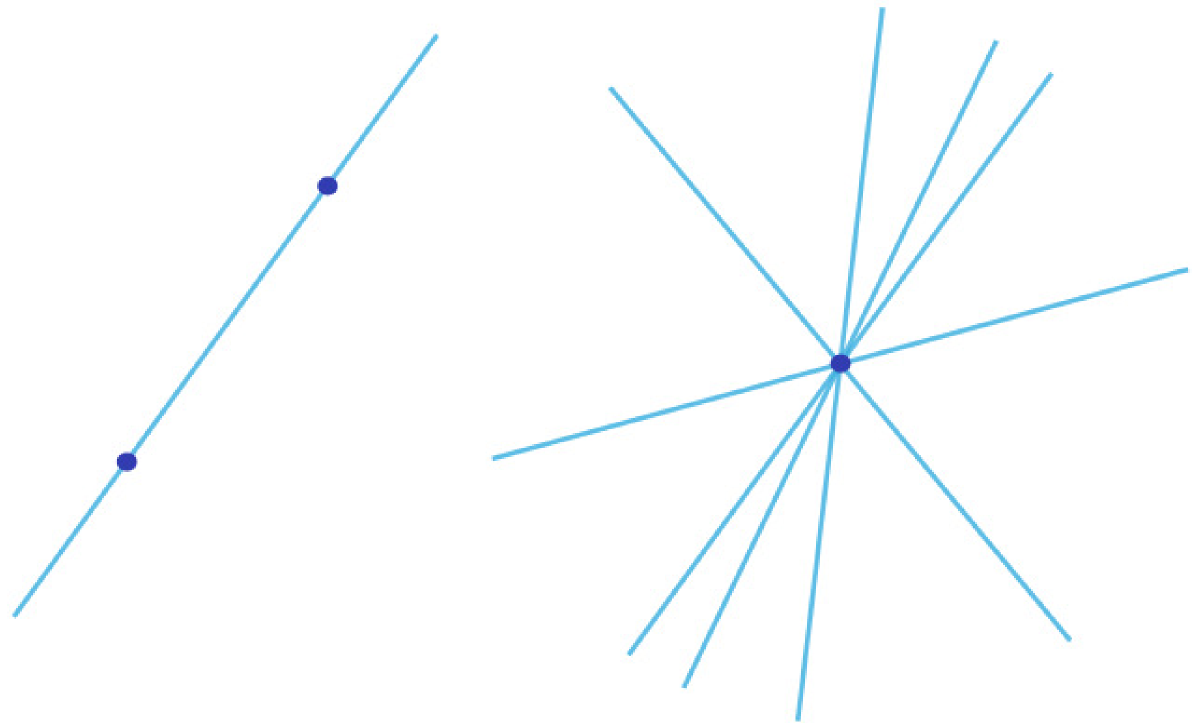


a classical problem in geometry is to count curves of given degree and genus (here, $d=3$ $g=0$) meeting given cyles (here, 8 points) in a variety X (here, plane)

the answer is the 3rd term in sequence 1, 1, 12, 620, 87304, 26312976, 14616808192, 13525751027392, 19385778269260800, 40739017561997799680,120278021410937387514880,482113680618029292368686080, 2551154673732472157928033617920, ...
[Kontsevich]

another popular sequence: 2875, 609250, 317206375, 242467530000, 229305888887625, 248249742118022000, 295091050570845659250, 375632160937476603550000, 503840510416985243645106250, 704288164978454686113488249750, 1017913203569692432490203659468875,1512323901934139334751675234074638000, ...[Givental....]

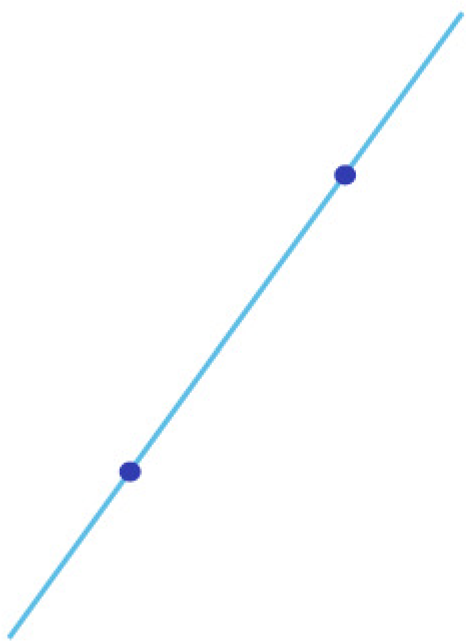
enumerative geometry is a land of large numbers and very complicated formulas

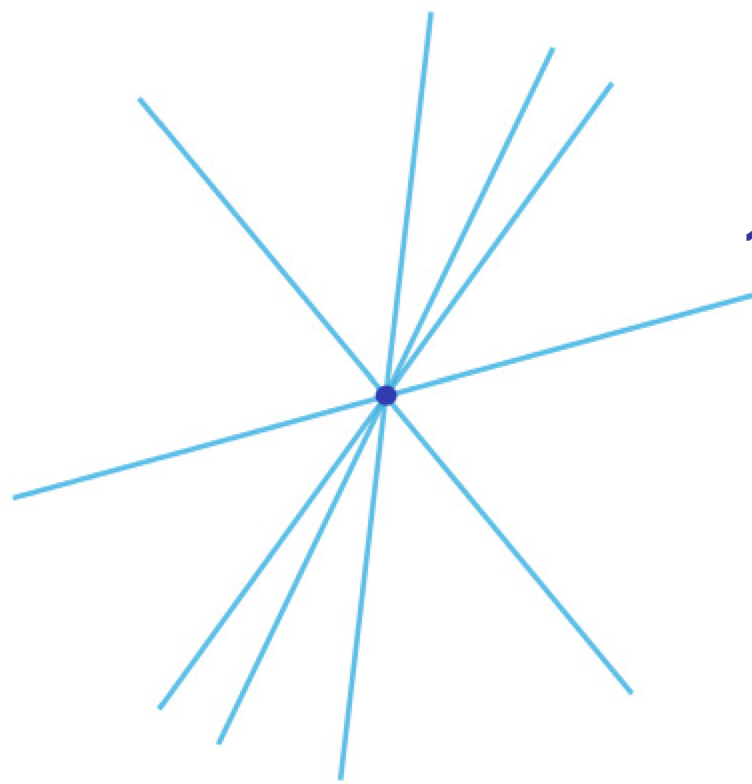


the 1st thing one learns about enumerative geometry: it is important to put things in general position

the 2nd thing one learns: don't be a slave of general position one should be able to count solutions even when they are not isolated ...

Maybe, we can take the Euler characteristic of the set of solutions? Fails badly in the very first example:





$$\chi(\mathbb{RP}^1) = 0$$

$$\chi(\mathbb{CP}^1) = 2$$

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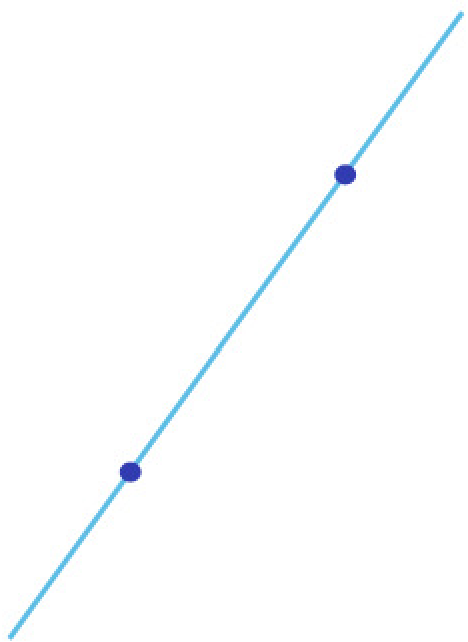
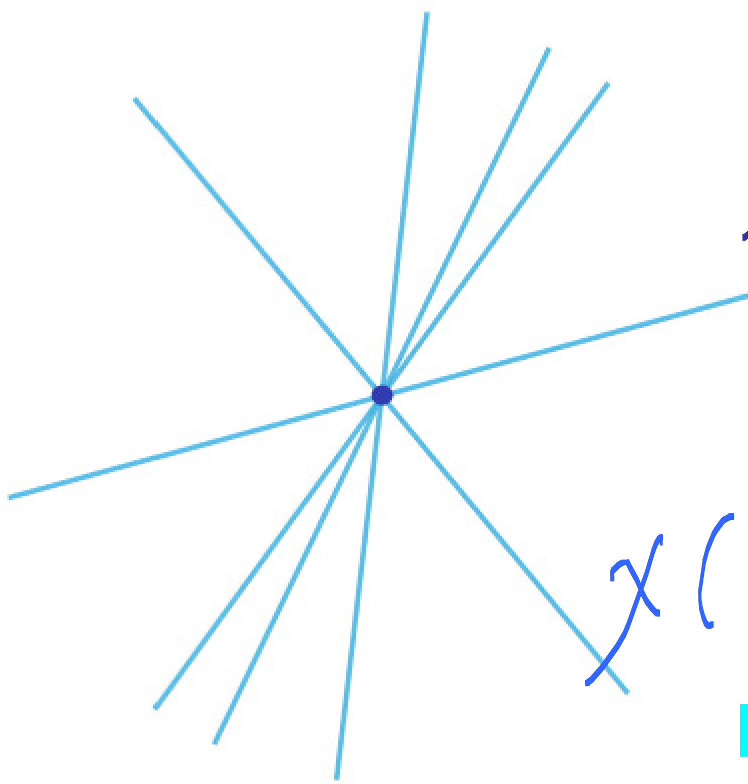




$$\chi(\mathbb{RP}^1) = 0$$

$$\chi(\mathbb{CP}^1) = 2$$

$$\chi(\mathcal{O}_{\mathbb{P}^1}) = \underline{1} - 0 = \underline{1}$$

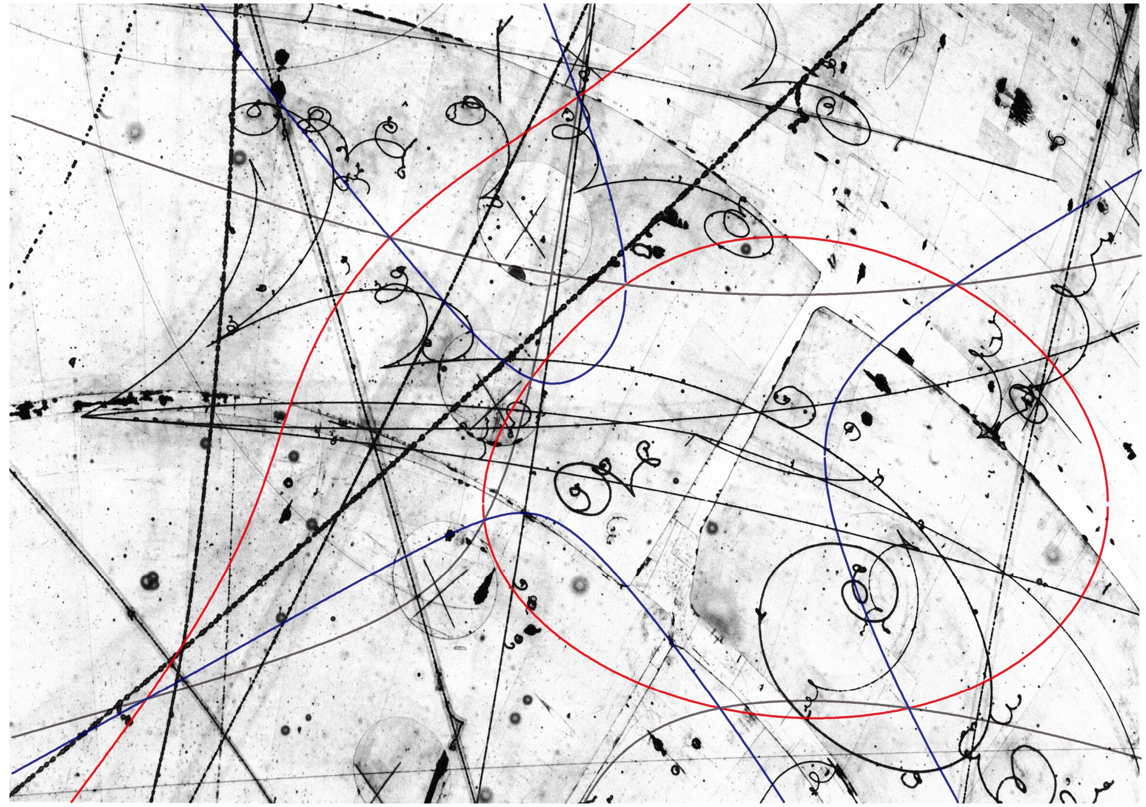
The right answer is that the enumerative constraints and deformations of curves in X put a certain sheaf $\bar{O}$ on the set of solutions, and we should take the Euler characteristic $\chi(\bar{O})$ of this sheaf.

$\bar{O} \approx$ polynomial functions (meaning, it is a coherent sheaf) and taking its Euler characteristic is a very standard thing to do in algebraic geometry.

Such formulation has many advantages, e.g. ...

If there is symmetry under the action of a group G, then $\chi(\bar{O})$ is a virtual representation of G. This is what it means to count in "equivariant K-theory".

Also, the problem and the answer make sense even if the classical enumerative problem is over/under determined.



Often, $\chi(\bar{O})$ has a direct interpretation in modern high energy physiscs as supertrace of a certain interesting operator over the Hilbert space of the theory

Today, we will talk about Donaldson-Thomas theory, which is an enumerative theory of curves in smooth algebraic 3-folds X, like the projective space $\mathbf{P}^3$. There is no need to assume X is Calabi-Yau, or anything like this, for the problem to be interesting and relevant.

In DT theory, one thinks about a curve C by thinking about equations that C satisfies. In other words, the DT moduli spaces are the Hilbert schemes (or closely related objects) which parametrize ideals the algebra of functions on X.

One can specify ideals by their generators, but can't get very far with the Hilbert scheme by working with such explicit data (Even the dimension of the Hilbert scheme is unknown !) Computations in DT theory are hard ...

... which is a good thing because most enumerative problems of interest, e.g. the two above, embed there

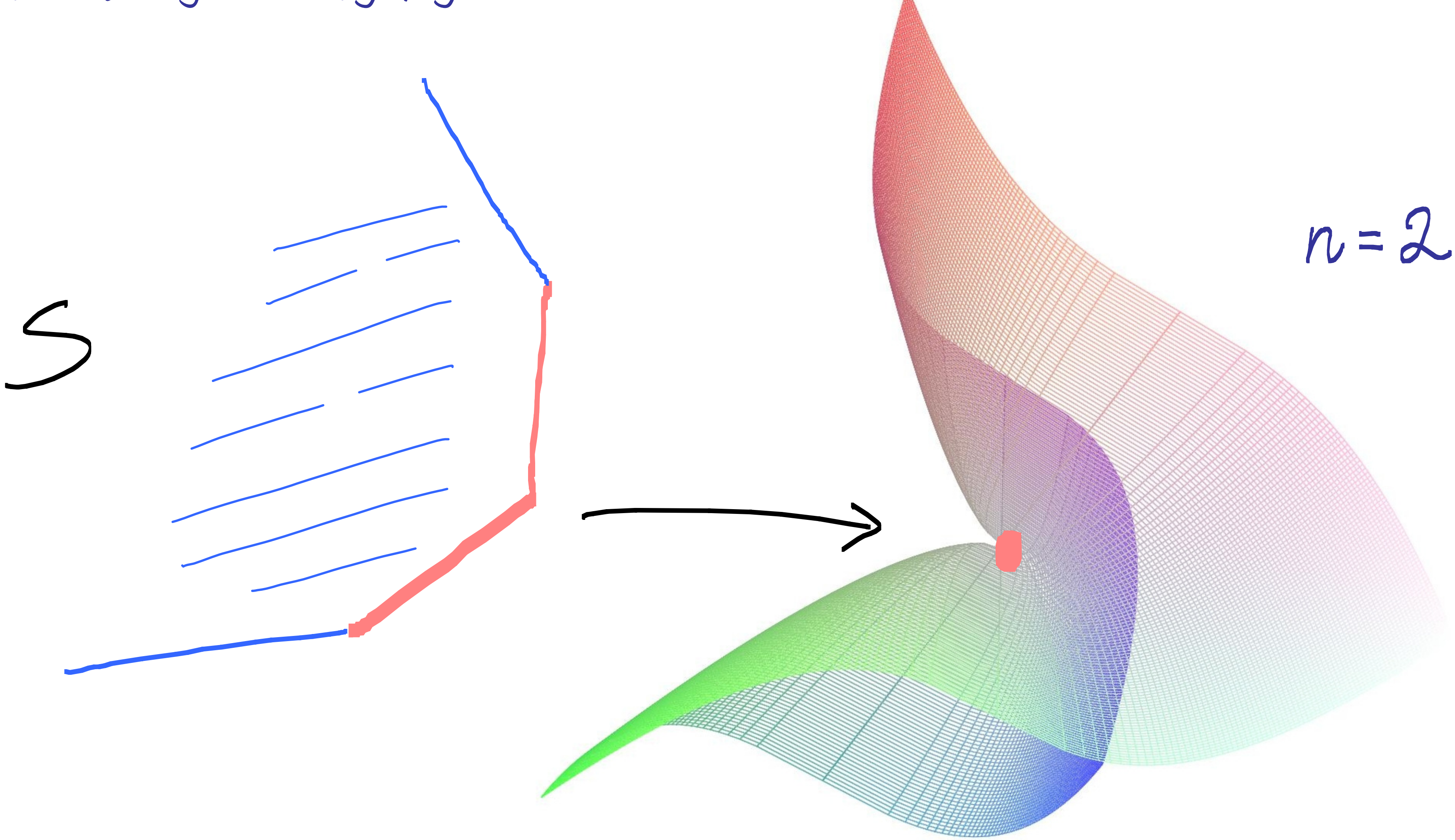
One gets most mileage out of breaking complicated curve counts into simpler standard pieces, which can be analyzed by specific means. Such reductions is an art, practiced e.g. by [MOOP], and elevated to near perfection by Pandharipande and Pixton

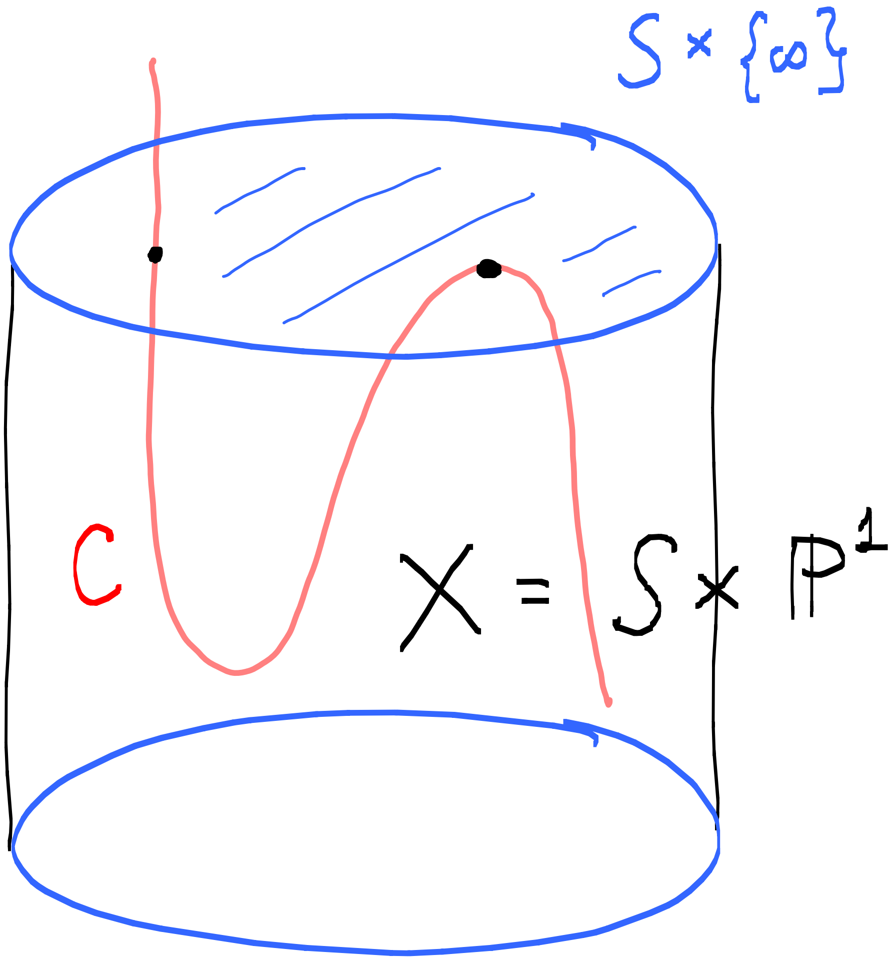
Today we will talk about those standard pieces. They are defined like this ...

Let S be an algebraic surface, such as

$$S = \mathbb{C}^2 = \text{coordinate plane} = \mathbf{A}_0$$

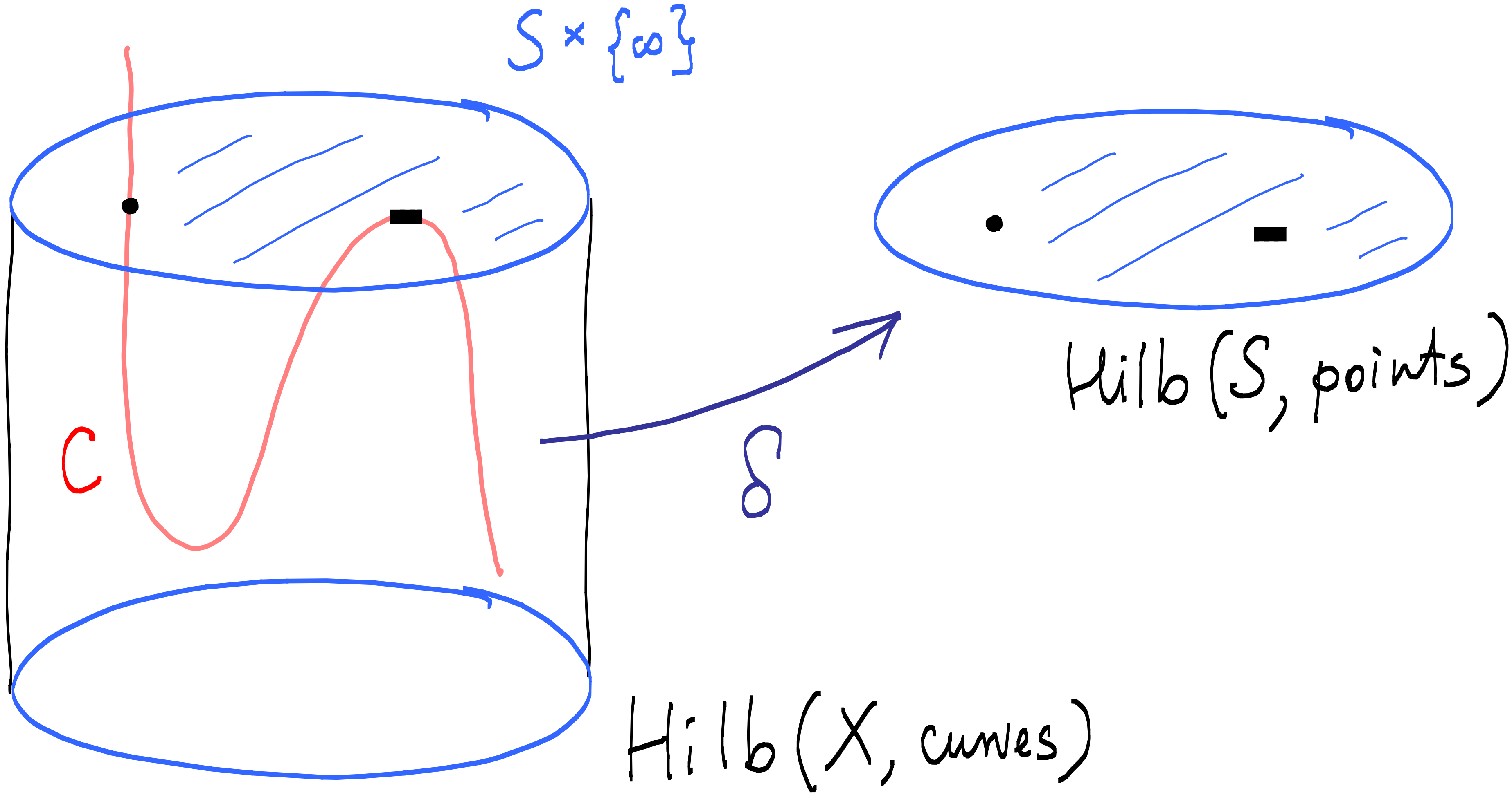
What one really needs are the surfaces $\mathbf{A}_n$ which resolve the singularity $xy = z^{n+1}$



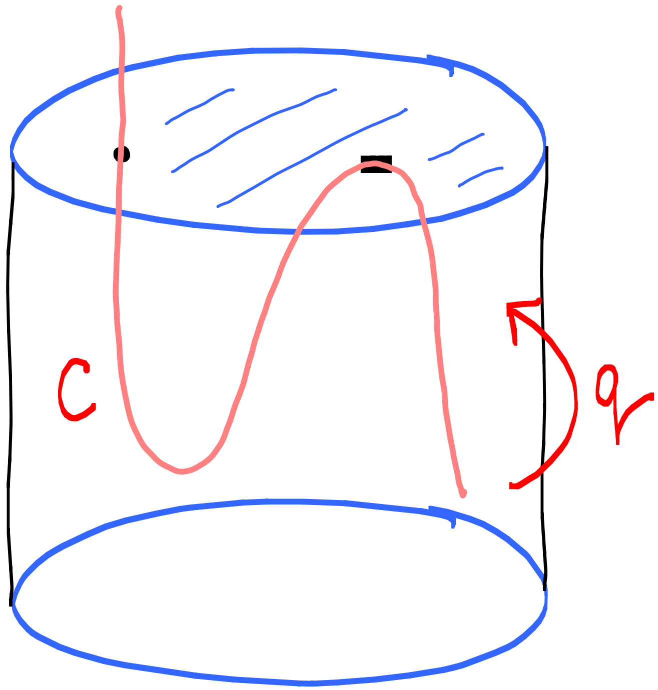


We take $X = S \times \mathbf{P}^1$ and consider an open set U in the Hilbert scheme of curves in X formed by curves C that intersect $S \times \{\infty\}$ in points.

On U, we have the intersection map δ



and we can push forward $\bar{O}$ under δ, i.e. count curves in the fibers of δ, as long as



1) we keep track of discrete invariants of C by weighting the count by $z^{[C]}$, where $[C]$ is the class

2) keep track of the action of $q \in GL(1)$ on the $\mathbf{P}^1$-direction in X

The result

$$V = \delta_*(\hat{\mathcal{O}}) \in K(\mathrm{Hilb}(S))[[z,q]]$$

has various names, including the "vertex" these vertices contain a profusion of enumerative information, the data of all possible ways a curve of some degree and genus can meet S in this geometry, like e.g. a 3fold tangency at some given point of S

remarkably, can be repackaged much more economically ...

Fundamental fact:

fundamental class

$$V = U^{-1}\, \mathcal{O}_{\mathrm{Hilb}(S)}$$

where U is the fundamental solution of a certain linear difference equation with regular singularities in z. This linear equation ("quantum connection") achieves a very dense packing of enumerative information.

$K(X)$

the shifts in this difference equations

$$z^{C} \longmapsto q^{(\mathcal{L}, C)}\, z^{C}, \qquad C \in H_2(\mathrm{Hilb}(S))$$

are line bundles on Hilb(S)

$$\mathcal{L} \in \mathrm{Pic}(\mathrm{Hilb}(S)) = \mathrm{Pic}(S) \oplus \mathbb{Z}$$

a poetic, but also technical, analogy:

quantum connection

3-folds

vertices

the computation of this quantum connection, in fact, for all Nakajima varieties and not just Hilbert schemes of ADE surfaces, is a recent result of Andrey Smirnov and the speaker.

in cohomology instead of K-thery, the computation of the corresponding differential equation is the main result of the book by [Maulik-O]. In turn, it generalizes ealier formulas of [OP] and [Maulik-Oblomkov], as well as key insights of [Nekrasov-Shatashvili] and Bezrukavnikov and his collaborators.

to write a difference equation, we need a supply of operators and, in fact, there is a geometrically defined action of a very big algebra

$$\mathcal{U}_\hbar\left(\widehat{\widehat{\mathfrak{gl}(n)}}\right) \curvearrowright K(\mathrm{Hilb}(A_{n-1}))$$

that extends the work of Nakajima. Here

$$\mathcal{U}_{\hbar}\left(\widehat{\widehat{\mathfrak{gl}(n)}}\right) = \text{a Hopf algebra deformation of } \mathcal{U} \text{ of } \mathfrak{gl}(n) \otimes \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$$

"Hopf algebra" means that there is tensor product on its representation, which, however, is not commutative !

This is what it means to be a "quantum group" in fact, geometrically, the fundamental object is precisely the braiding:

$$K(\mathrm{Hilb}(S)) \otimes K(\mathrm{Hilb}(S))$$

$$K(\mathrm{Hilb}(S)) \otimes K(\mathrm{Hilb}(S))$$

which is constructed first, and then it gives birth to everything else [Maulik-O]

E.g. one can take matrix elements of the braiding in one factor, to get operators in the other !

$$K(\mathrm{Hilb}(S)) \otimes K(\mathrm{Hilb}(S))$$

$$K(\mathrm{Hilb}(S)) \otimes K(\mathrm{Hilb}(S))$$

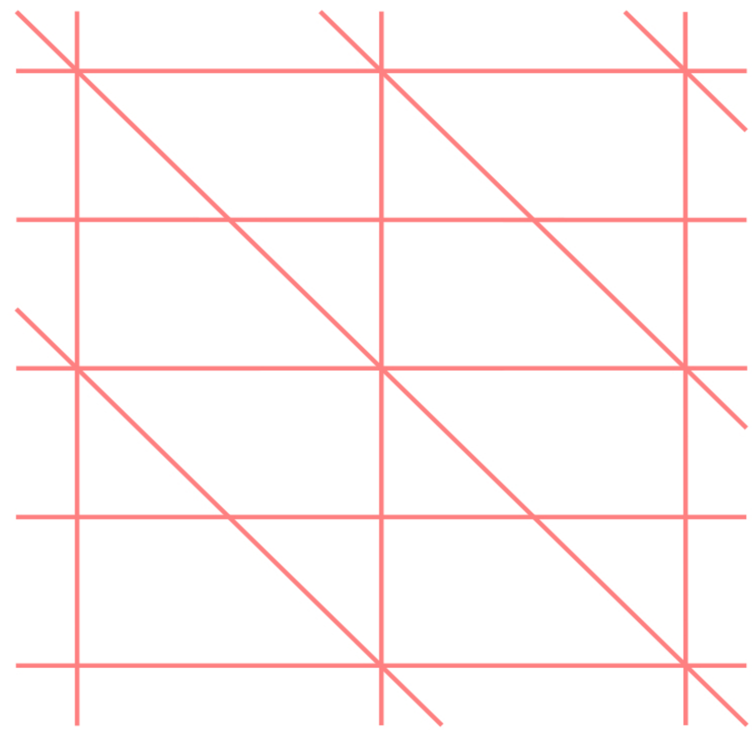
The braid relation (Yang-Baxter equation) then tells you how these operators commute.

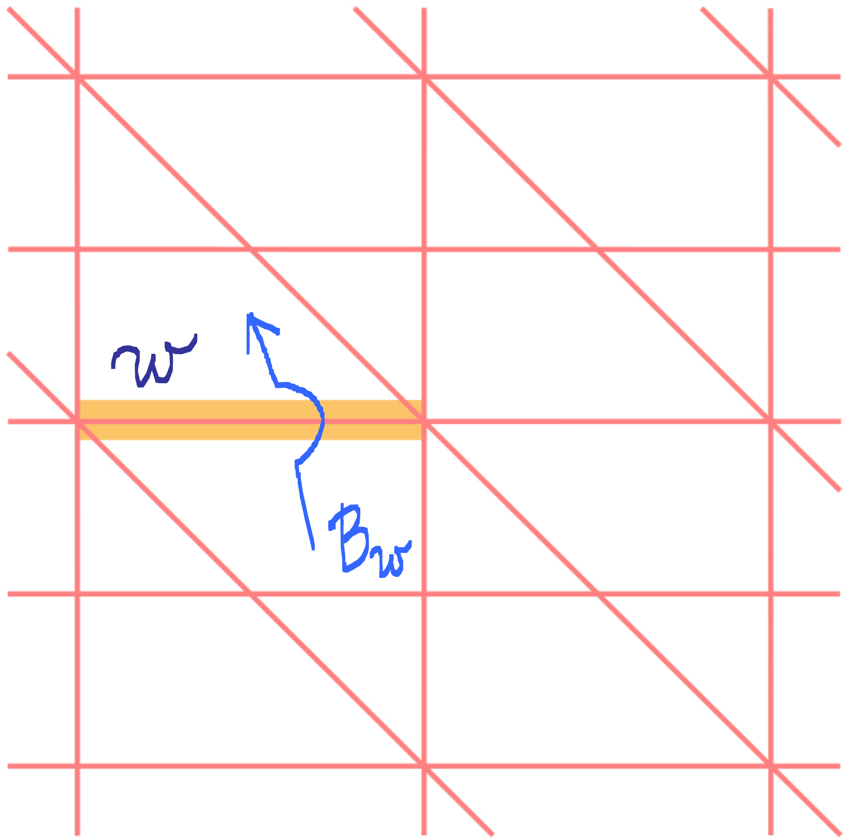
Especially convenient for construction actions of algebras whose generators and relations are not known ahead of time

In this quantum group, generalizing the work of Etingof and Varchenko for Kac-Moody Lie algebras, we construct an action of a "quantum dynamical affine Weyl group", which is really a braid grouppoid of a certain hyperplane arrangement, namely the arrangement given by the roots of

$$\hat{\mathfrak{g}} = \widehat{\widehat{\mathfrak{gl}(n)}},$$

or more general ....





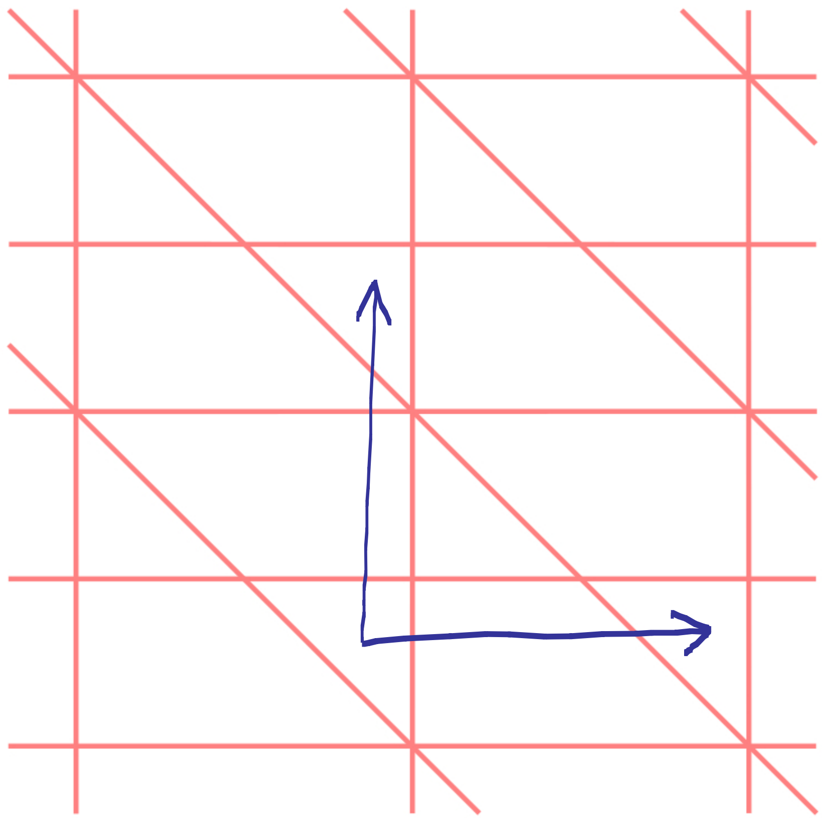
Every wall w in the arrangement of affine roots corresponds a rank=1 subalgebra

$$\mathcal{U}_q(\mathfrak{g}_w) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$$

We construct the "going over the wall" operators by associating a certain universal element to any braided Hopf algebra U

$$B_w \in U[[z]] \qquad U = \mathcal{U}_q(\mathfrak{g}_w)$$

dynamical, or degree, variables



The lattice Pic(Hilb(S)) lies in the dynamical braid grouppoid and we prove:

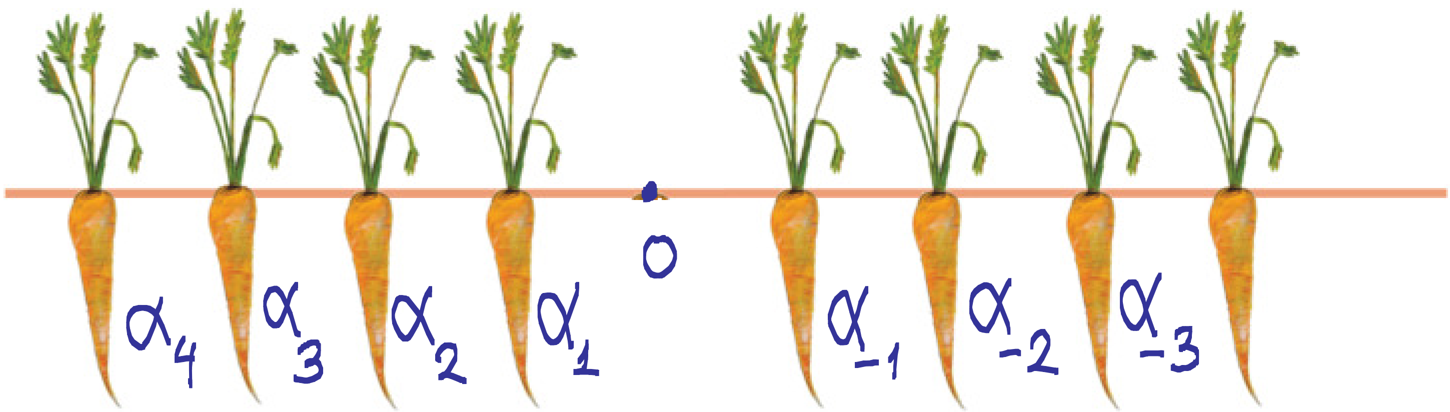
Theorem(O-Smirnov) The corresponding operators give the quantum difference connection, in fact, for any Nakajima variety

For example, for Hilb($\mathbb{C}^2$,points), we have

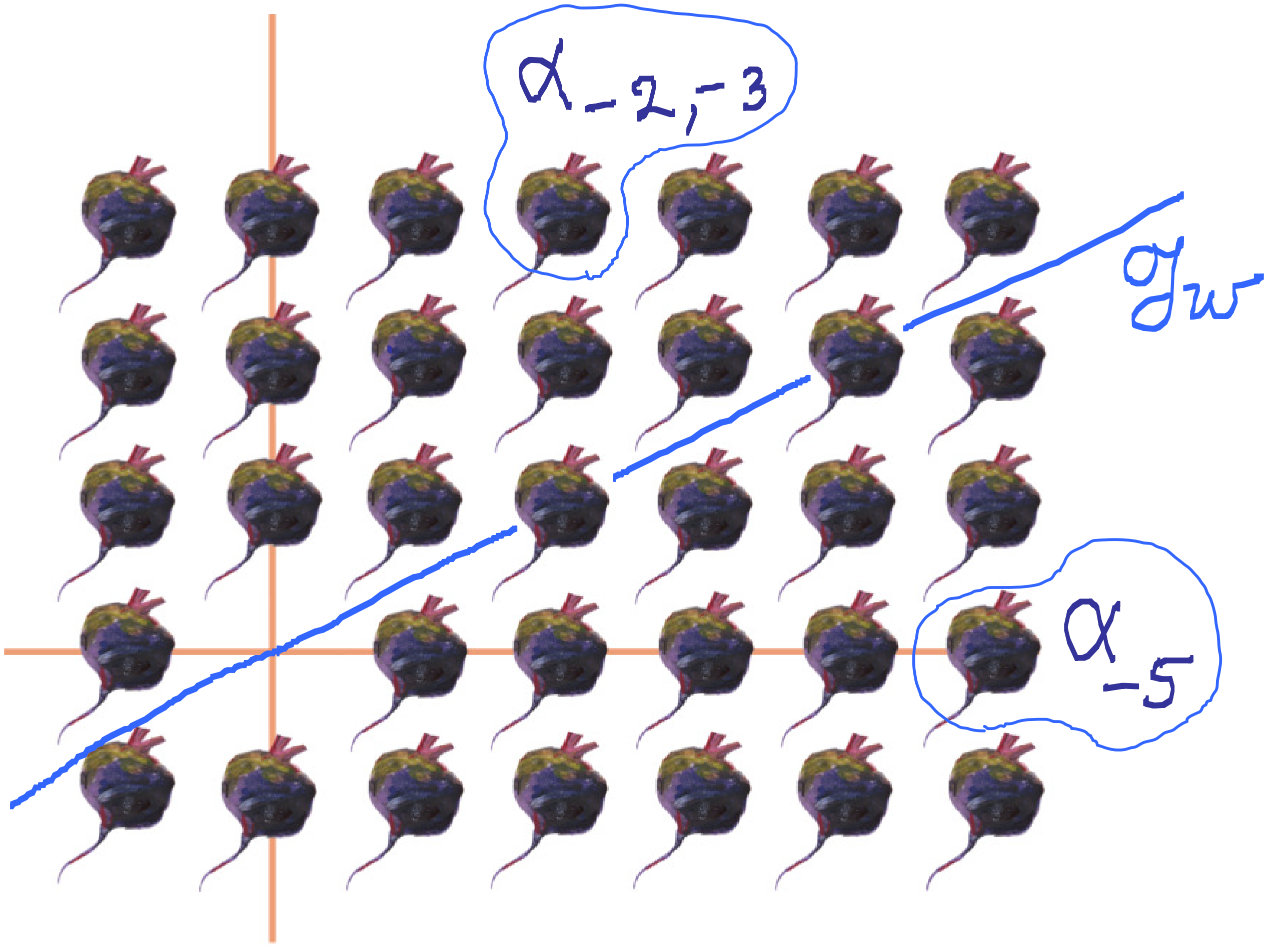
$$\mathfrak{g} = \widehat{\mathfrak{gl}(1)}$$

with roots $\mathbb{Z}\setminus\{0\}$ and the operators α_n

$$[\alpha_n, \alpha_m] = n\,\delta_{n+m}$$



acting by correspondences on Hilb(k) x Hilb(k+n) contructed by Nakajima



affine roots $= \mathbb{Z}^2 \setminus \{0\}$ and the corresponding operators on $K(\mathrm{Hilb})$ were studied by many authors with A.Negut giving perhaps the most geometric description

each root and its opposite belong to a Heisenberg subalgebra of the form

$$[\alpha_{n,m}, \alpha_{-n,-m}] = c_{n,m} - c_{n,m}^{-1}$$

central and grouplike

and the purely-quantum part of the difference connection becomes

$$\overleftarrow{\prod_{0 \le m < n}} \mathcal{B}_{n,m}$$

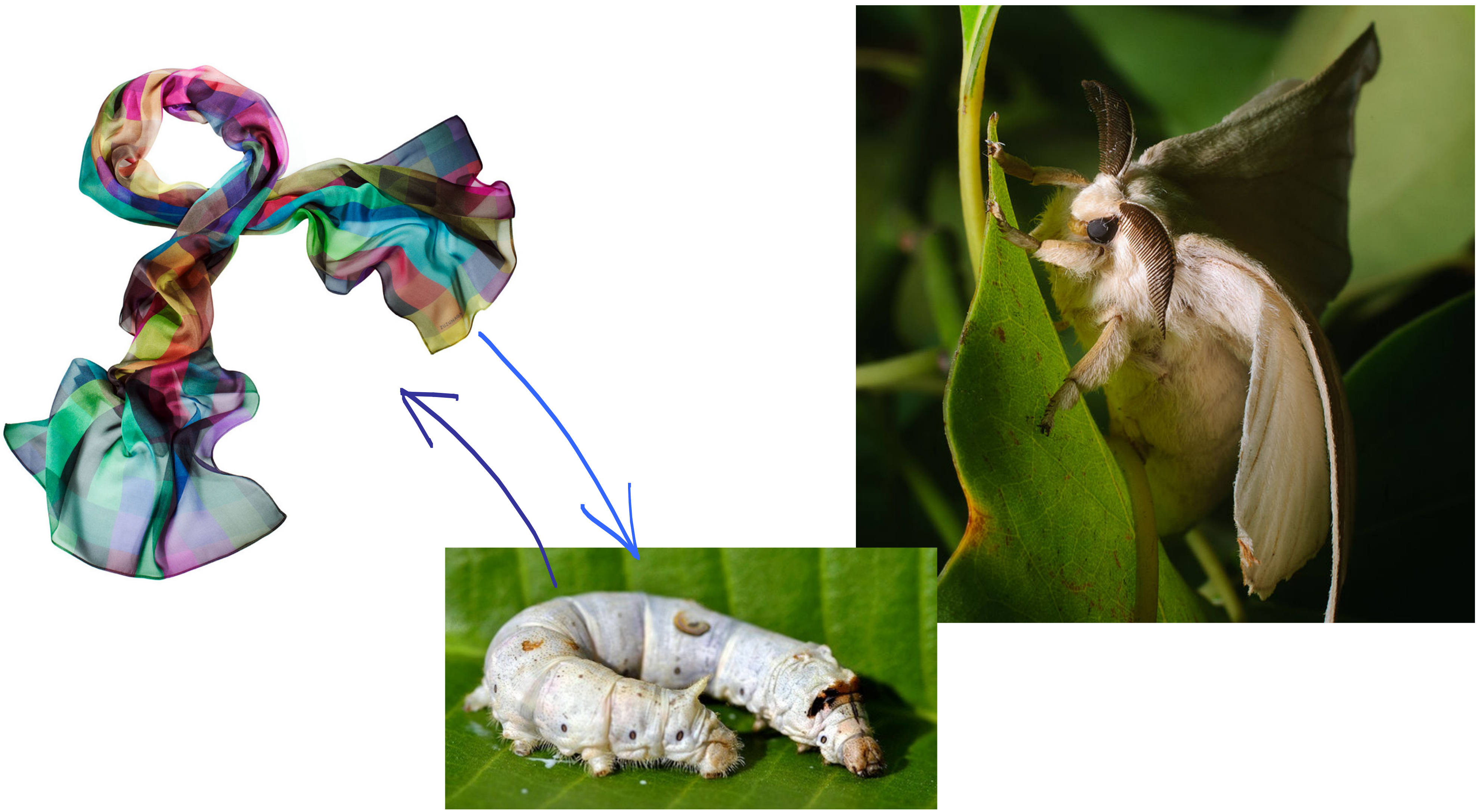
where

$$\mathcal{B}_{n,m} = :\exp\left(\frac{c_{n,m}}{1 - z^{-n} q^{-m}} \alpha_{-n,-m} \alpha_{n,m}\right):$$

the curve class variable

step of the q-difference equation $\in GL(1)$

In summary, I tried to explain that representation theory, in one of its forms, is a silkworm.



I did not have time to talk about the applications we have in mind, but ...

Just like the determination of quantum differential equation is a key step in the proof of the known cases of GW/DT correspondence, there is every reason to expect our result to be very relevant for

- conjectural equivalence of K-theoretic DT counts with the 5-dimensional membrane counting [Nekrasov-O], as well as
- conjectural equivalence of K-theoretic curve counts in symplectically dual varieties
- ...

a discussion of some of these conjecctures may be found here


Andrei Okounkov's web pa

www.math.columbia.edu/~okounkov/

Welcome to Andrei Okounkov's web page

here you can find the schedule of the Informal Mathematical Physics seminar

homework for my Lie Groups class: 1 ,

and materials from some of my talks:

Enumerative geometry and representation theory, Algebraic Geometry 2015, Part 1, Part 2, Part 3

Boxcounting, Lezione Leonardesca, June 2015

Quantum Groups and Quantum K-theory, UCLA February 2015

The index of M-theory, this version from the AMS-[illegible]U joint meeting, June 2014

Quantum Groups and Quantum cohomology, Princeton lectures, Spring 2014, courtesy of princetonmathematics and Changjian Su

Simons Lectures at MIT, "Applied noncommutative geometry", Lecture 1, Lecture 2, Lecture 3, May 2010.

Limit shapes: real and imagined, AMS Colloquium lectures, 2007


Thank you