INVARIANTS FROM TRIANGULATIONS OF HYPERBOLIC 3-MANIFOLDS

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Abstract. For any finite volume hyperbolic 3-manifold M we use ideal triangulation to define an invariant $\beta(M)$ in the Bloch group $\mathcal{B}(\mathbb{C})$. It actually lies in the subgroup of $\mathcal{B}(\mathbb{C})$ determined by the invariant trace field of M. The Chern-Simons invariant of M is determined modulo rationals by $\beta(M)$. This implies rationality and — assuming the Ramakrishnan conjecture — irrationality results for Chern Simons invariants.

1. Main Results

The *pre-Bloch group* $\mathcal{P}(k)$ of a field k is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(k - \{0,1\})$ by all instances of the following relations:

Misprint in (1) in the published paper corrected here 03/96

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}] = 0 \tag{1}$$

$$[x] = [1 - \frac{1}{x}] = [\frac{1}{1-x}] = -[\frac{1}{x}] = -[\frac{x}{x-1}] = -[1-x] \tag{2}$$

The *Bloch group* $\mathcal{B}(k)$ is the kernel of the map

$$\mu\colon \mathcal{P}(k) \to k^* \wedge_{\mathbb{Z}} k^* \quad \text{given by} \quad \mu([z]) = 2(z \wedge (1-z)).$$

(There are several variants of this definition in the literature. Dupont and Sah [6] show that the various definitions differ at most by torsion and that they agree with each other for algebraically closed fields. See also the discussion in [15].)

By results of Borel, Bloch, and Suslin [2, 1, 20] (see Theorem 4.1) the Bloch group $\mathcal{B}(k)$ of a number field is isomorphic modulo torsion to $\mathbb{Z}^{r_2}$, where r_2 is the number of complex embeddings of k (a complex embedding is an embedding $k \to \mathbb{C}$ with image not in $\mathbb{R}$). Thus $\mathcal{B}(k) \otimes \mathbb{Q} \cong \mathbb{Q}^{r_2}$. Moreover, if a specific complex embedding is chosen, that is, k is given as a subfield of $\mathbb{C}$, then the induced map $\mathcal{B}(k) \otimes \mathbb{Q} \to \mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$ is injective, so we may write $\mathcal{B}(k) \otimes \mathbb{Q} \subset \mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$.

Let $M = \mathbb{H}^3/\Gamma$ be an oriented complete hyperbolic manifold of finite volume (briefly just "hyperbolic 3-manifold" from now on). The *invariant trace field* $k(M) = k(\Gamma)$ is the field generated over $\mathbb{Q}$ by squares of traces of elements of Γ. It is the smallest field among trace fields of finite index subgroups of Γ ([19], see also [13]). It is a number field and comes with a specific embedding in $\mathbb{C}$.

Thurston has shown [21] that any hyperbolic 3-manifold M has a degree one ideal triangulation (see sect. 2) by ideal simplices $\Delta_1, \dots, \Delta_n$. Let $z_i \in \mathbb{C}$ be the parameter of the ideal simplex Δ_i for each i (cross-ratio of its four vertices). These z_i define an element $\beta(M) = \sum_{i=1}^n [z_i]$ in the pre-Bloch group $\mathcal{P}(\mathbb{C})$.

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Theorem 1.1. *This $\beta(M)$ depends only on M. It lies in the Bloch group $\mathcal{B}(\mathbb{C}) \subset \mathcal{P}(\mathbb{C})$. As an element of $\mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$, it lies in the subgroup $\mathcal{B}(k(M)) \otimes \mathbb{Q}$.*

One of the main ingredients for this theorem — Proposition 3.1 below — fills a gap in the literature. Statements have appeared in papers of at least two authors that implicitly assume it.

In the non-compact case one can find genuine (rather than just degree one) ideal triangulations of M (see [7]) and the simplex parameters z_i then lie in the invariant trace field $k(M)$ (see [13]). Thus $\beta(M)$ is the image of a class $\beta_k(M) := \sum[z_i] \in \mathcal{P}(k(M))$.

Theorem 1.2. *$\beta_k(M)$ lies in $\mathcal{B}(k(M))$ and is independent of triangulation.*

Thus, in the non-compact case the "$\otimes\mathbb{Q}$" of Theorem 1.1 can be deleted. We do not know if it can in the compact case, though we can describe an explicit integer $c = c(M) \geq 0$ such that $2^c\beta(M)$ is in the image of $\mathcal{B}(k(M)) \to \mathcal{B}(\mathbb{C})$.

The Bloch invariant $\beta(M)$ is intimately related to the volume $\operatorname{vol}(M)$ and the Chern-Simons invariant $\operatorname{CS}(M)$. Chern and Simons defined the latter invariant in ([3]) for any compact $(4n-1)$-dimensional Riemannian manifold. Meyerhoff [11] extended the definition in the case of hyperbolic 3-manifolds to allow noncompact ones, that is hyperbolic 3-manifolds with cusps. The Chern-Simons invariant $\operatorname{CS}(M)$ of such a hyperbolic 3-manifold M is an element in $\mathbb{R}/\pi^2\mathbb{Z}$. It is called *rational* (also called *torsion*) if it lies in $\pi^2\mathbb{Q}/\pi^2\mathbb{Z}$.

The relation with the Bloch invariant uses the "Bloch regulator map"

$$\rho\colon \mathcal{B}(\mathbb{C}) \longrightarrow \mathbb{C}/\mathbb{Q},$$

defined as follows. For $z \in \mathbb{C} - \{0,1\}$, define

$$\rho(z) = \frac{\log z}{2\pi i} \wedge \frac{\log(1-z)}{2\pi i} + 1 \wedge \frac{\mathcal{R}(z)}{2\pi^2},$$

where $\mathcal{R}(z)$ is the "Rogers dilogarithm function"

$$\mathcal{R}(z) = \frac{1}{2}\log(z)\log(1-z) - \int_0^z \frac{\log(1-t)}{t}dt.$$

See section 4 of [6] or [9] for details on how to interpret this formula. This ρ vanishes on the relations (1) and (2) which define $\mathcal{P}(\mathbb{C})$ and hence ρ induces a map

$$\rho\colon \mathcal{P}(\mathbb{C}) \longrightarrow \mathbb{C} \wedge_{\mathbb{Z}} \mathbb{C}.$$

This fits in a commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\mathbb{C}) & \xrightarrow{\mu} & \mathbb{C}^* \wedge \mathbb{C}^* \\ \downarrow\rho & & \downarrow= \\ \mathbb{C} \wedge \mathbb{C} & \xrightarrow{\epsilon} & \mathbb{C}^* \wedge \mathbb{C}^* \end{array}$$

where $\epsilon = 2(e \wedge e)$ with $e(z) = \exp(2\pi i z)$. The kernel of μ is $\mathcal{B}(\mathbb{C})$ and the kernel of ϵ is $\mathbb{C}/\mathbb{Q}$. Hence ρ restricts to give the desired map $\rho\colon \mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$.

Theorem 1.3. $\rho(\beta(M)) = \frac{1}{2\pi^2}(\operatorname{CS}(M) + i\operatorname{vol}(M)) \in \mathbb{C}/\mathbb{Q}$.

The volume part of this result is not hard (in part because "volume" is already well defined on $\mathcal{P}(\mathbb{C})$), but the part referring to Chern-Simons lies deeper. Modulo Proposition 3.1 below, the compact case is due to Dupont [5].

This theorem has various consequences for rationality and irrationality of the Chern-Simons invariant. For example, we show in [15]:

Theorem 1.4. *The Chern-Simons invariant* $\mathrm{CS}(M)$ *of a hyperbolic 3-manifold is rational if the invariant trace field* $k(M)$ *of* M*, as a subfield of* $\mathbb{C}$*, is an imaginary quadratic extension of a totally real field (briefly, "*k *is CM-embedded in* $\mathbb{C}$*").*

On the other hand we have the following irrationality conjecture for Chern-Simons invariant. We use $\overline{k}$ to mean the complex conjugate of the subfield $k \subset \mathbb{C}$.

Conjecture 1.5. *If the invariant trace field* $k = k(M)$ *satisfies* $k \cap \overline{k} \subset \mathbb{R}$ *then* $\mathrm{CS}(M)$ *is irrational. In particular,* $\mathrm{CS}(M)$ *is irrational if* $k(M)$ *has odd degree over* $\mathbb{Q}$*.*

We show in [15] that this conjecture would follow from a conjecture of Ramakrishnan [18]:

Conjecture 1.6. *For any number field* k *the Bloch map* ρ *restricted to* $\mathcal{B}(k) \otimes \mathbb{Q}$ *is injective.*

A number field k occurs as the invariant trace field of an *arithmetic* hyperbolic 3-manifold if and only if it has just one complex place (cf. e.g., [13]). It then either satisfies $k = \overline{k}$ and is CM-embedded or it satisfies $k \cap \overline{k} \subset \mathbb{R}$. Thus, in the arithmetic case Theorem 1.4 and Conjecture 1.5 would say that rationality or irrationality of the Chern-Simons invariant is completely determined by whether or not $k = \overline{k}$.

Although no example of irrationality of the Chern-Simons invariant of a hyperbolic 3-manifold has been proved, there is a lot of numerical evidence for the above conjecture. A similar comment applies to volume. We say more about computational aspects in the final section of this announcement. We also describe there a generalization of $\beta(M)$ to an invariant of a homomorphism $\Gamma = \pi_1(M) \to \mathrm{PGL}(2,\mathbb{C})$. In [16] we define an analogous invariant in any dimension, but its significance is not clear at this time.

Added May 1995. A.B. Goncharov has kindly shared with us his manuscript [8] in which he defines an invariant in $K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ for any hyperbolic $(2n-1)$-manifold of finite volume.

2. Background

2.1. **Ideal simplices and degree one ideal triangulations.** We shall denote the standard compactification of $\mathbb{H}^3$ by $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{CP}^1$. An ideal simplex Δ with vertices $z_1, z_2, z_3, z_4 \in \mathbb{CP}^1$ is determined up to congruence by the cross ratio

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

This z lies in the upper half plane of $\mathbb{C}$ if the orientation induced by the given ordering of the vertices agrees with the orientation of $\mathbb{H}^3$. Permuting the vertices by an even (i.e., orientation preserving) permutation replaces z by one of

$$z, \quad 1 - \frac{1}{z}, \quad \text{or} \quad \frac{1}{1-z},$$

while an odd permutation replaces z by

$$\frac{1}{z}, \quad \frac{z}{z-1}, \quad \text{or} \quad 1 - z.$$

We will also allow degenerate ideal simplices where the vertices lie in a plane, so the parameter z is real. However, we always require that the vertices are distinct.

Thus the parameter z of the simplex lies in $\mathbb{C}-\{0,1\}$ and every such z corresponds to an ideal simplex.

Let Y be a CW-complex obtained by gluing together finitely many 3-simplices by identifying the 2-faces in pairs. The complement of the 1-skeleton is then a 3-manifold, and if this 3-manifold is oriented we call Y a *3-cycle*. In this case the complement $Y - Y^{(0)}$ of the vertices is an oriented 3-manifold.

Suppose $M^3 = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold. A *degree one ideal triangulation of* M consists of a 3-cycle Y plus a map $f\colon Y - Y^{(0)} \to M$ satisfying

- f is degree one almost everywhere in M;
- for each 3-simplex S of Y there is a map f_S of S onto an ideal simplex in $\overline{\mathbb{H}}^3$, mapping vertices to ideal vertices, such that $f|S - S^{(0)}\colon S - S^{(0)} \to M$ is the composition $\pi \circ (f_S|S - S^{(0)})$, where $\pi\colon \mathbb{H}^3 \to M$ is the projection.

Thurston shows in [21] that any hyperbolic 3-manifold has degree one ideal triangulations. Such triangulations also arise "in practice" (e.g., in the program SNAPPEA for exploring hyperbolic manifolds — [23]) as follows. It follows from [7] that any non-compact M has a "genuine" ideal triangulation: one for which f is arbitrarily closely homotopic to a homeomorphism ([7] gives an ideal polyhedral subdivision and some flat simplices may be needed to subdivide the polyhedra consistently into ideal tetrahedra). The ideal simplices can be deformed to give degree one ideal triangulations (based on the same 3-cycle Y) on almost all manifolds obtained by Dehn filling cusps of M (see e.g., [17]).

2.2. **Bloch group.** We describe the geometric background for our definition of the Bloch group. For $k = \mathbb{C}$, the relations (2) express the fact that the pre-Bloch group $\mathcal{P}(\mathbb{C})$ may be thought of as being generated by congruence classes of ideal hyperbolic 3-simplices. The convex hull of five distinct points in the ideal boundary of $\mathbb{H}^3$ can be decomposed into ideal simplices in exactly two ways: once into two ideal simplices and once into three. The "five term relation" (1) expresses the fact that these two decompositions represent the same element in $\mathcal{P}(\mathbb{C})$.

As already mentioned, there are several different definitions of the Bloch group in the literature. By [6] they differ at most by torsion and agree with each other for algebraically closed fields. The version (1) of the five term relation we use is the one of Suslin [20]. Dupont and Sah use a slightly different one first written down by Bloch and Wigner:

$$[x]-[y]+[y/x]-[(1-y)/(1-x)]+[(1-y^{-1})/(1-x^{-1})]=0.$$

This is conjugate to Suslin's by the self-map $[z] \mapsto [z^{-1}]$ of $\mathbb{Z}(k-\{0,1\})$. See [15] or [20] for a discussion of the reason for Suslin's choice. Also, relation (2) is already implied modulo torsion by the five term relation (1). Omitting it gives the version of the Bloch group used by Dupont and Sah [6].

Remark. The above suggests that our invariant $\beta(M)$ captures a type of "ideal scissors congruence class" of the hyperbolic manifold M. In fact, $\mathcal{P}(\mathbb{C})$ is the "scissors congruence group" generated by hyperbolic polyhedra with only ideal vertices and triangular faces modulo the relations generated by cutting and pasting along such faces. From a scissors congruence point of view it is more natural not to restrict the faces to be triangular; we then obtain the quotient of $\mathcal{P}(\mathbb{C})$ by the subgroup generated by "flat" simplices, that is, the quotient of $\mathcal{P}(\mathbb{C})$ by the image of $\mathcal{P}(\mathbb{R})$. Hilbert's third problem for hyperbolic geometry is often interpreted as the problem of evaluating the analogous group for polyhedra without ideal vertices. Dupont and Sah [6] show that allowing only non-ideal vertices gives the same scissors congruence group up to 2-torsion as allowing both ideal and non-ideal vertices, and that the resulting group is the -1 co-eigenspace $\mathcal{P}_-(\mathbb{C})$ for the action of conjugation on $\mathcal{P}(\mathbb{C})$. The geometric background to this is that the group $\mathcal{P}(\mathbb{C})$ is orientation sensitive, while the non-ideal scissors congruence group is not. In fact, any ideal simplex Δ can be cut into three simplices which can be re-assembled to give the mirror image of Δ by dropping a perpendicular from a vertex of Δ to the opposite face.

The "imaginary part" of the Ramakrishnan conjecture (also conjectured in [24]) would imply that the imaginary part of $\rho(\beta(M))$, namely $\mathrm{vol}(M)$, is a complete scissors congruence invariant up to torsion for a hyperbolic manifold. This might be considered a weak positive answer to Hilbert's third problem for hyperbolic manifolds!

3. Sketch of Proofs

The consequences of our results for rationality and irrationality of Chern-Simons invariant follow in [15] from an analysis of the dimensions of the eigenspaces of the action of complex conjugation on the Bloch group $\mathcal{B}(K)$ for a number field $K = \overline{K} \subset \mathbb{C}$. We will not discuss this further here.

We first discuss the compact case of Theorem 1.1.

There is an exact sequence due to Bloch and Wigner (cf. [6])

$$0 \to \mu \to H_3(\mathrm{PGL}(2,\mathbb{C})^\delta;\mathbb{Z}) \xrightarrow{\sigma} \mathcal{B}(\mathbb{C}) \to 0, \tag{3}$$

where $\mu \subset \mathbb{C}^*$ is the group of roots of unity and the superscript δ means we are taking $\mathrm{PGL}(2,\mathbb{C})$ with discrete topology. If $M = \mathbb{H}^3/\Gamma$ is compact then the map $\Gamma \to \mathrm{PGL}(2,\mathbb{C})$ induces a map $H_3(\Gamma;\mathbb{Z}) \to H_3(\mathrm{PGL}(2,\mathbb{C})^\delta;\mathbb{Z})$. But $H_3(\Gamma;\mathbb{Z}) = H_3(M;\mathbb{Z}) \equiv \mathbb{Z}$. The image of a generator gives a "fundamental class" $[M] \in H_3(\mathrm{PGL}(2,\mathbb{C})^\delta;\mathbb{Z})$.

The independence of $\beta(M)$ on ideal triangulation is given by the following proposition.

Proposition 3.1. *$\beta(M) = \sigma([M]) \in \mathcal{B}(\mathbb{C})$.*

We next explain why $\beta(M)\otimes\mathbb{Q}$ lies in the image of $\mathcal{B}(k)\otimes\mathbb{Q} \to \mathcal{B}(\mathbb{C})\otimes\mathbb{Q}$, where k is the invariant trace field $k = k(\Gamma)$. We denote this subgroup by $\mathcal{B}(k)_\mathbb{Q}$.

If Γ has trace field K then one can find a quadratic extension K_1 of K so that the embedding $\Gamma \hookrightarrow \mathrm{PGL}(2,\mathbb{C})$ factors up to conjugacy through $\mathrm{PGL}(2,K_1)$. Now the exact sequence (3) holds modulo torsion for any field, in particular for K_1. We deduce that $\beta(M)\otimes\mathbb{Q}$ lives in $\mathcal{B}(K_1)_\mathbb{Q}$. By [14] there are infinitely many different fields K_1 for which we can do this. If we take two of them, say K_1 and K_2, we see that $\beta(M)\otimes\mathbb{Q}$ lives in $\mathcal{B}(K_1)_\mathbb{Q} \cap \mathcal{B}(K_2)_\mathbb{Q}$. This is $\mathcal{B}(K_1\cap K_2)_\mathbb{Q} = \mathcal{B}(K)_\mathbb{Q}$ by Proposition 2.1 of [15]. Finally, by replacing Γ by a subgroup of finite index we can arrange that its trace field is the invariant trace field k. Since this just multiplies $\beta(M)\otimes\mathbb{Q}$ by the index of the subgroup, it follows that $\beta(M)\otimes\mathbb{Q}$ is in $\mathcal{B}(k)_\mathbb{Q}$.

We prove Proposition 3.1 in [16] by factoring through a relative homology group for which the relationship between $[M]$ and $\beta(M)$ is easier to see. In the notation of [4] this relative group is $H_3(\mathrm{PGL}(2,\mathbb{C}),\mathbb{CP}^1;\mathbb{Z})$. It has a natural map to $\mathcal{P}(\mathbb{C})$. (Dupont and Sah [6] show — without using this notation — that this map is an isomorphism.)

In fact, our key lemma in both the compact and non-compact cases is

Lemma 3.2. *$H_3(\Gamma, \mathbb{CP}^1; \mathbb{Z})$ is infinite cyclic, generated by a "fundamental class" $[M]$. The composition $H_3(\Gamma, \mathbb{CP}^1; \mathbb{Z}) \to H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1; \mathbb{Z}) \to \mathcal{P}(\mathbb{C})$ maps $[M]$ to $\beta(M)$.*

In the non-compact case the fact that $\beta(M)$ lies in $\mathcal{B}(\mathbb{C})$ is the relation $\sum z_i \wedge (1-z_i) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*$ on the simplex parameters z_i. This has been attributed to Thurston (unpublished) by Gross [10]. It also follows easily from [17] (see also [12]). We give a cohomological proof in [16].

Finally, we discuss Theorem 1.3. We have already remarked that it is in the compact case, modulo Proposition 3.1, essentially a result of Dupont [5]. In general it follows from the simplicial formula for Chern-Simons invariant of [12]. That formula was deduced in the general case from the compact case and included an unknown constant which was claimed there to be a rational multiple of π^2. There was a gap in the proof of this rationality, which is filled by Proposition 3.1 above.

4. Final Remarks

4.1. **Computing the invariant $\beta(M)$.** Define the *Bloch-Wigner* function $D_2 \colon \mathbb{C} - \{0,1\} \to \mathbb{R}$ by (cf. [1])

$$D_2(z) = Im\,\ln_2(z) + \log|z| \arg(1-z), \quad z \in \mathbb{C} - \{0,1\}$$

where $\ln_2(z)$ is the classical dilogarithm function. The hyperbolic volume of an ideal tetrahedron Δ with cross ratio z is equal to $D_2(z)$. It follows that D_2 satisfies the functional equations given by the relations which define $\mathcal{P}(\mathbb{C})$, and therefore D_2 induces a map

$$D_2 \colon \mathcal{B}(\mathbb{C}) \longrightarrow \mathbb{R},$$

by defining $D_2[z] = D_2(z)$.

Given a number field k let $\sigma_1, \bar\sigma_1, \dots, \sigma_{r_2}, \bar\sigma_{r_2}$ denote the complex embeddings of k. One then has a map

$$\begin{array}{cccc} c_2: & \mathcal{B}(k) & \longrightarrow & \mathbb{R}^{r_2} \\ & \sum_i (n_i[z_i]) & \mapsto & (\sum_i n_i D_2(\sigma_1(z_i)), \dots, \sum_i n_i D_2(\sigma_{r_2}(z_i))). \end{array}$$

The following theorem is a re-interpretation by Bloch and Suslin of a theorem about K-groups of Borel.

Theorem 4.1. *The kernel of c_2 is exactly the torsion subgroup of $\mathcal{B}(k)$ and the image of c_2 is a maximal lattice in $\mathbb{R}^{r_2}$. In particular, the rank of $\mathcal{B}(k)$ is r_2.*

Using this theorem we can compute $\beta(M)$ up to torsion by computing its image $c_2(\beta(M))$ using the simplex parameters of an ideal triangulation. The program SNAPPEA works with ideal triangulations. However, in its current incarnation it computes the simplex parameters numerically rather than as exact algebraic numbers. A preliminary version of a modification of Snappea that computes to high precision and derives exact simplex parameters was written (mostly by Oliver Goodman) as part of an Australian Research Council project at Melbourne University. Using this, the element $c_2(\beta(M)) \in \mathbb{R}^{r_2}$ can be computed to high precision when the invariant trace field $k = k(M)$ does not have too high degree. Such calculations support the predictions of the Ramakrishnan conjecture.

These calculations have also yielded interesting examples. For example, there exist two compact arithmetic 3-manifolds of volume $1.83193119\dots$ defined over the field $\mathbb{Q}(i)$ but with different quaternion algebras (so they are non-commensurable) such that one can disassemble one of them into three ideal tetrahedra that can be reassembled into the other. The tetrahedral parameters lie in $K := \mathbb{Q}(i, \sqrt{4-2i})$, so the Bloch invariant is defined over this field. The examples have Chern-Simons invariants differing by $\pi^2/12$ modulo π^2. Thus Chern-Simons invariant is not determined by Bloch invariant if one does not take it modulo $\pi^2\mathbb{Q}$. Craig Hodgson found these examples and Alan Reid helped compute the quaternion algebras.

4.2. Generalization of $\beta(M)$.

Theorem 4.2. *If $M = \mathbb{H}^3/\Gamma$ is a finite volume hyperbolic manifold and $f\colon \Gamma \to \mathrm{PGL}(2,\mathbb{C})$ is any homomorphism then there is a natural invariant $\beta(f) \in \mathcal{P}(\mathbb{C})$. If f is the homomorphism corresponding to some Dehn filling M' of M then $\beta(f) = \beta(M')$. In particular, for the discrete embedding we have $\beta(f) = \beta(M)$. If each cusp subgroup of Γ has non-trivial elements γ with $f(\gamma)$ parabolic (or trivial) then $\beta(f) \in \mathcal{B}(\mathbb{C})$.*

This theorem gives a way of defining the "volume" of any homomorphism $f\colon \Gamma \to \mathrm{PGL}(2,\mathbb{C})$. The existence of such a volume was mentioned in [22]. In [16] a version of this theorem is proved for any dimension.

If $M = \mathbb{H}^3/\Gamma$ is compact then we can define $\beta(f)$ as the image of the fundamental class $[M] \in H_3(\Gamma;\mathbb{Z})$ under $H_3(\Gamma;\mathbb{Z}) \xrightarrow{f_*} H_3(\mathrm{PGL}(2,\mathbb{C})^\delta;\mathbb{Z}) \to \mathcal{B}(\mathbb{C})$. This clearly generalizes the invariant $\beta(M)$, which is the value of $\beta(f)$ for the discrete embedding. If M is non-compact it is harder to define $\beta(f)$. We cannot directly follow the program of the previous section since $H_3(\Gamma, \mathbb{CP}^1;\mathbb{Z})$ will depend on the action of Γ on $\mathbb{CP}^1$ and hence on the homomorphism f. We use instead $H_3(\Gamma, \mathcal{C};\mathbb{Z})$, where $\mathcal{C}$ is the union of Γ/P as P runs through a set of representatives for the conjugacy classes of cusp subgroups of Γ. This homology group is cyclic generated by a class $[M]$. There are, in general, several maps $\mathcal{C} \to \mathbb{CP}^1$ which are equivariant with respect to $f\colon \Gamma \to \mathrm{PGL}(2,\mathbb{C})$, so f induces several maps $H_3(\Gamma, \mathcal{C};\mathbb{Z}) \to H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1;\mathbb{Z}) \to \mathcal{P}(\mathbb{C})$. One can define a simplicial version of $\beta(M)$ and show that it is the image of $[M]$ under any one of these maps. Thus the maps are all the same and the image of $[M]$ agrees again with the simplicial version of the invariant.

References

[1] S. Bloch: *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves,* Lecture notes U.C. Irvine (1978).

[2] A. Borel: *Cohomologie de SL_n et valeurs de fonction zeta aux points entiers*, Ann. Sci. Ecole Norm. Sup. (4) 7, (1974), 613–636.

[3] S. Chern, J. Simons, *Some cohomology classes in principal fiber bundles and their application to Riemannian geometry*, Proc. Nat. Acad. Sci. U.S.A. 68 (1971), 791-794.

[4] W. Dicks, M. J. Dunwoody: *Groups acting on graphs*, Camb. Studies in Adv. Math. 17 (1989)

[5] J. Dupont: *The dilogarithm as a characteristic class for flat bundles*, J. Pure and App. Algebra 44 (1987), 137–164.

[6] J. Dupont, H. Sah: *Scissors congruences II*, J. Pure and App. Algebra 25 (1982), 159–195.

[7] D. B. A. Epstein, R. Penner: *Euclidean decompositions of non-compact hyperbolic manifolds,* J. Diff. Geom. 27 (1988), 67–80.

[8] A.B. Goncharov: *Volumes of hyperbolic manifolds and mixed Tate motives*, (in preparation).

[9] R. Hain: *Classical polylogarithms*, Motives, Proc. Symp. Pure Math, 55 (1994), Part 2, 3–42

[10] B. Gross: *On the values of Artin L-functions*, preprint (Brown, 1980).

[11] R. Meyerhoff: *Hyperbolic 3-manifolds with equal volumes but different Chern-Simons invariants*, in Low-dimensional topology and Kleinian groups, edited by D. B. A. Epstein, London Math. Soc. lecture notes series, 112 (1986) 209–215.

[12] W. D. Neumann: *Combinatorics of triangulations and the Chern Simons invariant for hyperbolic 3-manifolds*, in *Topology 90, Proceedings of the Research Semester in Low Dimensional Topology at Ohio State* (Walter de Gruyter Verlag, Berlin - New York 1992), 243–272.

[13] W. D. Neumann, A. W. Reid: *Arithmetic of hyperbolic manifolds*, *Topology 90, Proceedings of the Research Semester in Low Dimensional Topology at Ohio State* (Walter de Gruyter Verlag, Berlin - New York 1992), 273–310.

[14] W. D. Neumann, A. W. Reid: *Amalgamation and the invariant trace field of a Kleinian group*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 509–515.

[15] W. D. Neumann, J. Yang: *Rationality problems for K-theory and Chern-Simons invariants of hyperbolic 3-manifolds*, Enseignement Mathématique (to appear).

[16] W. D. Neumann, J. Yang: *Bloch invariants of hyperbolic 3-manifolds*, in preparation.

[17] W. D. Neumann, D. Zagier: *Volumes of hyperbolic 3-manifolds*, Topology 24 (1985), 307–332.

[18] D. Ramakrishnan: *Regulators, algebraic cycles, and values of L-functions*, Contemp. Math. 83 (1989), 183–310.

[19] A. W. Reid: *A note on trace-fields of Kleinian groups.* Bull. London Math. Soc. 22(1990), 349–352.

[20] A. A. Suslin: *Algebraic K-theory of fields,* Proc. Int. Cong. Math. Berkeley 86, vol. 1 (1987), 222-244.

[21] W. P. Thurston: *Hyperbolic structures on 3-manifolds I: Deformation of acylindrical manifolds.* Annals of Math. 124 (1986), 203-246

[22] W. P. Thurston: *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982), 357–381.

[23] J. Weeks: *Snappea*, Software for hyperbolic 3-manifolds, available at ftp://ftp.geom.umn.edu/pub/software/snappea

[24] D. Zagier: *Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields*, In "Arithmetic Algebraic Geometry" (G. v.d. Geer, F. Oort, J. Steenbrink, eds.), Prog. in Math. 89, (Birkäuser, Boston 1991), 391–430.

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