QUASI-ISOMETRIC CLASSIFICATION OF GRAPH MANIFOLD GROUPS

JASON A. BEHRSTOCK AND WALTER D. NEUMANN

ABSTRACT. We show that the fundamental groups of any two closed irreducible non-geometric graph-manifolds are quasi-isometric. This answers a question of Kapovich and Leeb. We also classify the quasi-isometry types of fundamental groups of graph-manifolds with boundary in terms of certain finite two-colored graphs. A corollary is the quasi-isometric classification of Artin groups whose presentation graphs are trees. In particular any two right-angled Artin groups whose presentation graphs are trees of diameter greater than 2 are quasi-isometric, answering a question of Bestvina; further, this quasi-isometry class does not include any other right-angled Artin groups.

A finitely generated group can be considered geometrically when endowed with a word metric—up to quasi-isometric equivalence, such metrics are unique. (Henceforth only finitely generated groups will be considered.) Given a collection of groups, $\mathcal{G}$, Gromov proposed the fundamental questions of identifying which groups are quasi-isometric to those in $\mathcal{G}$ (*rigidity*) and which groups in $\mathcal{G}$ are quasi-isometric to each other (*classification*) [10].

In this paper, we focus on the classification question for graph manifold groups and right-angled Artin groups.

A compact 3-manifold M is called *geometric* if $M \setminus \partial M$ admits a geometric structure in the sense of Thurston (i.e., a complete locally homogeneous Riemannian metric of finite volume). The Geometrization Conjecture [19, 20, 21] provides that every irreducible 3-manifold of zero Euler characteristic (i.e., with boundary consisting only of tori and Klein bottles) admits a decomposition along tori and Klein bottles into geometric pieces, the minimal such decomposition is called the *geometric decomposition*.

There is a considerable literature on quasi-isometric rigidity and classification of 3-manifold groups. The rigidity results can be briefly summarized:

Theorem. *If a group G is quasi-isometric to the fundamental group of a 3-manifold M with zero Euler characteristic, then G is weakly commensurable*[1] *with $\pi_1(M')$ for some such 3-manifold M'. Moreover, M' is closed resp. irreducible resp. geometric if and only if M is.*

This quasi-isometric rigidity for 3-manifold groups is the culmination of the work of many authors, key steps being provided by Gromov-Sullivan, Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb, Rieffel, Schwartz [4, 6, 9, 12, 22, 23]. The

2000 *Mathematics Subject Classification.* Primary 20F65; Secondary 57N10, 20F36.

Key words and phrases. graph manifold, quasi-isometry, commensurability, right-angled Artin group.

Research supported under NSF grants no. DMS-0083097 and DMS-0604524.

[1]Two groups are said to be *weakly commensurable* if they have quotients by finite normal subgroups which have isomorphic finite index subgroups.

reducible case reduces to the irreducible case using Papasoglu and Whyte [18] and the irreducible non-geometric case is considered by Kapovich and Leeb [12].

The classification results in the geometric case can be summarized by the following; the first half is an easy application of the Milnor-Švarc Lemma [15], [25]:

Theorem. *There are exactly seven quasi-isometry classes of fundamental groups of closed geometric 3-manifolds, namely any such group is quasi-isometric to one of the eight Thurston geometries (*$\mathbb{S}^3$*,* $\mathbb{S}^2\times\mathbb{E}^1$*,* $\mathbb{E}^3$*,* Nil*,* $\mathbb{H}^2\times\mathbb{E}^1$*,* $\widetilde{\mathrm{PSL}}$*,* Sol*,* $\mathbb{H}^3$*) but the two geometries* $\mathbb{H}^2\times\mathbb{E}^1$ *and* $\widetilde{\mathrm{PSL}}$ *are quasi-isometric.*

If a geometric manifold M has boundary, then it is either Seifert fibered and its fundamental group is quasi-isometric (indeed commensurable) with $F_2\times\mathbb{Z}$ [12], or it is hyperbolic, in which case quasi-isometry also implies commensurability [23].

A *graph manifold* is a 3-manifold that can be decomposed along embedded tori and Klein bottles into finitely many Seifert manifolds; equivalently, these are exactly the class of manifolds with no hyperbolic pieces in their geometric recomposition. Since the presence of a hyperbolic piece can be quasi-isometrically detected [7] [13] [1], this implies that the class of fundamental groups of graph manifolds is rigid. We answer the classification question for graph manifold groups. Before discussing the general case we note the answer for closed non-geometric graph manifolds, resolving a question of Kapovich and Leeb [13].

Theorem 2.1. *Any two closed non-geometric graph manifolds have bilipschitz homeomorphic universal covers. In particular, their fundamental groups are quasi-isometric.*

This contrasts with commensurability of closed graph manifolds: already in the case that the graph manifold is composed of just two Seifert pieces there are infinitely many commensurability classes (they are classified in that case but not in general, see Neumann [16]).

We also classify compact graph manifolds with boundary. To describe this we need some terminology. We associate to the geometric decomposition of a non-geometric graph manifold M its *decomposition graph* $\Gamma(M)$ which has a vertex for each Seifert piece and an edge for each decomposing torus or Klein bottle. We color the vertices of $\Gamma(M)$ **b**lack or **w**hite according to whether the Seifert piece includes a boundary component of M or not (**b**ounded or **w**ithout boundary). We call this the *two-colored decomposition graph*. We can similarly associate a two-colored tree to the decomposition of the universal cover $\tilde{M}$ into its fibered pieces. We call this infinite valence two-colored tree $BS(M)$, since it is the Bass-Serre tree corresponding to the graph of groups decomposition of $\pi_1(M)$.

The Bass-Serre tree $BS(M)$ can be constructed directly from the decomposition graph $\Gamma=\Gamma(M)$ by first replacing each edge of Γ by a countable infinity of edges with the same endpoints and then taking the universal cover of the result. If two two-colored graphs Γ_1 and Γ_2 lead to isomorphic two-colored trees by this procedure we say Γ_1 and Γ_2 are *bisimilar*. In Section 4 we give a simpler, algorithmically checkable, criterion for bisimilarity[2] and show that each bisimilarity class contains a unique minimal element.

Our classification theorem, which includes the closed case (Theorem 2.1), is:

[2]We thank Ken Shan for pointing out that our equivalence relation is a special case of the computer science concept bisimilarity, related to bisimulation.

Theorem 3.2. *If M and M' are non-geometric graph manifolds then the following are equivalent:*

(1) *$\tilde{M}$ and $\tilde{M}'$ are bilipschitz homeomorphic.*
(2) *$\pi_1(M)$ and $\pi_1(M')$ are quasi-isometric.*
(3) *$BS(M)$ and $BS(M')$ are isomorphic as two-colored trees.*
(4) *The minimal two-colored graphs in the bisimilarity classes of the decomposition graphs $\Gamma(M)$ and $\Gamma(M')$ are isomorphic.*

One can list minimal two-colored graphs of small size, yielding, for instance, that there are exactly $2, 6, 26, 199, 2811, 69711, 2921251, 204535126, \dots$ quasi-isometry classes of fundamental groups of non-geometric graph manifolds that are composed of at most $1, 2, 3, 4, 5, 6, 7, 8, \dots$ Seifert pieces.

For closed non-geometric graph manifolds we recover that there is just one quasi-isometry class (Theorem 2.1): the minimal two-colored graph is a single white vertex with a loop. Similarly, for non-geometric graph manifolds that have boundary components in every Seifert component there is just one quasi-isometry class (the minimal two-colored graph is a single black vertex with a loop).

For graph manifolds with boundary the commensurability classification is also rich, but not yet well understood. If M consists of two Seifert components glued to each other such that M has boundary components in both Seifert components one can show that M is commensurable with any other such M, but this appears to be already no longer true in the case of three Seifert components.

We end by giving an application to the quasi-isometric classification of Artin groups. The point is that if the presentation graph is a tree then the group is a graph-manifold group, so our results apply. In particular, we obtain the classification of right-angled Artin groups whose presentation graph is a tree, answering a question of Bestvina. We also show rigidity of such groups amongst right angled Artin groups.

We call a right-angled Artin group whose presentation graph is a tree a *right-angled tree group*. If the tree has diameter ≤ 2 then the group is $\mathbb{Z}, \mathbb{Z}^2$ or (free)$\times\mathbb{Z}$. We answer Bestvina's question by showing that right-angled tree groups with presentation graph of diameter > 2 are all quasi-isometric to each other. In fact:

Theorem 5.3. *Let G' be any Artin group and let G be a right-angled tree group whose tree has diameter > 2. Then G' is quasi-isometric to G if and only if G' has presentation graph an even-labeled tree of diameter ≥ 2 satisfying: (i) all interior edges have label 2; and (ii) if the diameter is 2 then at least one edge has label > 2. (An "interior edge" is an edge that does not end in a leaf of the tree.)*

The commensurability classification of right-angled tree groups is richer: Any two whose presentation graphs have diameter 3 are commensurable, but it appears that there are already infinitely many commensurability classes for diameter 4.

Theorem 5.3 also has implications for quasi-isometric rigidity phenomena in relatively hyperbolic groups. For such applications see Behrstock–Druţu–Mosher [1], where it is shown that graph manifolds, and thus tree groups, can only quasi-isometrically embed in relatively hyperbolic groups in very constrained ways.

In the course of proving Theorem 5.3 we classify which Artin groups are quasi-isometric to 3-manifold groups. This family of groups coincides with those proven by Gordon to be isomorphic to 3-manifold groups [8].

Acknowledgments. We express our appreciation to Mladen Bestvina, Mohamad Hindawi, Tadeusz Januszkiewicz and Misha Kapovich for useful conversations and the referee for pertinent comments.

1. Quasi-isometry of fattened trees

Let T be a tree all of whose vertices have valence in the interval $[3, K]$ for some K. We fix a positive constant L, and assume T has been given a simplicial metric in which each edge has length between 1 and L. Now consider a "fattening" of T, where we replace each edge E by a strip isometric to $E \times [-\epsilon, \epsilon]$ for some $\epsilon > 0$ and replace each vertex by a regular polygon around the boundary of which the strips of incoming edges are attached in some order. Call this object X. Let X_0 be similarly constructed, but starting from the regular 3-valence tree with all edges having length 1, and with $\epsilon = 1/2$.

We first note the easy lemma:

Lemma 1.1. *There exists C, depending only on K, L, ϵ, such that X is C-bilipschitz homeomorphic to X_0.* □

Note that if S is a compact riemannian surface with boundary having Euler characteristic < 0 then its universal cover $\tilde{S}$ is bilipschitz homeomorphic to a fattened tree as above, and hence to X_0. We can thus use X_0 as a convenient bilipschitz model for any such $\tilde{S}$.

Let X be a manifold as above, bilipschitz equivalent to X_0 (so X may be a fattened tree or an $\tilde{S}$). X is a 2-manifold with boundary, and its boundary consists of infinitely many copies of $\mathbb{R}$.

Theorem 1.2. *Let X be as above with a chosen boundary component $\partial_0 X$. Then there exists K and a function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that for any K' and any K'-bilipschitz homeomorphism Φ_0 from $\partial_0 X$ to a boundary component $\partial_0 X_0$ of the "standard model" X_0, Φ_0 extends to a $\phi(K')$-bilipschitz homeomorphism $\Phi\colon X \to X_0$ which is K-bilipschitz on every other boundary component.*

Proof. If true for some X, then the theorem will be true (with K replaced by KL) for any X' L-bilipschitz homeomorphic to X, so we may assume X is isometric to our standard model X_0. In this case we will see that K can be arbitrarily close to 1 (with very slightly more effort one can make $K = 1$).

We will construct the homeomorphism in two steps. The first step will be to extend near $\partial_0 X$ and the second to extend over the rest of X.

We consider vertices of the underlying tree adjacent to the boundary component $\partial_0 X$. These will have a certain "local density" along $\partial_0 X$ given by the number of them in an interval of a given length, measured with respect to the metric on $\partial_0 X$ that pulls back from $\partial_0 X_0$ by Φ_0. We first describe how to modify this local density using a $(1, L)$-quasi-isometric bilipschitz homeomorphism with $L = O(|\log(D)|)$, where D is the factor by which we want to modify density. We increase density locally by moves on the underlying tree in which we take a vertex along $\partial_0 X$ and a vertex adjacent to it not along $\partial_0 X$ and collapse the edge between them to give a vertex of valence 4, which we then expand again to two vertices of valence 3, now both along $\partial_0 X$ (see Fig. 1). This can be realized by a piecewise-linear homeomorphism. Since it is an isometry outside a bounded set, it has a finite bilipshitz bound k say. To increase density along an interval by at most a factor of D we need to repeat this process at most $\log_2(D)$ times, so we get a bilipshitz homeomorphism whose bilipschitz bound is bounded in terms of D. Similarly we can decrease density (using the inverse move) by a bilipschitz map whose bilipschitz bound is bounded in terms of D.

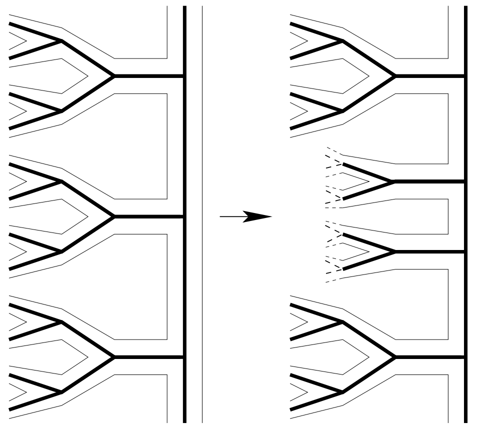


FIGURE 1. Increasing number of vertices along $\partial_0 X$ by a *depth one splitting*

One can then apply this process simultaneously on disjoint intervals to change the local density along disjoint intervals. For instance, applying the above doubling procedure to all vertices along $\partial_0 X$ doubles the density, further, since it affects disjoint bounded sets, we still have bilipshitz bound k. Similarly, given two disjoint intervals, one could, for instance, increase the local density by a factor of D on one of the intervals and decrease it on the other by a different factor D'; since the intervals are disjoint the bilipschitz bound depending only on the largest factor, which is uniformly bounded by K', the bilipschitz bound for Φ_0.

By these means we can, by replacing X by its image under a bilipshitz map with bilipshitz bound bounded in terms of K' and which is an isometry on $\partial_0 X$, assure that the number of vertices along $\partial_0 X$ and $\partial_0 X_0$ matches to within a fixed constant over any interval in $\partial_0 X$ and the corresponding image under Φ_0. We now construct a bilipshitz map from this new X to X_0 by first extending Φ_0 to a 1-neighborhood of $\partial_0 X$ and then extending over the rest of X by isometries of the components of the complement of this neighborhood. By composing the two bilipshitz maps we get a bilipshitz homeomorphism Ψ from the original X that does what we want on $\partial_0 X$ while on every other boundary component $\partial_i X$ it is an isometry outside an interval of length bounded in terms of K'.

Now choose arbitrary $K > 1$. On $\partial_i X$ we can find an interval J of length bounded in terms of K' and K that includes the interval J_0 on which our map is not an isometry, and whose length increases or decreases under Ψ by a factor of at most K (specifically, if the length of J_0 was multiplied by s, choose J of length $\lambda\ell(J_0)$ with $\lambda \geq \max(\frac{K-Ks}{K-1}, \frac{s-1}{K-1})$). Let Ψ' be the map of J that is a uniform stretch or shrink by the same factor (so the images of Ψ' and $\Psi|J$ are identical). The following self-map α of a collar neighborhood $J \times [0, \epsilon]$ restricts to $\Psi' \circ \Psi^{-1}$ on the left boundary $J \times \{0\}$ and to the identity on the rest of the boundary:

$$\alpha(x, t) = \frac{\epsilon - t}{\epsilon} \Psi' \circ \Psi^{-1}(x) + \frac{t}{\epsilon} x.$$

This α has bilipshitz constant bounded in terms of the bound on the left boundary and the length of J, hence bounded in terms of K and K'. By composing Ψ with α on a collar along the given interval we adjust $\Psi|\partial_i X$ to be a uniform stretch or shrink along this interval. We can do this on each boundary component other than $\partial_0 X_0$. The result is a bilipschitz homeomorphism whose bilipschitz bound L is still bounded in terms of K' and K and which satisfies the conditions of the theorem. □

We now deduce an analogue of the above Theorem in the case where the boundary curves ∂X are each labelled by one of a finite number of colors, C, and the maps are required to be color preserving. We call a labelling a *bounded coloring* if there is a uniform bound, so that given any point in X and any color there is a boundary component of that color a uniformly bounded distance away. The lift of a coloring on a compact surface yields a bounded coloring. We now fix a bounded coloring on our "standard model" X_0, further, we choose this coloring so that it satisfies the following regularity condition which is stronger than the above hypothesis: for every point on a boundary component and every color in C, there is an adjacent boundary component with that color a bounded distance from the given point. Call the relevant bound B.

Theorem 1.3. *Let X be as above with a chosen boundary component $\partial_0 X$ and fix a bounded coloring on the elements of ∂X. Then there exists K and a function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that for any K' and any color preserving K'-bilipschitz homeomorphism Φ_0 from $\partial_0 X$ to a boundary component $\partial_0 X_0$ of the "standard model" X_0, Φ_0 extends to a $\phi(K')$-bilipschitz homeomorphism $\Phi\colon X \to X_0$ which is K-bilipschitz on every other boundary component and which is a color preserving map from ∂X to ∂X_0.*

Proof. As in the proof of Theorem 1.2 we may assume X is isometric to our standard model X_0 and then we proceed in two steps, first extending near $\partial_0 X$, then extending over the rest of X.

To extend near $\partial_0 X$ we proceed as in the proof of Theorem 1.2, except now we need to not only match density, but colors as well. Instead of using only a *depth one splitting* as in Figure 1, one may perform a *depth n splitting* by choosing a vertex at distance n from $\partial_0 X$ and then moving that vertex to be adjacent to $\partial_0 X$; this bilipschitz map increases the density of vertices along a given boundary component. Note that a depth n move (and its inverse) can be obtained as a succession of depth 1 moves and their inverses, so using such moves is only to yield a more concise language. Since the coloring of X is a bounded coloring, from any point on $\partial_0 X$, there is a uniform bound on the distance to a vertex adjacent to a boundary component of any given color. Thus, with a bounded bilipschitz constant we may alter the density and coloring as needed.

As in the previous proof, we may extend to a map which does what is required on $\partial_0 X$ and which is an isometry, but not preserving boundary colors, outside a neighborhood of $\partial_0 X$, and which is a K-bilipschitz map on the boundary components other than $\partial_0 X$ with K close to 1.

Step two will be to apply a further bilipschitz map that fixes up colors on these remaining boundary components.

Consider a boundary component $\partial_1 X$ adjacent to $\partial_0 X$. We want to make colors correct on boundary components adjacent to $\partial_1 X$. They are already correct on $\partial_0 X$ and the boundary components adjacent on each side of this. Call these $\partial_2 X$

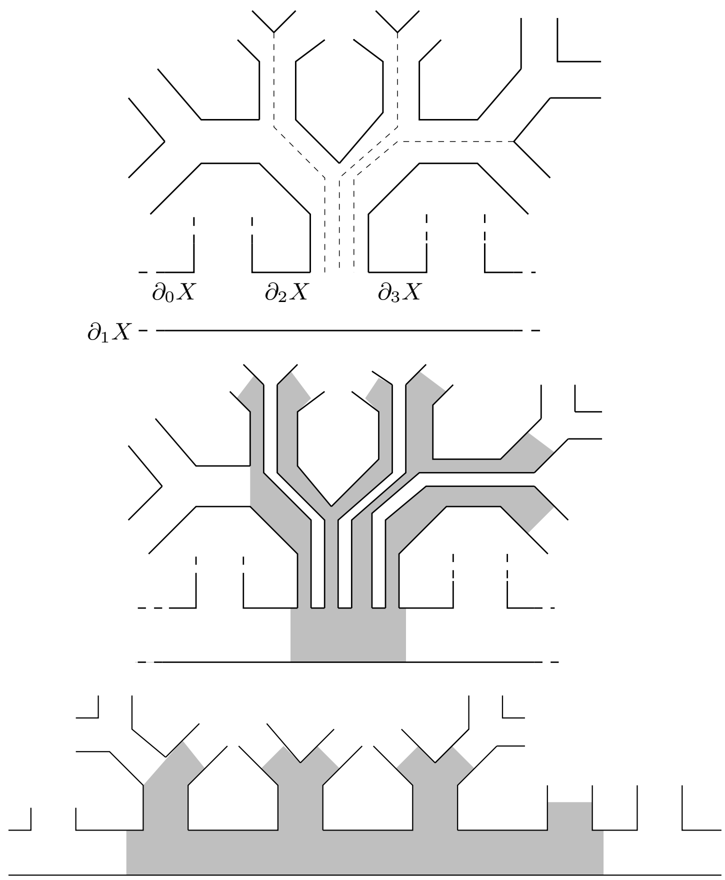


FIGURE 2. Adjusting colors along $\partial_1 X$. Three depth 3 moves are illustrated. The shaded region shows where metric is adjusted. The final metric is shown at the bottom. In the first and last pictures all edges should have the same length since both are isometric to the standard model (some distortion was needed to draw them).

and $\partial_2' X$. As we move along $\partial_1 X$ looking at boundary components, number the boundary components $\partial_0 X$, $\partial_2 X$, $\partial_3 X, \dots$ until we come to a $\partial_{j+1} X$ which is the wrong color. We will use splitting moves to bring new boundary components of the desired colors in to be adjacent to $\partial_1 X$ between $\partial_j X$ and $\partial_{j+1} X$. By our regularity assumption on X_0 we will need to add at most $2B$ new boundary components before the color of $\partial_{j+1} X$ is needed; thus we need to perform at most $2B$ splitting moves. Moreover, the bounded coloring hypothesis implies that each of these splitting moves can be chosen to be of a uniformly bounded depth (note that the bounded coloring assumption implies that at any point of X and any direction in the underlying tree there will be any desired color a uniformly bounded distance away). We repeat this process along all of $\partial_1 X$ in both directions to make colors correct. The fact that we do at most $2B$ such moves for each step along $\partial_1 X$ means that we affect the bilipschitz constant along $\partial_1 X$ by at most a factor of $2B + 1$. Since bounded depth splitting moves have compact support and since there are at most $2B$ of these performed between any pair $\partial_j X$, $\partial_{j+1} X$, we see that the bilipschitz constant need to fix this part of $\partial_1 X$ is bounded in terms of B and the bounded coloring constant. Since for $i \neq j$ the neighborhoods affected by fixing the part of $\partial_1 X$ between $\partial_j X$, $\partial_{j+1} X$ are disjoint from those affected by fixing between $\partial_i X$, $\partial_{i+1} X$, we see that fixing all of $\partial_1 X$ requires a bilipschitz bound depending only on B and the bounded coloring constant; let us call this bilipschitz bound C.

We claim that repeating this process boundary component by boundary component one can keep the bilipschitz constant under control and thus prove the theorem. To see this, consider Figure 2, which illustrates a typical set of splitting moves and shows the neighborhoods on which the metric has been altered. Since we can assure an upper bound on the diameter of these neighborhoods, we can bound the bilipshitz constants of the modifications, this is the constant C above. Only a bounded number of the neighborhoods needed for later modifications will intersect these neighborhoods, this yields an overall bilipschitz bound which is at most a bounded power of C. $\square$

2. Closed graph manifolds

The following theorem answers Question 1.2 of Kapovich–Leeb [13]. It is a special case of Theorem 3.2, but we treat it here separately since its proof is simple and serves as preparation for the general result.

Theorem 2.1. *Any two closed non-geometric graph manifolds have bilipschitz homeomorphic universal covers. In particular, their fundamental groups are quasi-isometric.*

Let us begin by recalling:

Lemma 2.2 (Kapovich-Leeb [13]; Neumann [16])**.** *Any non-geometric graph manifold has an orientable finite cover where all Seifert components are circle bundles over orientable surfaces of genus ≥ 2. Furthermore, one can arrange that the intersection numbers of the fibers of adjacent Seifert components are ± 1.*

If we replace our graph manifold by a finite cover as in the above lemma then we have a trivialization of the circle bundle on the boundary of each Seifert piece using the section given by a fiber of a neighboring piece. The fibration of this piece then has a relative Euler number.

Lemma 2.3 (Kapovich–Leeb [13])**.** *Up to a bilipschitz homeomorphism of the universal cover, we can assume all the above relative Euler numbers are 0.*

A graph manifold G as in the last lemma is what Kapovich and Leeb call a "flip-manifold." It is obtained by gluing together finitely many manifolds of the form (surface)$\times S^1$ by gluing using maps of the boundary tori that exchange base and fiber coordinates. We can give it a metric in which every fiber S^1 (and hence every boundary circle of a base surface) has length 1.

A topological model for the universal cover $\tilde{G}$ can be obtained by gluing together infinitely many copies of $X_0 \times \mathbb{R}$ according to a tree, gluing by the "flip map" $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ when gluing boundary components. We call the resulting manifold Y.

We wish to show that $\tilde{G}$ is bilipschitz homeomorphic to Y.

Proof of Theorem 2.1. The universal cover of each Seifert component of G is identified with $\tilde{S}_i \times \mathbb{R}$, where S_i is one of a finite collection of compact surfaces with boundary. Choose a number K sufficiently large that Theorem 1.2 applies for each of them. Choose a bilipschitz homeomorphism from one piece $\tilde{S}_i \times \mathbb{R}$ of $\tilde{G}$ to a piece $X_0 \times \mathbb{R}$ of Y, preserving the (surface)$\times\mathbb{R}$ product structure. We want to extend to a neighboring piece of $\tilde{G}$. On the common boundary $\mathbb{R} \times \mathbb{R}$ we have a map that is of the form $\phi_1 \times \phi_2$ with ϕ_1 and ϕ_2 both bilipschitz. By Theorem 1.2 we can extend over the neighboring piece by a product map, and on the other boundaries of this piece we then have maps of the form $\phi_1' \times \phi_2$ with ϕ_1' K-bilipschitz. We do this for all neighboring pieces of our starting piece. Because of the flip, when we extend over the next layer we have maps on the outer boundaries that are K-bilipschitz in both base and fiber. We can thus continue extending outwards inductively to construct our desired bilipschitz map. □

3. Graph manifolds with boundary

A non-geometric graph manifold M has a minimal decomposition along tori and Klein bottles into geometric (Seifert fibered) pieces, called the *geometric decomposition*. The cutting surfaces are then π_1 injective. In this decomposition one cuts along one-sided Klein bottles; this differs from JSJ, where one would cut along the torus neighborhood boundaries of these Klein bottles. (See, e.g., Neumann–Swarup [17] Section 4.)

We associate to this decomposition its *decomposition graph*, which is the graph with a vertex for each Seifert component of M and an edge for each decomposing torus or Klein bottle. If there are no one-sided Klein bottles then this graph is the graph of the associated graph of groups decomposition of $\pi_1(M)$. (If there are decomposing Klein bottles, the graph of groups has, for each Klein bottle, an edge to a new vertex rather than a loop. This edge corresponds to an amalgamation to a Klein bottle group along a $\mathbb{Z} \times \mathbb{Z}$, and corresponds also to an inversion for the action of $\pi_1(M)$ on the Bass-Serre tree. Using a loop rather than an edge makes the Bass-Serre tree a weak covering of the decomposition graph.)

We color vertices of the decomposition graph **b**lack or **w**hite according to whether the Seifert piece includes a boundary component of M or not (**b**ounded or **w**ithout boundary).

A second graph we consider is the two-colored decomposition graph for the decomposition of the universal cover $\tilde{M}$ into its fibered pieces. We denote it $BS(M)$ and call it the *two-colored Bass-Serre tree,* since it is the Bass-Serre tree for our graph of groups decomposition. It can be obtained from the two-colored decomposition graph by replacing each edge by a countable infinity of edges between its endpoints, and then taking the universal cover of the resulting graph.

A *weak covering map* from a two-colored graph Γ to a two-colored graph Γ' is a color-preserving graph homomorphism $\phi \colon \Gamma \to \Gamma'$ with the property that for any vertex v of Γ and every edge e' at $\phi(v)$, there is at least one edge e at v mapping to e'. An example of such a map is the map that collapses any multiple edge of Γ to a single edge. Any covering map of non-geometric graph manifolds induces a weak covering map of their two-colored decomposition graphs.

Note that if a weak covering map exists from Γ to Γ' then Γ and Γ' will have isomorphic two-colored Bass-Serre trees. The equivalence relation on two-colored graphs generated by the relation of existence of a weak covering map will be called *bisimilarity*. We shall prove in the next section:

Proposition 3.1. *If we restrict to countable connected graphs then each equivalence class of two-colored graphs includes two characteristic elements: a unique tree that weakly covers every element in the class (the Bass-Serre tree); and a unique minimal element, which is weakly covered by all elements in the class.*

For example, if all the vertices of a graph have the same color, then the minimal graph for its bisimilarity class is a single vertex with a loop attached and the Bass-Serre tree is the single-colored regular tree of countably infinite degree.

Our main theorem is:

Theorem 3.2. *If M and M' are non-geometric graph manifolds then the following are equivalent:*

(1) *$\tilde{M}$ and $\tilde{M}'$ are bilipschitz homeomorphic.*
(2) *$\pi_1(M)$ and $\pi_1(M')$ are quasi-isometric.*
(3) *$BS(M)$ and $BS(M')$ are isomorphic as two-colored trees.*
(4) *The minimal two-colored graphs in the bisimilarity classes of the decomposition graphs $\Gamma(M)$ and $\Gamma(M')$ are isomorphic.*

Proof. Clearly (1) implies (2). The equivalence of (3) and (4) is Proposition 3.1. Kapovich and Leeb [13] proved that any quasi-isometry essentially preserves the geometric decomposition of Haken manifolds, and therefore induces an isomorphism between their Bass-Serre trees. To prove the theorem it remains to show that (3) or (4) implies (1).

Suppose therefore that M and M' are non-geometric graph manifolds that satisfy the equivalent conditions (3) and (4). Let Γ be the minimal graph in the bisimilarity class of $\Gamma(M)$ and $\Gamma(M')$. It suffices to show that each of $\tilde{M}$ and $\tilde{M}'$ is bilipschitz homeomorphic to the universal cover of some standard graph manifold associated to Γ. There is therefore no loss in assuming that M' *is* such a standard graph manifold; "standard" will mean that $\Gamma(M') = \Gamma$ and that each loop at a vertex in Γ corresponds to a decomposing Klein bottle (i.e., a boundary torus of the corresponding Seifert fibered piece that is glued to itself by a covering map to the Klein bottle).

Denote the set of pairs consisting of a vertex of Γ and an outgoing edge at that vertex by C. Since the decomposition graphs $\Gamma(M)$, $\Gamma(M')$, $BS(M)$, and $BS(M')$ for M, M', $\tilde{M}$, and $\tilde{M}'$ map to Γ, we can label the boundary components of the geometric pieces of these manifolds by elements of C.

Our desired bilipschitz map can now be constructed inductively as in the proof of Theorem 2.1, at each stage of the process having extended over some submanifold Y of $\tilde{G}$. The difference from the situation there is that now when we extend the map from Y over a further fibered piece $X \times \mathbb{R}$, we must make sure that we are mapping boundary components to boundary components with the same C–label. That this can be done is exactly the statement of Theorem 1.3. ☐

Remark 3.3. With some work, Theorem 3.2 can be generalized to cover many situations outside of the context of 3-manifolds; such a formulation will appear in a forthcoming paper.

4. Two-colored graphs

Definition 4.1. A *graph* Γ consists of a vertex set $V(\Gamma)$ and an edge set $E(\Gamma)$ with a map $\epsilon\colon E(\Gamma) \to V(\Gamma)^2/C_2$ to the set of unordered pairs of elements of $V(\Gamma)$.

A *two-colored graph* is a graph Γ with a "coloring" $c\colon V(\Gamma) \to \{\mathbf{b}, \mathbf{w}\}$.

A *weak covering* of two-colored graphs is a graph homomorphism $f\colon \Gamma \to \Gamma'$ which respects colors and has the property: for each $v \in V(\Gamma)$ and for each edge $e' \in E(\Gamma')$ at $f(v)$ there exists an $e \in E(\Gamma)$ at v with $f(e) = e'$.

From now on, all graphs we consider will be assumed to be connected. It is easy to see that a weak covering is then surjective. The graph-theoretic results are valid for n–color graphs, but we only care about $n = 2$.

Definition 4.2. Two-colored graphs Γ_1, Γ_2 are *bisimilar*, written $\Gamma_1 \sim \Gamma_2$, if Γ_1 and Γ_2 weakly cover some common two-colored graph.

The following proposition implies, among other things, that this definition agrees with our earlier version.

Proposition 4.3. *The bisimilarity relation $\sim$ is an equivalence relation. Moreover, each equivalence class has a unique minimal element up to isomorphism.*

Lemma 4.4. *If a two-colored graph Γ weakly covers each of a collection of graphs $\{\Gamma_i\}$ then the Γ_i all weakly cover some common Γ'.*

Proof. The graph homomorphism that restricts to a bijection on the vertex set but identifies multiple edges with the same ends to a single edge is a weak covering. Moreover, if we do this to both graphs Γ and Γ_i of a weak covering $\Gamma \to \Gamma_i$ we still have a weak covering. So there is no loss in assuming all our graphs have no multiple edges. A graph homomorphism $\Gamma \to \Gamma_i$ is then determined by its action on vertices. The induced equivalence relation $\equiv$ on vertices of Γ satisfies the property:

> If $v \equiv v_1$ and e is an edge with $\epsilon(e) = \{v, v'\}$ then there exists an edge e_1 with $\epsilon(e_1) = \{v_1, v_1'\}$ and $v' \equiv v_1'$.

Conversely, an equivalence relation on vertices of Γ with this property induces a weak covering. We must thus just show that if we have several equivalence relations on $V(\Gamma)$ with this property, then the equivalence relation $\equiv$ that they generate still has this property. Suppose $v \equiv w$ for the generated relation. Then we have $v = v_0 \equiv_1 v_1 \equiv_2 \cdots \equiv_n v_n = w$ for some n, where the equivalence relations $\equiv_i$ are chosen from our given relations. Let e_0 be an edge at $v = v_0$ with other end at v_0'. Then the above property guarantees inductively that we can find an edge e_i at v_i for $i = 1, 2, \ldots, n$, with other end at v_i' and with $v_{i-1}' \equiv_i v_i'$. Thus we find an edge e_n at $w = v_n$ whose other end v_n' satisfies $v_0' \equiv v_n'$. $\square$

Proof of Proposition 4.3. We must show that $\Gamma_1 \sim \Gamma_2 \sim \Gamma_3$ implies $\Gamma_1 \sim \Gamma_3$. Now Γ_1 and Γ_2 weakly cover a common Γ_{12} and Γ_2 and Γ_3 weakly cover some Γ_{23}. The lemma applied to $\Gamma_2, \{\Gamma_{12}, \Gamma_{23}\}$ gives a graph weakly covered by all three of $\Gamma_1, \Gamma_2, \Gamma_3$, so $\Gamma_1 \sim \Gamma_3$.

The minimal element in a bisimilarity class is found by applying the lemma to an element Γ and the set $\{\Gamma_i\}$ of all two-colored graphs that Γ weakly covers. $\square$

Proposition 4.5. *If we restrict to two-colored graphs all of whose vertices have countable valence (so the graphs are also countable, by our connectivity assumption), then each bisimilarity class contains a tree T, unique up to isomorphism, that weakly* *covers every element of the class. It can be constructed as follows: If Γ is in the bisimilarity class, duplicate every edge of Γ a countable infinity of times, and then take the universal cover of the result (in the topological sense).*

Note that uniqueness of T in the above proposition depends on the fact that T is a tree; there are many different two-colored graphs that weakly cover every two-colored graph in a given bisimilarity class.

Proof of Proposition 4.5. Given a two-colored graph Γ, we can construct a tree T as follows: Start with one vertex x, labeled by a vertex v of Γ. Then for each vertex w of Γ connected to v by an edge, add infinitely many edges at x leading to vertices labeled w. Then repeat the process at these new vertices and continue inductively. Finally forget the Γ–labels on the resulting tree and only retain the corresponding $\{\mathbf{b},\mathbf{w}\}$–labels.

If Γ weakly covers a graph Γ', then using Γ' instead of Γ to construct the above tree T makes no difference to the inductive construction. Thus T is an invariant for bisimilarity. It clearly weakly covers the original Γ, and since Γ was arbitrary in the bisimilarity class, we see that T weakly covers anything in the class.

To see uniqueness, suppose T' is another tree that weakly covers every element of the bisimilarity class. Then T' weakly covers the T constructed above from Γ. Composing with $T \to \Gamma$ gives a weak covering $f\colon T' \to \Gamma$ for which infinitely many edges at any vertex $v \in V(T')$ lie over each edge at the vertex $f(v) \in V(\Gamma)$. It follows that T' itself can be constructed from Γ as in the first paragraph of this proof, so T' is isomorphic to T. $\square$

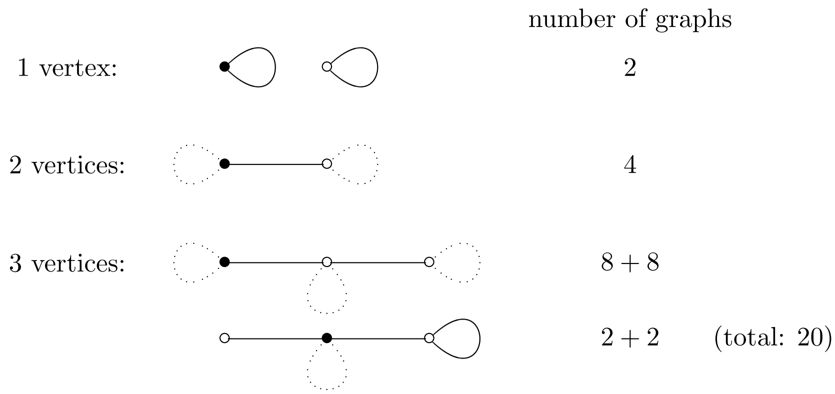
Using a computer we have found (in about 5 months of processor time):

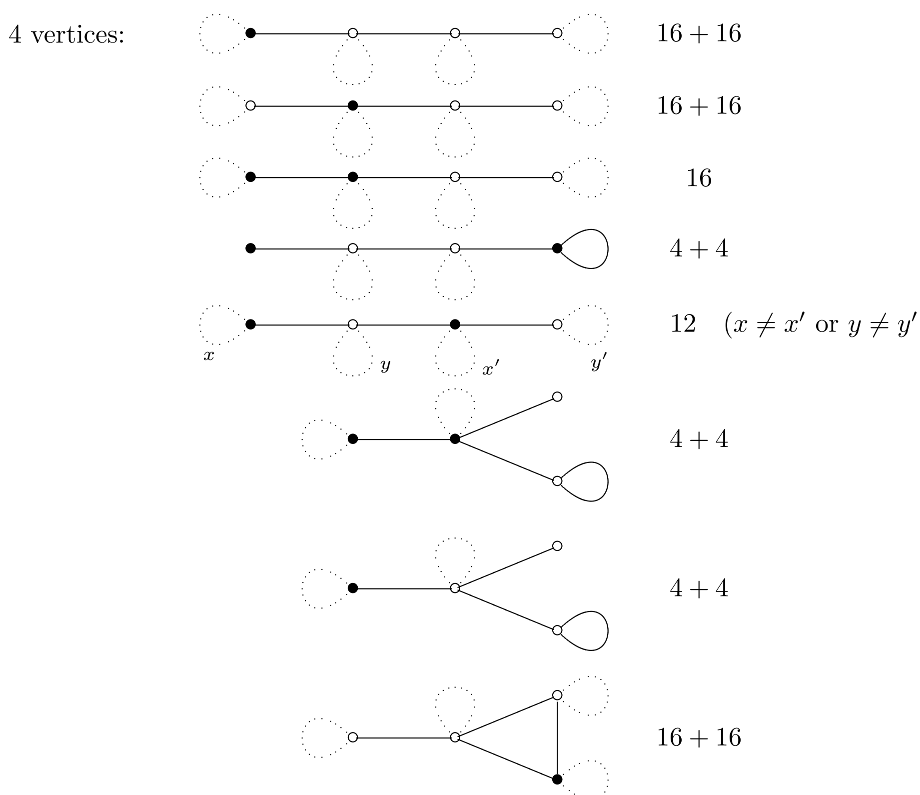
Proposition 4.6. *The number of connected minimal two-colored graphs with n vertices of which exactly b are black (excluding the two 1–vertex graphs with no edges) is given by the table:*

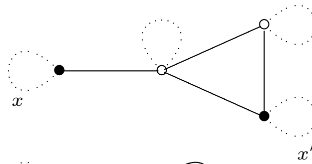
n	$b:0$	1	2	3	4	5	6	7	*total*
1	1	1	0	0	0	0	0	0	2
2	0	4	0	0	0	0	0	0	4
3	0	10	10	0	0	0	0	0	20
4	0	56	61	56	0	0	0	0	173
5	0	446	860	860	446	0	0	0	2612
6	0	6140	17084	20452	17084	6140	0	0	66900
7	0	146698	523416	755656	755656	523416	146698	0	2851540
8	0	6007664	25878921	44839104	48162497	44839104	25878921	6007664	201613875

The proposition shows, for example, that there are 199 quasi-isometry classes for non-geometric graph manifolds having four or fewer Seifert pieces ($199 = 2+4+20+173$). In the next subsection we list the corresponding 199 graphs. These were found by hand before programming the above count. This gives some confidence that the computer program is correct.

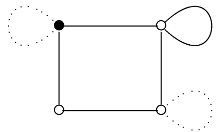
4.1. **Enumeration of minimal two-colored graphs up to 4 vertices.** We only consider connected graphs and we omit the two 1-vertex graphs with no edges. In the following table "number of graphs $n+n$" means n graphs as drawn and n with $\mathbf{b}$ and $\mathbf{w}$ exchanged. Dotted loops in the pictures represent loops that may or may not be present and sometimes carry labels $x, x', \dots$ referring to the two-element set {"present", "absent"}.


number of graphs
1 vertex: 2
2 vertices: 4
3 vertices: $8+8$
$2+2$ (total: 20)

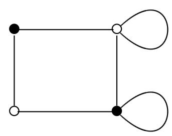

4 vertices: $16+16$
$16+16$
16
$4+4$
12 ($x \neq x'$ or $y \neq y'$)
x
y
x'
y'
$4+4$
$4+4$
$16+16$



$8+8 \quad (x \neq x')$



$4+4$



1

Total for 4 vertices: 173

4.2. **Algorithm for finding the minimal two-colored graph.** Let Γ be a connected two-colored graph. We wish to construct the minimal two-colored graph Γ_0 for which there is a weak covering $\Gamma \to \Gamma_0$. Note that any coloring $c\colon V(\Gamma) \to C$ of the vertices of Γ induces a graph homomorphism to a graph Γ_c with vertex set C and with an edge connecting the vertices $w_1, w_2 \in C$ if and only if there is some edge connecting a $v_1, v_2 \in V(\Gamma)$ with $c(v_i) = w_i$, $i = 1, 2$.

We start with C containing just our original two colors, which we now call $0, 1$, and gradually enlarge C while modifying c until the the map $\Gamma \to \Gamma_c$ is a weak covering. For a vertex v let Adjacent(v) be the set of colors of vertices connected to v by an edge (these may include v itself). We shall always call our coloring c, even as we modify it.

(1) CurrentColor = 0; MaxColor = 1;
(2) **While** CurrentColor $\leq$ MaxColor;
  (a) **If** there are two vertices v_1, v_2 with $c(v_i)$ = CurrentColor that have different Adjacent(v_i)'s;
  (b) **Then** increment MaxColor and add it to the set C, change the color of each v with $c(v)$ = CurrentColor and Adjacent(v) = Adjacent(v_1) to MaxColor, and then set CurrentColor = 0;
  (c) **Else** increment CurrentColor;
  (d) **End If**;
(3) **End While**.

We leave it to the reader to verify that this algorithm terminates with $\Gamma \to \Gamma_c$ the weak covering to the minimal two-colored graph (in step (2b) we could add a new color for each new value of Adjacent(v) with $v \in \{v : c(v) = \text{CurrentColor}\}$ rather than for just one of them; this seems *a priori* more efficient but proved hard to program efficiently). The algorithm is inspired by Brendan McKay's "nauty" [14]; we are grateful to Dylan Thurston for the suggestion.

Counting the number of minimal two-colored graphs with b black vertices and w white vertices is now easy. We order the vertices $1, \dots, b, \dots, b+w$ and consider all connected graphs on this vertex set. For each we check by the above procedure if it is minimal and if so we count it. Finally, we divide our total count by $b!w!$ since each graph has been counted exactly that many times (a minimal two-colored graph has no automorphisms).

5. Artin groups

An *Artin group* is a group given by a presentation of the following form:

$$A = \langle x_1, ..., x_n \mid (x_i, x_j)_{m_{ij}} = (x_j, x_i)_{m_{ji}} \rangle$$

where, for all $i \neq j$ in $\{1, \ldots, n\}$, $m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\}$ with $(x_i, x_j)_{m_{ij}} = x_i x_j x_i ...$ (m_{ij} letters) if $m_{ij} < \infty$ and when $m_{ij} = \infty$ we do not add a defining relation between x_i and x_j. A concise way to present such a group is as a finite graph labeled by integers greater than 1: such a graph has n vertices, one for each generator, and a pair of vertices are connected by an edge labeled by m_{ij} if $m_{ij} < \infty$.

An important class of Artin groups is the class of *right-angled Artin group*. These are Artin groups with each m_{ij} either 2 or ∞, i.e., the only defining relations are commutativity relations between pairs of generators. These groups interpolate between the free group on n generators (n vertices and no edges) and $\mathbb{Z}^n$ (the complete graph on n vertices).

We shall call a presentation tree *big* if it has diameter ≥ 3 or has diameter 2 and at least one weight on it is > 2. An Artin group given by a non-big tree has infinite center and is virtually (free)$\times\mathbb{Z}$. The Artin groups given by non-big presentation trees thus fall into three quasi-isometry classes ($\mathbb{Z}$, $\mathbb{Z}^2$, $F_2 \times \mathbb{Z}$, where F_2 is the 2-generator free group) and are not quasi-isometric to any Artin group with big presentation trees (this follows, for instance, from [12]). We shall therefore only be concerned with Artin groups whose presentation trees are big. For right-angled Artin groups this just says the presentation tree has diameter larger than 2.

We use the term *tree group* to refer to any Artin group whose presentation graph is a big tree. Any right-angled tree group is the fundamental group of a flip graph manifold: this is seen by identifying each diameter 2 region with a (punctured surface) $\times\, \mathbb{S}^1$ and noting that pairs of such regions are glued together by switching fiber and base directions.

Since any right-angled tree group corresponds to a graph manifold with boundary components in each Seifert piece, Theorem 3.2 yields immediately the following answer to Bestvina's question about their quasi-isometry classification:

Theorem 5.1. *Any pair of right-angled tree groups are quasi-isometric.*

This raises the following natural question:

Question 5.2. When is a finitely generated group G quasi-isometric to a right-angled tree group?

The simple answer is that G must be weakly commensurable with the fundamental group of a non-geometric graph manifold with boundary components in every Seifert component, this follows from our Theorem 3.2 and Kapovich–Leeb's quasi-isometric rigidity result for non-geometric 3-manifolds [12]. But it is natural to ask the question within the class of Artin groups, where this answer is not immediately helpful. We give the following answer, which in particular shows that right-angled tree groups are quasi-isometrically rigid in the class of right-angled Artin groups.

Theorem 5.3. *Let G' be any Artin group and let G be a right-angled tree group. Then G' is quasi-isometric to G if and only if G' has presentation graph a big* *even-labeled tree with all interior edges labeled 2. (An "interior edge" is an edge that does not end in a leaf of the tree.)*

We first recall two results relevant to Artin groups given by trees. The first identifies which Artin groups are 3-manifold groups and the second says what those 3-manifolds are.

Theorem 5.4 (Gordon; [8])**.** *The following are equivalent for an Artin group A:*

(1) *A is virtually a 3-manifold group.*
(2) *A is a 3-manifold group.*
(3) *Each connected component of its presentation graph is either a tree or a triangle with each edge labeled 2.*

Theorem 5.5 (Brunner [3], Hermiller-Meier [11])**.** *The Artin group associated to a weighted tree T is the fundamental group of the complement of the following connected sum of torus links. For each n-weighted edge of T associate a copy of the $(2, n)$-torus link and if n is even associate each end of the edge with one of the two components of this link; if n is odd associate both ends of the edge with the single component (a $(2, n)$-knot). Now take the connected sum of all these links, doing connected sum whenever two edges meet at a vertex, using the associated link components to do the sum.*

(In Theorem 5.5 the fact that for an odd-weighted edge the $(2.n)$ torus knot can be associated with either end of the edge shows that one can modify the presentation tree without changing the group. This is a geometric version of the "diagram twisting" of Brady, McCammond, Mühlherr, Neumann [2].)

Proof of Theorem 5.3. Let G' be an Artin group that is quasi-isometric to a right-angled tree group. Right-angled tree groups are one-ended and hence G, and thus G' as well, is not freely decomposable. Thus the presentation graph for G' is connected.

By the quasi-isometric rigidity Theorem for 3-manifolds, as stated in the introduction, we know that G' is weakly commensurable to a 3-manifold group.

Unfortunately it is not yet known if every Artin group is torsion free. If we knew G' is torsion free then we could argue as follows. First, since G' is torsion free, it follows that G' is commensurable with a 3-manifold group. Thus by Theorem 5.4 it *is* a 3-manifold group and is a tree group. By Theorem 3.2 the corresponding graph manifold must have boundary components in every Seifert component. Using Theorem 5.5 it is then easy to see that this gives precisely the class of trees of the theorem. We say more on this in Theorem 5.7 below.

Since we only know that the quotient of G' by a finite group, rather than G' itself, is commensurable with a 3-manifold group we cannot use Gordon's result (Theorem 5.4) directly. But we will follow its proof.

Gordon rules out most Artin groups being fundamental groups of 3-manifolds by proving that they contain finitely generated subgroups which are not finitely presented (i.e., they are not *coherent*). Since Scott [24] proved 3-manifold groups are coherent, and since coherence is a commensurability invariant, such Artin groups are not 3-manifold groups. Since coherence is also a weak commensurability invariant, this also rules out these Artin groups in our situation.

The remaining Artin groups which Gordon treats with a separate argument are those that include triangles with labels $(2, 3, 5)$ or $(2, 2, m)$. The argument given by Gordon for these cases also applies for weak commensurability. (A simpler argument than Gordon's in the $(2,2,m)$ case is that A then contains both a $\mathbb{Z}^3$ subgroup and a non-abelian free subgroup, which easily rules out weak commensurability with a 3-manifold group.) $\square$

The above argument leads also to the following generalization of Gordon's theorem.

Theorem 5.6. *An Artin group A is quasi-isometric to a 3-manifold group if and only if it is a 3-manifold group (and is hence as in Theorem 5.4).*

Proof. Fix an Artin group A which is quasi-isometric to a 3-manifold group. By Papasoglu-Whyte [18], the reducible case reduces to the irreducible case, so we assume the Artin group has a connected presentation graph.

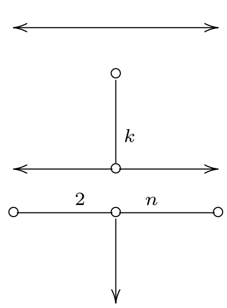
The quasi-isometric rigidity Theorem for 3-manifolds implies that A is weakly commensurable (or in some cases even commensurable) with a 3-manifold group, so as in the previous proof an easy modification of Gordon's argument applies. $\square$

We can, in fact, more generally describe the quasi-isometry class of any tree group A in terms of Theorem 5.3. That is, we can describe the two-colored decomposition graph for the graph manifold G whose fundamental group is A.

Theorem 5.7. *The colored decomposition graph is obtained from the presentation tree of the Artin group by the following sequence of moves:*

(1) *Color all existing vertices black.*
(2) *For each odd-weighted edge, collapse the edge, thus identifying the vertices at its ends, and add a new edge from this vertex to a new leaf which is colored white.*
(3) *Remove any 2-weighted edge leading to a leaf, along with the leaf; on each 2-weighted edge which does not lead to a leaf, simply remove the weight.*
(4) *The only weights now remaining are even weights > 2. If such a weight is on an edge to a leaf, just remove the weight. If it is on an edge joining two nodes, remove the weight and add a white vertex in the middle of the edge.*

Proof. By Theorem 5.5 our graph manifold G is a link complement. Eisenbud and Neumann in [5] classify link complements (in arbitrary homology spheres) in terms of "splice diagrams." We first recall from [5] how to write down the splice diagram in our special case. The splice diagram for the $(2,n)$–torus link, in which arrowheads correspond to components of the link, is as follows:

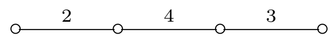


$n = 2$

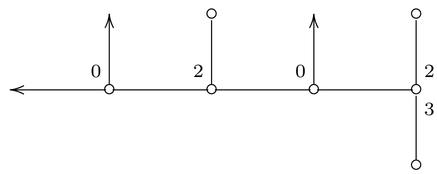
$n = 2k > 2$

$n = 2k + 1 > 2$

(Omitted splice diagram weights are 1.) The splice diagram for a connected sum of two links is obtained by joining the splice diagrams for each link at the arrowheads corresponding to the link components along which connected sum is performed, changing the merged arrowhead into an ordinary vertex, and adding a new 0–weighted arrow at that vertex. For example the splice diagram corresponding to the Artin presentation graph



would be



Now the nodes of the splice diagram correspond to Seifert pieces in the geometric decomposition of the graph manifold. Thus the colored decomposition graph is obtained by taking the full subtree on the nodes of the diagram with nodes that had arrowheads attached colored black and the others colored white. This is as described in the theorem. □

References

[1] J. Behrstock, C. Druţu, and L. Mosher. Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity. Preprint, arXiv:math.GT/0512592, 2005.

[2] Noel Brady, Jonathan P. McCammond, Bernhard Mühlherr, Walter D. Neumann. Rigidity of Coxeter groups and Artin groups. In *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000)*. Geom. Dedicata **94** (2002), 91–109.

[3] A. M. Brunner, Geometric quotients of link groups. Topology Appl. **48** (1992), 245–262.

[4] J. Cannon and D. Cooper. A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three. Trans. AMS, **330** (1992), 419–431.

[5] D. Eisenbud and Walter D. Neumann. *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities*. Annals of Math. Studies **110** (Princeton Univ. Press 1985).

[6] Alex Eskin, David Fisher, and Kevin Whyte. Quasi-isometries and rigidity of solvable groups. Preprint, arXiv:math.GR/0511647, 2005.

[7] S. M. Gersten. Divergence in 3-manifold groups. Geom. Funct. Anal. **4** (1994), 633–647.

[8] C. McA. Gordon. Artin groups, 3-manifolds and coherence. (Bol. Soc. Mat. Mexicana (3) **10** (2004), Special Issue in Honor of Francisco "Fico" Gonzalez-Acuna, 193–198.

[9] M. Gromov. Groups of polynomial growth and expanding maps. IHES Sci. Publ. Math., **53** (1981) 53–73.

[10] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric Group Theory, Vol. 2 (Sussex, 1991)* **182** *LMS Lecture Notes*, (Cambridge Univ. Press, 1993) 1–295.

[11] Susan Hermiller and John Meier. Artin groups, rewriting systems and three-manifolds. J. Pure Appl. Algebra. **136** (1999), 141–156.

[12] M. Kapovich and B. Leeb. Quasi-isometries preserve the geometric decomposition of Haken manifolds. Invent. Math. **128** (1997), 393–416.

[13] M. Kapovich and B. Leeb. 3-manifold groups and nonpositive curvature. Geom. Funct. Anal. **8** (1998), 841–852.

[14] Brendan McKay, "nauty": a program for isomorphism and automorphism of graphs, http://cs.anu.edu.au/~bdm/nauty/

[15] J. Milnor. A note on curvature and the fundamental group. J. Diff. Geom. **2** (1968), 1–7.

[16] Walter D. Neumann. Commensurability and virtual fibration for graph manifolds. Topology **36** (1997), 355–378.

[17] Walter D. Neumann and G.A. Swarup. Canonical decompositions of 3-manifolds. Geometry and Topology **1** (1997), 21–40.

[18] P. Papasoglu and K. Whyte. Quasi-isometries between groups with infinitely many ends. *Comment. Math. Helv.*, **77 (1)** (2002) 133–144.

[19] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. Preprint, arXiv:math.DG/0211159, 2002.

[20] G. Perelman. Ricci flow with surgery on three-manifolds. Preprint, arXiv:math.DG/0303109, 2003.

[21] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. Preprint, arXiv:math.DG/0307245, 2003.

[22] E. Rieffel. Groups quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$. *J. London Math. Soc. (2)*, **64 (1)** (2001) 44–60.

[23] R. Schwartz. The quasi-isometry classification of rank one lattices. *IHES Sci. Publ. Math.*, **82** (1996) 133–168.

[24] G. P. Scott. Finitely generated 3-manifold groups are finitely presented. J. London Math. Soc. (2) **6** (1973), 437–440.

[25] A.S. Švarc. Volume invariants of coverings. Dokl. Akad. Nauk. SSSR **105** (1955), 32–34.

Department of Mathematics, The University of Utah, Salt Lake City, UT 84112, USA
E-mail address: jason@math.utah.edu

Department of Mathematics, Barnard College, Columbia University, New York, NY 10027, USA
E-mail address: neumann@math.columbia.edu