ON LEIGHTON'S GRAPH COVERING THEOREM

WALTER D. NEUMANN

Abstract. We give short expositions of both Leighton's proof and the Bass-Kulkarni proof of Leighton's graph covering theorem, in the context of colored graphs. We discuss a further generalization, needed elsewhere, to "symmetry-restricted graphs." We can prove it in some cases, for example, if the "graph of colors" is a tree, but we do not know if it is true in general. We show that Bass's Conjugation Theorem, which is a tool in the Bass-Kulkarni approach, does hold in the symmetry-restricted context.

Leighton's graph covering theorem says:

Theorem (Leighton [5]). *Two finite graphs which have a common covering have a common finite covering.*

It answered a conjecture of Angluin and Gardiner who had proved the case that both graphs are k–regular [1]. Leighton's proof is short (two pages), but has been considered by some to lack transparency. It was reframed in terms of Bass-Serre theory by Bass and Kulkarni [2, 3], expanding its length considerably but providing group-theoretic tools which have other uses.

The general philosophy of the Bass-Kulkarni proof is that adding more structure helps. Let us illustrate this by giving a very short proof of Angluin and Gardiner's original k–regular case.

We assume all graphs considered are connected. "Graph" will thus mean a connected 1–complex. "Covering" means covering space in the topological sense. Two graphs are isomorphic if they are isomorphic as 1–complexes (i.e., homeomorphic by a map which is bijective on the vertex and edge sets). We want to show that if G and G' are finite k-regular graphs (i.e., all vertices have valence k) then they have a common finite covering.

Proof of the k–regular case. Replace G and G' by oriented "fat graphs"—thicken edges to rectangles of length 10 and width 1, say, and replace vertices by regular planar k-gons of side length 1, to which the rectangles are glued at their ends (see Fig. 1; the underlying space of the fat graph is often required to be orientable as a 2-manifold but we don't need this). G and G' both have universal covering the k–regular fat tree T_k, whose isometry group Γ acts properly discontinuously (the orbit space T_k/Γ is the 2–orbifold pictured in Fig. 2). The covering transformation groups for the coverings $T_k \to G$ and $T_k \to G'$ are finite index subgroups Λ and Λ' of Γ. So $\Lambda \cap \Lambda'$ is finite index in each of Λ and Λ' and the quotient $T_k/(\Lambda \cap \Lambda')$ is a common finite covering of G and G'. □

We return now to unfattened graphs. In addition to the simplicial view of graphs, it is helpful to consider in parallel a combinatorial point of view, in which an edge

2010 *Mathematics Subject Classification.* 20F65, 05C70, 20E08.

Key words and phrases. Leighton's theorem, graph covering, tree lattice.

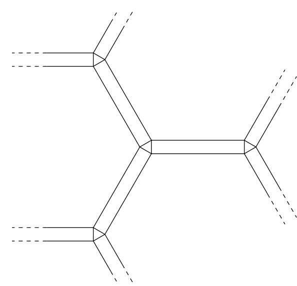

FIGURE 1. 3-regular fat graph

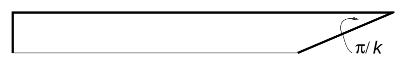

FIGURE 2. $T_k/\operatorname{Aut}(T_k)$; bold edges are mirror edges.

of an undirected graph consists of a pair $(e, \bar{e})$ of directed edges. From this point of view a graph G is defined by a vertex set $\mathrm{V}(G)$ and directed edge set $\mathrm{E}(G)$, an involution $e \mapsto \bar{e}$ on $\mathrm{E}(G)$, and maps ∂_0 and ∂_1 from $\mathrm{E}(G)$ to $\mathrm{V}(G)$ satisfying $\partial_0\bar{e} = \partial_1 e$ for all $e \in \mathrm{E}(G)$. One calls $\partial_0 e$ and $\partial_1 e$ the *tail* and *head* of e.

The combinatorial point of view is especially convenient for quotients of graphs by groups of automorphisms: if a group of automorphisms inverts some edge, the corresponding edge in the quotient graph will be a directed loop (an edge satisfying $e = \bar{e}$; in the simplicial quotient this is a "half-edge"—an orbifold with underlying space an interval having a vertex at one end and a mirror at the other).

A *coloring* of a graph G will mean a graph-homomorphism of G to a fixed *graph of colors*. The vertex and edge sets of this graph are the *vertex-colors* and *edge-colors* respectively. By a graph-homomorphism of a colored graph we always mean one which preserves colors; in particular, covering maps should preserve colors, and for a colored graph G, $\operatorname{Aut}(G)$ will always mean the group of colored graph automorphisms.

It is an exercise to derive from Leighton's theorem the version for colored graphs. But it is also implicit in Leighton's proof, so we will describe this in Section 1. This paper was motivated by the desire in [4] of a yet more general version, which we describe in Section 2, and prove in a special case in Section 4, using the Bass-Kulkarni approach, which we expose in Section 3.

The *universal covering* $\tilde{G}$ of a colored graph G is its universal covering in the topological sense, i.e., of the underlying undirected graph as a simplicial complex. This is a colored tree, with the coloring induced from that of G. If $\operatorname{Aut}(\tilde{G})$ does not act transitively on the set of vertices or edges of $\tilde{G}$ of each color, we can refine the colors to make it so, by replacing the graph of colors by the *refined graph of colors* $C := \tilde{G}/\operatorname{Aut}(\tilde{G})$. This does not change $\operatorname{Aut}(\tilde{G})$. We will usually use refined colors, since graphs which have a common covering have the same universal covering and therefore have the same refined colors.

1. Leighton's theorem for colored graphs

We give Leighton's proof, mildly modified to clarify its structure and to make explicit the fact that it handles colored graphs. To ease comparison with his version, we have copied some of his notation.

Theorem 1.1. *Two finite colored graphs G and G' which have a common covering have a common finite covering.*

Proof. We can assume we are working with refined colors, so $C := \widetilde{G}/\operatorname{Aut}(\widetilde{G})$ is our graph of colors. We denote the sets of vertex and edge colors by $I = \mathrm{V}(C)$, $K = \mathrm{E}(C)$. For $k \in K$ we write $\partial k = ij$ if $\partial_0 k = i$ and $\partial_1 k = j$. An *i–vertex* is one with color i and a *k–edge* is one with color k.

Denote by n_i and m_k the numbers of i–vertices and k-edges of G. For $k \in K$ with $\partial k = ij$ denote by r_k the number of k–edges from any fixed i–vertex v of G. Clearly

$$n_i r_k = m_k = m_{\bar{k}} = n_j r_{\bar{k}}\,.$$

Let s be a common multiple of the m_k's. Put $a_i := \frac{s}{n_i}$ and $b_k := \frac{a_i}{r_k} = \frac{s}{m_k}$. Then

$$b_k = \frac{a_i}{r_k} = \frac{a_j}{r_{\bar{k}}} = b_{\bar{k}}\,. \tag{1}$$

The a_i and b_k can be defined by equations (1), without reference to G (or G'). For if positive integers a_i ($i \in I$) and b_k ($k \in K$) satisfy (1) whenever $\partial k = ij$, then $a_i = \frac{r_k a_j}{r_{\bar{k}}}$, so $n_i a_i = \frac{n_i r_k a_j}{r_{\bar{k}}} = \frac{n_j r_{\bar{k}} a_j}{r_{\bar{k}}} = n_j a_j$, so $s := n_i a_i$ is independent of i. This s is divisible by every m_k and $a_i = \frac{s}{n_i}$.

For $i \in I$ choose a set A_i of size a_i. For $k \in K$ choose a group Π_k of size r_k, a set $\mathrm{B}_k = \mathrm{B}_{\bar{k}}$ of size b_k and a bijection $\phi_k \colon \Pi_k \times \mathrm{B}_k \to \mathrm{A}_{\partial_0 k}$.

For each i–vertex v of G choose a bijection ψ_{vk} of the set of k–edges at v to Π_k. Do the same for the graph G'.

Define a graph H as follows (v and v' will refer to vertices of G and G' respectively, and similarly for edges e, e'):

$$\begin{aligned}
\mathrm{V}(H) &:= \{(i, v, v', \alpha) : v, v' \text{ of color } i,\ \alpha \in \mathrm{A}_i\} \\
\mathrm{E}(H) &:= \{(k, e, e', \beta) : e, e' \text{ of color } k,\ \beta \in \mathrm{B}_k\} \\
\partial_0(k, e, e', \beta) &:= \big(\partial_0 k, \partial_0 e, \partial_0 e', \phi_k(\psi_{vk}(e)\psi_{v'k}(e')^{-1}, \beta)\big) \\
\overline{(k, e, e', \beta)} &:= (\bar{k}, \bar{e}, \bar{e}', \beta), \quad \text{so} \\
\partial_1(k, e, e', \beta) &= \big(\partial_1 k, \partial_1 e, \partial_1 e', \phi_{\bar{k}}(\psi_{v\bar{k}}(\bar{e})\psi_{v'\bar{k}}(\bar{e}')^{-1}, \beta)\big)\,.
\end{aligned}$$

We claim the projection map $H \to G$ given by second coordinate is a covering. So let v be a i–vertex of G and e a k–edge at v and (i, v, v', α) a vertex of H lying over v. We must show there is exactly one edge of H at this vertex lying over e. The edge must have the form (k, e, e', β) with $\phi_k(\psi_{vk}(e)\psi_{v'k}(e')^{-1}, \beta) = \alpha$. Since ϕ_k is a bijection, this equation determines β and $\psi_{vk}(e)\psi_{v'k}(e')^{-1}$ uniquely, hence also $\psi_{v'k}(e')$, which determines e'. This proves the claim. By symmetry, H also covers G', so the colored Leighton's theorem is proved. □

Remark 1.2. Leighton's original proof is essentially the above proof with A_i the cyclic group $\mathbb{Z}/a_i$, B_k its cyclic subgroup of order b_k, and Π_k the quotient group $A_i/B_k \cong \mathbb{Z}/r_k$.

2. Symmetry-restricted graphs

We define a concept of a "symmetry-restricted graph." The underlying data consist of a graph of colors C together with, for each vertex-color $i \in I$, a finite permutation group Δ_i together with a bijection between the set of orbits of Δ_i and the edge colors k with $\partial_0 k = i$.

Definition 2.1. A *symmetry-restricted graph* for these data is a C-colored graph G and for each i–vertex $v \in \mathrm{V}(G)$ a representation of Δ_i as a color-preserving permutation group on the set $\mathrm{star}(v)$ of edges departing v (thus $|\mathrm{star}(v)|$ equals the degree of the permutation group Δ_i). A *morphism* of symmetry-restricted graphs $\phi\colon G \to G'$ is a colored graph homomorphism ϕ which restricts to a weakly equivariant isomorphism from $\mathrm{star}(v)$ to $\mathrm{star}(\phi(v))$ for each v. (A map $\phi\colon X \to Y$ of Δ–sets is *weakly equivariant* if it is equivariant up to conjugation, i.e., there is a $\gamma \in \Delta$ such that $\phi(\delta x) = \gamma\delta\gamma^{-1}\phi(x)$ for each $x \in X$ and $\delta \in \Delta$.) Note that a morphism is a covering map; if it is bijective it is an *automorphism*.

An example of a symmetry-restricted graph in this sense is a k-regular oriented fat graph; we have just one vertex color and the group Δ is a cyclic group of order k acting transitively on each $\mathrm{star}(v)$. Another example is the following:

Example 2.2. Consider a "graph" G in which each vertex is a small dodecahedron or cube, and each corner of a dodecahedron is connected by an edge to a corner of a cube and vice versa. The graph of colors is a graph C with $\mathrm{V}(C) = \{d, c\}$, $\mathrm{E}(C) = \{e, \bar{e}\}$, $\partial_0 e = d$, $\partial_1 e = c$. The groups Δ_d and Δ_c are the symmetry groups of the dodecahedron and cube respectively, acting as permutation groups of the 20 corners of the dodecahedron and the 8 corners of the cube. The graph G is thus bipartite, with 20 edges at each d–vertex and 8 edges at each c–vertex.

The desired application in [4] of Leighton's theorem for symmetry-restricted graphs often needs a more general concept of symmetry restriction, which we describe, although we have no significant results for it. We first reformulate the definition of a symmetry-restricted graph.

Let T be any infinite tree and Γ a subgroup of its full automorphism group such that $C := T/\Gamma$ is finite. For a vertex v of T, denote by $\Gamma_{(v)}$ the restriction to the star of v of the vertex group Γ_v (the isotropy group of v); this is a finite permutation group on the edges at v which we call the *restricted vertex group* for the action. Up to isomorphism as a permutation group, this group only depends on the image of v in C, so it can be taken as the datum Δ_i for symmetry restriction, where $i \in C$ is the color of v.

Now assume that Γ is maximal among groups that act on T with quotient C and with prescribed restricted vertex groups. Then Γ is the symmetry-restricted automorphism group $\mathrm{Aut}^s(T)$ of T (the superscript reminds to consider only symmetry-restricted automorphisms). Any T/Γ_0, where $\Gamma_0 \le \Gamma = \mathrm{Aut}^s(T)$ is a subgroup which acts freely on T, is a symmetry-restricted graph for the given data.

Now, for an edge e of T define the *restricted edge group* $\Gamma_{(e)}$ to be the restriction to the star of e (e together with adjacent edges) of the isotropy group Γ_e. The version of symmetry restriction needed in [4] is as follows:

Definition 2.3. Suppose Γ is maximal among groups that act on T with quotient C and with restricted vertex *and* edge groups equal to those of Γ. The quotients

T/Γ_0, where $\Gamma_0 \leq \Gamma$ acts freely on T, are symmetry-restricted graphs for the data (T, C, Γ).

Note that $\Gamma_{(e)}$ is a subgroup of $(\Gamma_{(\tau e)})_e \times (\Gamma_{(\iota e)})_{\bar{e}}$, or $((\Gamma_{(\tau e)})_e \times (\Gamma_{(\iota e)})_{\bar{e}}) \rtimes \mathbb{Z}/2$ if Γ inverts e. The "vertex-only" definition of symmetry restriction (Definition 2.1) is the special case when the restricted edge groups are as large as possible: $(\Gamma_{(\tau e)})_e \times (\Gamma_{(\iota e)})_{\bar{e}}$ or $((\Gamma_{(\tau e)})_e \times (\Gamma_{(\iota e)})_{\bar{e}}) \rtimes \mathbb{Z}/2$.

We'd like to know if Leighton's theorem extends to this setting. More generally one could ask if Leighton's theorem extends when symmetry is restricted on possibly larger finite portions of T. Unfortunately, the only case in which we can give any answers is the "vertex-only" version (Definition 2.1) of symmetry restriction.

Theorem 2.4. *For the "vertex-only" version of symmetry restriction suppose the graph of colors is a tree. Then any two finite symmetry-restricted graphs G and G' which have a common covering have a common finite covering.*

For Example 2.2 one has a simple geometric proof similar to the fat-graph proof for k–regular graphs. Create a 3–dimensional "fat graph" from G by truncating the corners of the dodecahedra and cubes to form small triangles and thickening each edge of G to be a rod with triangular cross-section joining these triangles. The rods should have a fixed length and thickness, and be attached rigidly to the truncated polyhedra which play the role of vertices. Then the universal covering is a 3-dimensional fat tree whose isometry group acts properly discontinuously, so the result follows as before.

But if we have a graph G made, say, of icosahedra connected to cubes by edges, then it is less obvious how to create a rigid fat-graph version, since the vertex degrees of icosahedron and cube are 5 and 3, which do not match.

To prove the above theorem we will need the graph of groups approach of Bass and Kulkarni.

3. The Bass-Kulkarni proof

We give a simplified version of the Bass-Kulkarni proof of Leighton's theorem in its colored version, Theorem 1.1.

We retain the notation of Section 1. In particular, $C = \tilde{G}/\operatorname{Aut}(\tilde{G})$ is the (refined) graph of colors, with vertex set I and edge set K. For the moment we assume that $\operatorname{Aut}(\tilde{G})$ acts without inversions, so C has no edge with $k = \bar{k}$.

We use this graph as the underlying graph for a graph of groups, associating a group A_i of size a_i to each vertex i and a group $\mathrm{B}_k = \mathrm{B}_{\bar{k}}$ of size b_k to each edge k, along with an injection $\phi_k \colon \mathrm{B}_k \to \mathrm{A}_{\partial_0 k}$. Of course we have to choose our groups so that B_k embeds in $\mathrm{A}_{\partial_0 k}$ for each k; one such choice is the one of Remark 1.2.

Let Γ be the fundamental group of this graph of groups and T the Bass-Serre tree on which Γ acts; this action has quotient $T/\Gamma = C$, vertex stabilizers A_i, and edge stabilizers B_k. Then T is isomorphic to the tree $\tilde{G}$. Now Γ acts properly discontinuously on T. So, if we can express G and G' as quotients T/Λ and T/Λ' with Λ and Λ' in Γ, then Λ and Λ' are finite index in Γ so $T/(\Lambda \cap \Lambda')$ is the desired common covering.

To complete the proof we must show that such Λ and Λ' exist in Γ. This is the content of Bass's Conjugacy Theorem ([3], see also [6]). We replace it for now by a "fat graph" argument (but see Theorem 4.2).

For each finite group Δ choose a finite complex $B\Delta$ with fundamental group Δ and denote its universal covering by $E\Delta$. We also assume that any inclusion $\phi\colon \Phi \to \Delta$ of finite groups which we consider can be realized as the induced map on fundamental groups of some map $B\phi\colon B\Phi \to B\Delta$ (this is always possible, for example, if $B\Phi$ is a presentation complex for a finite presentation of Φ). We now create a "fat graph" version of our graph of groups by replacing vertex i by $B\mathrm{A}_i$, edges k and $\bar{k}$ by $B\mathrm{B}_k \times [0,1]$ (with the parametrization of the interval $[0,1]$ reversed when associating this to $\bar{k}$), and gluing each $B\mathrm{B}_k \times [0,1]$ to $B\mathrm{A}_{\partial_0 k}$ by the map $B\phi_k\colon B\mathrm{B}_k \times \{0\} \to B\mathrm{A}_{\partial_0 k}$ which realizes the inclusion $\phi_k\colon \mathrm{B}_k \to \mathrm{A}_{\partial_0 k}$.

This is a standard construction which replaces the graph of groups by a finite complex K whose fundamental group is Γ. The universal covering of K is a fat-graph version $\mathcal{T}$ of the tree T, obtained by replacing i–vertices by copies of $E\mathrm{A}_i$ and k–edges by copies of $E\mathrm{B}_k \times [0,1]$. The "fat edges" $E\mathrm{B}_k \times [0,1]$ are glued to the "fat vertices" $E\mathrm{A}_i$ by the lifts of the maps $B\phi_k$. An automorphism of $\mathcal{T}$ will be a homeomorphism which is an isomorphism on each piece $E\mathrm{A}_i$ and $E\mathrm{B}_k \times [0,1]$ (where the only isomorphisms allowed on an $E\Delta$ are covering transformations for the covering $E\Delta \to K\Delta$).

We can similarly construct fat versions of the graphs G and G', replacing each i–vertex by a copy of $E\mathrm{A}_i$ and each k–edge by a copy of $E\mathrm{B}_k \times [0,1]$. There is choice in this construction: if $\partial_0 k = i$ then at the $E\mathrm{A}_i$ corresponding to an i–vertex v there are r_k edge-pieces $E\mathrm{B}_k \times [0,1]$ to glue to $E\mathrm{A}_i$ and r_k "places" on $E\mathrm{A}_i$ to do the gluing, and we can choose any bijection between these edge-pieces and places; moreover, each gluing is then only determined up to the action of B_k. Nevertheless, however we make these choices, we have:

Lemma 3.1. *The above fattened graphs have universal covering isomorphic to $\mathcal{T}$.*

Proof. We construct an isomorphism of $\mathcal{T}$ to the universal covering of the fattened G inductively over larger and larger finite portions. The point is that if one has constructed the isomorphism on a finite connected portion of $\mathcal{T}$, when extending to an adjacent piece (either an $E\mathrm{A}_i$ or an $E\mathrm{B}_k \times [0,1]$), the choice in the gluing map for that piece is an element of a B_k, which extends over the piece, so the isomorphism can be extended over that piece. □

Since the fattened versions of G and G' each have universal covering $\mathcal{T}$, they are each given by an action of a subgroup of $\Gamma = \mathrm{Aut}(\mathcal{T})$, as desired, thus completing the proof of the colored Leighton's theorem for the case that Γ has no inversions.

If Γ does invert some edge, so C has an edge $k = \bar{k}$, then the edge stabilizer is an extension $\bar{\mathrm{B}}_k$ of the cyclic group C_2 by B_k. We can assume that the inclusion $\mathrm{B}_k \subset \bar{\mathrm{B}}_k$ is represented by a double covering $K\mathrm{B}_k \to K\bar{\mathrm{B}}_k$. The complex $(K\mathrm{B}_k \times [0,1])/C_2$ (diagonal action of C_2) is then the object that the "half-edge" k of C should be replaced by in fattening C. The proof then goes through as before. □

Our earlier fat graph proofs for the k–regular case and for Example 2.2 are special cases of the proof we have just given if we generalize the proof to allow orbifolds.

4. Proofs for symmetry-restricted graphs

Proof of Theorem 2.4. Recall the situation of Theorem 2.4: we have a graph of colors C defining the set of vertex colors $I = \mathrm{V}(C)$, and for each $i \in I$ we have a finite permutation group Δ_i which acts as a permutation group of $\mathrm{star}(v)$ for each

i–vertex v of our colored graphs. Theorem 2.4 also required the graph of colors C to be a tree; for the moment we will not assume this.

Consider an edge-color $k \in K = \mathrm{E}(C)$ with $\partial k = ij$. For a k–edge e of G the stabilizer e in the Δ_i action on $\mathrm{star}(\partial_0 e)$ will be denoted Δ_k; it is a subgroup of Δ_i which is determined up to conjugation, so we make a choice.

Suppose that for every k we have $\Delta_k \cong \Delta_{\bar{k}}$. This is the case, for example, for the dodecahedron-cube graphs of Example 2.2, where these stabilizer groups are dihedral of order 6. In this case the proof of the previous section works with essentially no change, using $\mathrm{A}_i = \Delta_i$ and $\mathrm{B}_k = \Delta_k$. The only change is that when fattening the r_k k–edges at a fattened i–vertex v, our freedom of choice in attaching the r_k edge-pieces $E\mathrm{B}_k \times [0,1]$ to r_k places on $E\mathrm{A}_i$ is now restricted: we must attach them equivariantly with respect to the action of $\mathrm{A}_i = \Delta_i$ on $\mathrm{star}(v)$ (this still leaves some choice). This proves 2.4 for this case.

Now assume C is a tree. We can reduce the general case to the above special case as follows: For any vertex color i define $K_i := \{k \in K : k \text{ points towards } i\}$ and then replace each Δ_i by $\bar{\Delta}_i := \Delta_i \times \prod_{k \in K_i} \Delta_k$, acting on $\mathrm{star}(i)$ via the projection to Δ_i. Then the stabilizer $\bar{\Delta}_k$ is $\Delta_k \times \prod_{k \in K_{\partial_0 k}} \Delta_k$, which equals $\bar{\Delta}_{\bar{k}}$, so we are in the situation of the previous case. □

It is not hard to extend the above proof to prove the following theorem, which we leave to the reader.

Theorem 4.1. *Suppose that for every closed directed path $(k_1, k_2, \dots, k_r)$ in the graph of colors we have $\prod_{i=1}^r \Delta_{k_i} \cong \prod_{i=1}^r \Delta_{\bar{k}_i}$ (note that these groups have the same order). Then any two finite symmetry-restricted graphs with these data which have a common covering have a common finite covering.* □

One of the two ingredients of the original Bass-Kulkarni proof of Leighton's theorem is Bass's Conjugacy Theorem. This theorem holds for symmetry-restricted graphs (see below), but this appears not to help extend the above results. The other ingredient in the Bass-Kulkarni proof is to find a subgroup of $\mathrm{Aut}(T)$ which acts properly discontinuously on T with quotient C. Such a group would necessarily be given by a graph of finite groups with underlying graph C, and we are back in the situation of the proof we have already given, which appears to need strong conditions on C.

The Conjugacy Theorem says, in our language, that if T is a colored tree whose colored automorphism group acts without inversions (i.e., the graph of colors $C = T/\mathrm{Aut}(T)$ has no loops $k = \bar{k}$), and $H \subset \mathrm{Aut}(T)$ is a subgroup with $T/H = C$, then any Γ which acts freely on T can be conjugated into H by an element of $\mathrm{Aut}(T)$. In the symmetry-restricted setting we write $\mathrm{Aut}^s(T)$ to remind that we mean symmetry-restricted automorphisms.

Theorem 4.2 (Conjugacy Theorem)**.** *Fix data for symmetry-restricted graphs ("vertex-only" version), and assume the graph of colors C has no loops. If T is the symmetry-restricted tree for this data (it is unique) and $H \subset \mathrm{Aut}^s(T)$ a subgroup with $T/H = C$, then for any $\Gamma \subset \mathrm{Aut}^s(T)$ which acts freely on T, there exists $g \in \mathrm{Aut}^s(T)$ with $g\Gamma g^{-1} \subset H$.*

Proof. In [3] Bass includes a short proof of his Conjugacy Theorem proposed by the referee. That proof constructs the conjugating element g directly, and one verifies by inspection that g is a symmetry-restricted automorphism. The point is that g

is the identity on the stars of a representative set S of vertices for orbits of the Γ action. If v is any vertex of T let $\gamma \in \Gamma$ be the element that takes v to a vertex in S and $h = g\gamma g^{-1} \in H$. Restricted to the star of v the map g is $h^{-1}\gamma$, which is in $\mathrm{Aut}^s(T)$. □

It is worth noting that the "fat graph argument" in Section 3 proves Bass's Conjugacy Theorem in its original form (see also [6]). However, neither that proof nor Bass's proof can be applied to the symmetry-restricted case, so it is somewhat remarkable that the above proof works.

The above approach to extend Leighton's theorem involved extending each vertex group to make it act non-effectively on the star of the vertex; we used trivial extensions (direct products). By using other extensions we can prove isolated additional cases. But this approach has no hope of working if C has a closed directed path $\{k_1, \dots, k_n\}$ for which $\sum_{i=1}^r [\Delta_{k_i}] \neq \sum_{i=1}^r [\Delta_{\bar{k}_i}]$, where $[\Delta]$ means the class of Δ in the Grothendieck group of finite groups modulo the relations given by short exact sequences (so a group is equivalent to the sum of its composition factors).

In the application to [4], the groups Δ_k and $\Delta_{\bar{k}}$ are both extensions of a finite cyclic group $F_k = F_{\bar{k}}$ of order 1, 2, 3, 4 or 6 by a finite 2-generator abelian group, so the above the obstruction does not arise. Nevertheless, we have been unable to resolve the question in general for this case, even if the F_k are trivial. And when an F_k is non-trivial we have a corresponding edge restriction (Definition 2.3), namely the subgroup of $(a, b) \in \Delta_k \times \Delta_{\bar{k}}$ for which a and b have the same image in F_k, so we are outside the cases of symmetry restriction where we have any results.

References

[1] Dana Angluin and A. Gardner, Finite common coverings of pairs of regular graphs, J. Combin. Theory Ser. B **30** (1981), 184–187.
[2] Hyman Bass and Ravi Kulkarni, Uniform tree lattices, J. Amer. Math. Soc. **3** (1990), 843–902.
[3] Hyman Bass, Covering theory for graphs of groups, J. Pure Appl. Algebra **89** (1993), 3–47.
[4] Jason Behrstock and Walter D Neumann, Quasi-isometry classification of non-geometric 3-manifold groups. arXiv:1001.0212
[5] Frank Thomson Leighton, Finite Common Coverings of Graphs, J. Comb. Theory, Series B **33** (1982), 231–138.
[6] Seonhee Lim, Anne Thomas, Covering theory for complexes of groups, J. Pure Appl. Alg. **212** (2008), 1632–1663.