Mathematical Sciences Research Institute

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Seminar

GRTA Applications and Open Problems: Characters, and other special functions, from the point of view of the enumerative geometry

May 03, 2018 (10:30 AM PDT - 12:00 PM PDT)

PARENT PROGRAM: 1. GROUP REPRESENTATION THEORY AND APPLICATIONS

LOCATION: MSRI: SIMONS AUDITORIUM

Speaker(s)

Andrei Okounkov (Columbia University)

Description

No Description

Video

No Video Uploaded

Abstract/Media

Characters of Lie algebras and related algebras (both in zero and prime characteristic) fit into a larger class of special functions of, essentially, q-hypergeometric type, that is, solutions of certain regular q-difference equations. Basic phenomena of representation theory, like the appearance of a submodule under a specialization of parameters, have analytic counterparts in this broader setting. My goal in this talk is to explain the enumerative geometry perspective on both the representations and q-difference equations in question, following ideas from joint projects with Roman Bezrukavnikov and Mina Aganagic.

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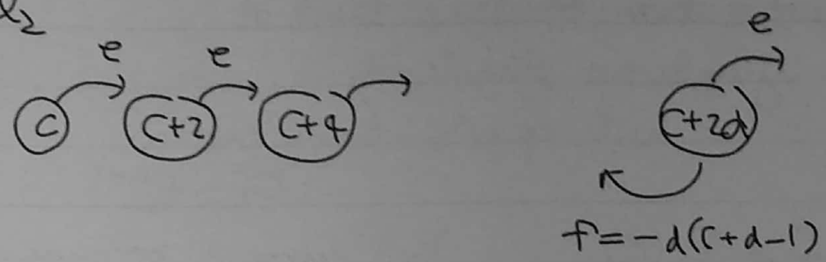
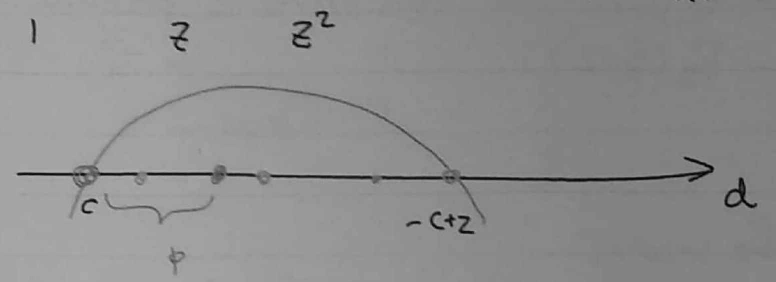
Notes by H. Nakajima

sl_2



(Bezrukavnikov
Aganagic



q-hypergeometric function

$$F\left[\begin{matrix} \cdot\ \cdot \\ c \end{matrix} ; z\right] = \sum_d z^d \frac{\cdots}{(q)_d (c)_d}$$

$(x)_d = (1-x)(1-qx)\cdots$
$\cdots(1-q^{d-1}x)$

↑ solution on 2^{nd} order q-diff. eq. in z,
but also in c, also t, $t = q$ then

$\frac{\cdots}{\cdots} = 1$

get the usual character

$(f(qz) = A(z) f(z))$

$$\frac{1}{(1-q)\cdots(1-q^d)(1-c)\cdots(1-q^{d-1}c)}$$

$q^p = 1$

↑ zero if $p | d$

⇓

pole

→ solution breaks to pole part & reg. part

→ Verma + irreducible

($c = q^{\tilde{c}}$ similar to $\tilde{c} + d - 1 \equiv 0$ mod p

algebra $\hat{X}$ = a quantization of some kind of symplectic singularity such as A_n-singularities $ef = h^{n+1}$

$ef = P_2(h)$
$fe = P_2(h-2)$

P_2 : degree 2 polynomial in h

h: Cartan

branches of the moduli spaces of vacua in certain 3d theories, such as

$X_0 = \mu^{-1}(0)/G$

$\nearrow$

$X = \mu^{-1}(0) /\!\!/_{\alpha} G$

difference equation = quantum difference equations for X

(Known for $\forall$ Nakajima varieties in terms of certain quantum groups)

gKZ

Main important feature:
these singularities should come in pairs $X \leftrightarrow X^\vee$
have same quantum difference equations but with exchange of param.

$z \leftrightarrow a^\vee$

$a \leftrightarrow z^\vee$

$\left(z \leftarrow \text{variable in the Cartan torus} \subset \mathrm{Aut}(X, \omega) \right)$

$\mathrm{Aut}(X,\omega) \supset \text{max torus} = \mathrm{Pic}(X^\vee) \otimes \mathbb{G}_m$

characters like z in F

deformation like C

Main idea in the subject:

$\mathfrak{X} \curvearrowright K_{T,\mathrm{crit}}$ (suitable moduli space of holo maps  $\xrightarrow{f} X^\vee$)

by Hecke modification at a point of B

$B \leftarrow$ space

$q \in \mathbb{C}^\times \curvearrowright$

In particular $B =$  $\circlearrowleft_q$ or $B = \mathbb{C}$

quasimaps for $X^\vee$
$\parallel$
$lci /\!/ G$

$QM(\mathbb{P}^1 \to X^\vee) \supset$ nonsingular at $\infty \in \mathbb{P}^1$

analog of F

$$= \sum_{d=\deg f} z^d \chi(QM \text{ of degree } d, \hat{\mathcal{O}}_{vir})$$

fully equiv. K-theory

$\hat{\mathcal{O}}_{vir}$ like $(-1)^i \Omega^i$ moduli

Substitution

$\hbar = q$ means restriction to CY torus $\subset T$

For $B =$ disk, $\mathbb{C}$, $\mathbb{P}^1$ nonsing at ∞

$$K_{\mathrm{crit}, T_{CY}}(\text{moduli}) = \text{Verma module for } \hat{\mathfrak{X}}$$

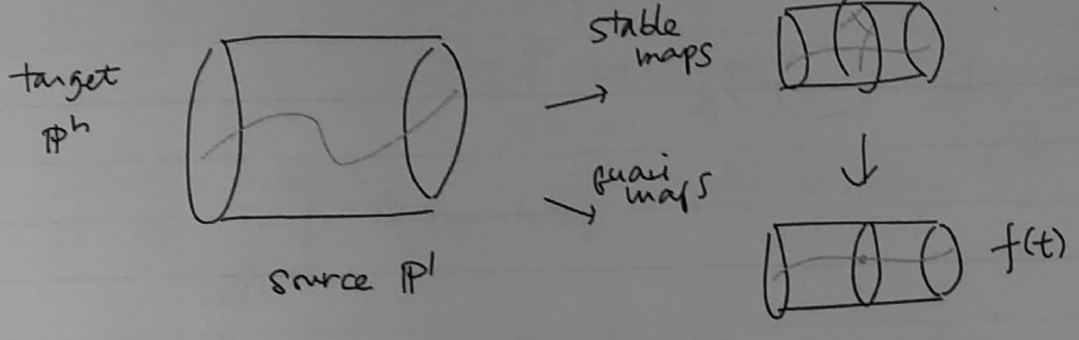
(Bullimore, Dimofte, Gaiotto, Hilburn, Kim)

<u>Main feature</u>: Deformation variable like c become equivariant variables for $X^\vee$

$\Rightarrow$ specialization of equiv. variable $\rightsquigarrow$ analysis of fixed pts

$\hat{A}_n \curvearrowright$ quasimaps to $T^*\mathbb{P}^n$

(maps to $\mathbb{P}^n$)



only remember the location of singularities and mult.

$f(t) = [P_0(t) : P_1(t) : \cdots : P_n(t)]$ $\quad f(\infty) = [1:0:\cdots:0]$

(can be different fixed pts $\Updownarrow$ diff. highest wt)

$P_0(t) = t^d + \cdots$

$\cdots\ P_i(t) = 0 \cdot t^d + * t^{d-1} + \cdots$

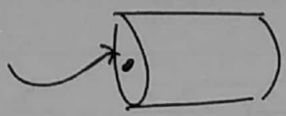
So QM is vector space of dim $d(n+1)$

$\begin{bmatrix} 1 & & & 0 \\ & a_1 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} \curvearrowright \mathbb{P}^n$

$\mathbb{C}^\times_\hbar : t \mapsto t\hbar$ rescale on the domain

character

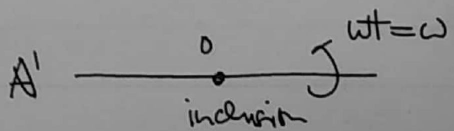
$$\begin{bmatrix} \hbar & \hbar^2 & \cdots & \hbar^d \\ a_1\hbar & a_1\hbar^2 & \cdots & a_1\hbar^d \\ \vdots & \vdots & & \vdots \\ a_n\hbar & a_n\hbar^2 & \cdots & a_n\hbar^d \end{bmatrix} \rightsquigarrow \text{cf. } (\hbar)_d \text{ in the beginning}$$

Hecke modification $\pm$ at 

$\mathcal{M}_d \subset \mathcal{M}_{d+1}$ via ι

$e = \iota_*$
$f = \iota^*$

$$\begin{bmatrix} f & f^2 & \cdots & f^d \\ a_1 f & a_1 f^2 & \cdots & a_1 f^d \\ \vdots & \vdots & & \vdots \\ a_n f & a_n f^2 & \cdots & a_n f^d \end{bmatrix}$$



$\iota^* \iota_* = (1-\omega^{-1})$
mult. by

$\mathcal{O}_0 = (1-\omega^{-1})\, \mathcal{O}_{\mathbb{A}^1}$

$ef = \text{polynomial}_{n+1}(h)$ $h = f^{-d}$
$= \prod (1 - a_i^{-1} h)$ $(a_0 = 1)$

$fe = \text{poly}_{n+1}(f^{-1} h)$
↑ same polynomial

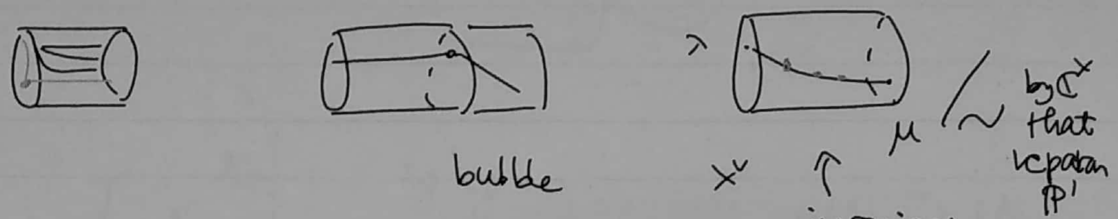
(NB. same polynomial) $\iota_* \iota^*$ same

h comes from tautological b'dle $|_0$

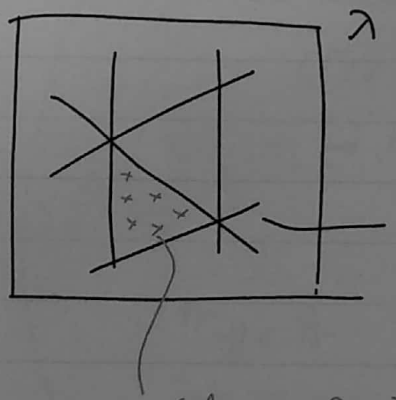
$f \to 1$ $ef = fe = h^{n+1}$ (A_n singularity)

Geometric analogs of KL numbers

"rubber"



$\zeta = e^{2\pi i/p}$ $\quad a = \zeta^\lambda$ $\circlearrowleft$ a commuting ζ-diff.



get a solution of ζ-diff. equ in $\mathbb{Z}$ for every simple modules

$\circlearrowright$ moved around by the conn. in $\underline{\underline{a}}$

Rems

$p \to \infty$, $\zeta \to 1$ $\quad$ ζ-diff. conn. $\Rightarrow$ to the quantum differential equ.