BI-LIPSCHITZ GEOMETRY OF WEIGHTED HOMOGENEOUS SURFACE SINGULARITIES

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ABSTRACT. We show that a weighted homogeneous complex surface singularity is metrically conical (i.e., bi-Lipschitz equivalent to a metric cone) only if its two lowest weights are equal. We also give an example of a pair of weighted homogeneous complex surface singularities that are topologically equivalent but not bi-Lipschitz equivalent.

1. Introduction and main results

A natural question of metric theory of singularities is the existence of a metrically conical structure near a singular point of an algebraic set. For example, complex algebraic curves, equipped with the inner metric induced from an embedding in $\mathbb{C}^N$, always have metrically conical singularities. It was discovered recently (see [1]) that weighted homogeneous complex surface singularities are not necessarily metrically conical. In this paper we show that they are rarely metrically conical.

Let (V,p) be a normal complex surface singularity germ. Any set $z_1,\dots,z_N$ of generators for $\mathcal{O}_{(V,p)}$ induces an embedding of germs $(V,p)\to(\mathbb{C}^N,0)$. The Riemannian metric on $V-\{p\}$ induced by the standard metric on $\mathbb{C}^N$ then gives a metric space structure on the germ (V,p). This metric space structure, in which distance is given by arclength within V, is called the *inner metric* (as opposed to *outer metric* in which distance between points of V is distance in $\mathbb{C}^N$).

It is easy to see that, up to bi-Lipschitz equivalence, this inner metric is independent of choices. It depends strongly on the analytic structure, however, and may not be what one first expects. For example, we shall see that if (V,p) is a quotient singularity $(V,p)=(\mathbb{C}^2/G,0)$, with $G\subset U(2)$ finite acting freely, then this metric is usually not bi-Lipschitz equivalent to the conical metric induced by the standard metric on $\mathbb{C}^2$.

If M is a smooth compact manifold then a *cone on M* will mean the cone on M with a standard Riemannian metric off the cone point. This is the metric completion of the Riemannian manifold $\mathbb{R}_+\times M$ with metric given (in terms of an element of arc length) by $ds^2=dt^2+t^2ds_M^2$ where t is the coordinate on $\mathbb{R}_+$ and ds_M is given by any Riemannian metric on M. It is easy to see that this metric completion simply adds a single point at $t=0$ and, up to bi-Lipschitz equivalence, the metric on the cone is independent of choice of metric on M.

If M is the link of an isolated complex singularity (V,p) then the germ (V,p) is homeomorphic to the germ of the cone point in a cone CM. If this homeomorphism

Key words and phrases. bi-Lipschitz, complex surface singularity.
Research supported under CNPq grant no 300985/93-2.
Research supported under CNPq grant no 300393/2005-9.
Research supported under NSA grant H98230-06-1-011 and NSF grant no. DMS-0206464.

can be chosen to be bi-Lipschitz we say, following [3], that the germ (V,p) is *metrically conical*. In [3] the approach taken is to consider a semialgebraic triangulation of V and consider the star of p according to this triangulation. The point p is metrically conical if the intersection $V \cap B_\epsilon[p]$ is bi-Lipschitz homeomorphic to the star of p, considered with the standard metric of the simplicial complex.

Suppose now that (V,p) is weighted homogeneous. That is, V admits a good $\mathbb{C}^*$–action (a holomorphic action with positive weights: each orbit $\{\lambda x \mid \lambda \in \mathbb{C}^*\}$ approaches zero as $\lambda \to 0$). The weights $v_1,\dots,v_r$ of a minimal set of homogeneous generators of the graded ring of V are called the *weights of* V. We shall order them by size, $v_1 \ge \dots \ge v_r$, so v_{r-1} and v_r are the two lowest weights.

If (V,p) is a cyclic quotient singularity $V = \mathbb{C}^2/\mu_n$ (where μ_n denotes the n–th roots of unity) then it does not have a unique $\mathbb{C}^*$–action. In this case we use the $\mathbb{C}^*$–action induced by the diagonal action on $\mathbb{C}^2$.

If (V,p) is homogeneous, that is, the weights $v_1,\dots,v_r$ are all equal, then it is easy to see that (V,p) is metrically conical.

Theorem 1. *If the two lowest weights of V are unequal then (V,p) is not metrically conical.*

For example, the Kleinian singularities $A_k, k\ge 1$, $D_k, k \ge 4$, E_6, E_7, E_8 are the quotient singularities $\mathbb{C}^2/G$ with $G \subset SU(2)$ finite. The diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^2$ induces an action on $\mathbb{C}^2/G$, so they are weighted homogeneous. They are the weighted homogeneous hypersurface singularities:

	equation	**weights**
A_k :	$x^2+y^2+z^{k+1}=0$	$(k+1,k+1,2)$ or $(\frac{k+1}{2},\frac{k+1}{2},1)$
D_k:	$x^2+y^2z+z^{k-1}=0$	$(k-1,k-2,2)$, $k\ge 4$
E_6:	$x^2+y^3+z^4=0$	$(6,4,3)$
E_7:	$x^2+y^3+yz^3=0$	$(9,6,4)$
E_8:	$x^2+y^3+z^5=0$	$(15,10,6)$

By the theorem, none of them is metrically conical except for the quadric A_1 and possibly[1] the quaternion group quotient D_4.

The general cyclic quotient singularity is of the form $V=\mathbb{C}^2/\mu_n$ where the n–th roots of unity act on $\mathbb{C}^2$ by $\xi(u_1,u_2) = (\xi^q u_1, \xi u_2)$ for some q prime to n with $0<q<n$; the link of this singularity is the lens space $L(n,q)$. It is homogeneous if and only if $q=1$.

Theorem 2. *A cyclic quotient singularity is metrically conical if and only if it is homogeneous.*

Many non-homogeneous cyclic quotient singularities have their two lowest weights equal, so the converse to Theorem 1 is not generally true.

We can also sometimes distinguish weighted homogeneous singularities with the same topology from each other.

Theorem 3. *Let (V,p) and (W,q) be two weighted homogeneous normal surface singularities, with weights $v_1\ge v_2\ge\dots\ge v_r$ and $w_1\ge w_2\ge\dots\ge w_s$ respectively. If either $\frac{v_{r-1}}{v_r} > \frac{w_1}{w_s}$ or $\frac{w_{s-1}}{w_s} > \frac{v_1}{v_r}$ then (V,p) and (W,q) are not bi-Lipschitz homeomorphic.*

[1] We have a tentative proof that the quaternion quotient is metrically conical, see [2].

Corollary 4. *Let $V, W \subset \mathbb{C}^3$ be defined by $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \ :\ z_1^2 + z_2^{51} + z_3^{102} = 0\}$ and $W = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \ :\ z_1^{12} + z_2^{15} + z_3^{20} = 0\}$. Then, the germs $(V, 0)$ and $(W, 0)$ are homeomorphic, but they are not bi-Lipschitz homeomorphic.*

The corollary follows because in both cases the link of the singularity is an S^1 bundle of Euler class -1 over a curve of genus 26; the weights are $(51, 2, 1)$ and $(5, 4, 3)$ respectively and Theorem 2 applies since $\frac{2}{1} > \frac{5}{3}$.

The idea of the proof of Theorem 1 is to find an essential closed curve in $V - \{p\}$ with the property that as we shrink it towards p using the $\mathbb{R}^*$ action, its diameter shrinks faster than it could if V were bi-Lipschitz equivalent to a cone. Any essential closed curve in $V - \{p\}$ that lies in the hyperplane section $z_r = 1$ will have this property, so we must show that the hyperplane section contains such curves. The proofs of Theorems 2 and 3 are similar.

2. Proofs

Let $z_1, \dots, z_r$ be a minimal set of homogeneous generators of the graded ring of V, with z_i of weight v_i and $v_1 \geq v_2 \geq \dots v_{r-1} \geq v_r$. Then $x \mapsto (z_1(x), \dots, z_r(x))$ embeds V in $\mathbb{C}^r$. This is a $\mathbb{C}^*$–equivariant embedding for the $\mathbb{C}^*$–action on $\mathbb{C}^r$ given by $z(z_1, \dots, z_r) = (z^{v_1} z_1, \dots, z^{v_r} z_r)$

Consider the subset $V_0 := \{x \in V \mid z_r(x) = 1\}$ of V. This is a nonsingular complex curve.

Lemma 2.1. *Suppose (V, p) is not a homogeneous cyclic quotient singularity. Then for any component V_0' of V_0 the map $\pi_1(V_0') \to \pi_1(V - \{p\})$ is non-trivial.*

Proof. Denote $v = lcm(v_1, \dots, v_r)$. A convenient version of the link of the singularity is given by

$$M = S \cap V \quad \text{with} \quad S = \{z \in \mathbb{C}^r \mid |z_1|^{2v/v_1} + \dots + |z_r|^{2v/v_r} = 1\}\,.$$

The action of $S^1 \subset \mathbb{C}^*$ restricts to a fixed-point free action on M. If we denote the quotient $M/S^1 = (V - \{p\})/\mathbb{C}^*$ by P then the orbit map $M \to P$ is a Seifert fibration, so P has the structure of an orbifold. The orbit map induces a surjection of $\pi_1(V - \{p\}) = \pi_1(M)$ to the orbifold fundamental group $\pi_1^{orb}(P)$ (see eg [5, 6]) so the lemma will follow if we show the image of $\pi_1(V_o')$ in $\pi_1^{orb}(P)$ is nontrivial.

Denote $V_r := \{z \in V \mid z_r \neq 0\}$ and $P_r := \{[z] \in P \mid z_r \neq 0\}$ and $\pi \colon V \to P$ the projection. Each generic orbit of the $\mathbb{C}^*$–action on V_r meets V_0 in v_r points; in fact the $\mathbb{C}^*$–action on V_r restricts to an action of μ_{v_r} (the v_r–th roots of unity) on V_0, and $V_0/\mu_{v_r} = V_r/\mathbb{C}^* = P_r$. Thus $V_0 \to P_r$ is a cyclic cover of orbifolds, so the same is true for any component V_0' of V_0. Thus $\pi_1(V_0') \to \pi_1^{orb}(P_r)$ maps $\pi_1(V_0')$ injectively to a normal subgroup with cyclic quotient. On the other hand $\pi_1^{orb}(P_r) \to \pi_1^{orb}(P)$ is surjective, since P_r is the complement of a finite set of points in P. Hence, the image of $\pi_1(V_0')$ in $\pi_1^{orb}(P)$ is a normal subgroup with cyclic quotient. Thus the lemma follows if $\pi_1^{orb}(P)$ is not cyclic.

If $\pi_1^{orb}(P)$ is cyclic then P is a 2–sphere with at most two orbifold points, so the link M must be a lens space, so (V, p) is a cyclic quotient singularity, say $V = \mathbb{C}^2/\mu_n$. Here μ_n acts on $\mathbb{C}^2$ by $\xi(u_1, u_2) = (\xi^q u_1, \xi u_2)$ with $\xi = e^{2\pi i/n}$, for some $0 < q < n$ with q prime to n.

Recall that we are using the diagonal $\mathbb{C}^*$–action. The base orbifold is then $(\mathbb{C}^2/\mu_n)/\mathbb{C}^* = (\mathbb{C}^2/\mathbb{C}^*)/\mu_n = P^1\mathbb{C}/\mu_n$. Note that μ_n may not act effectively on

$P^1\mathbb{C}$; the kernel of the action is

$$\begin{aligned}\mu_n \cap \mathbb{C}^* &= \{(\xi^{qa}, \xi^a\ ) \mid \xi^{qa} = \xi^a\} \\ &= \{(\xi^{qa}, \xi^a\ ) \mid \xi^{(q-1)a} = 1\} \\ &= \mu_d \quad \text{with } d = \gcd(q-1, n)\,.\end{aligned}$$

So the actual action is by a cyclic group of order $n' := n/d$ and the orbifold P is $P^1\mathbb{C}/(\mathbb{Z}/n')$, which is a 2–sphere with two degree n' cone points.

The ring of functions on V is the ring of invariants for the action of μ_n on $\mathbb{C}^2$, which is generated by functions of the form $u_1^a u_2^b$ with $qa + b \equiv 0 \pmod n$. The minimal set of generators is included in the set consisting of u_1^n, u_2^n, and all $u_1^a u_2^b$ with $qa + b \equiv 0 \pmod n$ and $0 < a, b < n$. If $q = 1$ these are the elements $u_1^a u_2^{n-a}$ which all have equal weight, and (V, p) is homogeneous and a cone; this case is excluded by our assumptions. If $q \neq 1$ then a generator of least weight will be some $u_1^a u_2^b$ with $a + b < n$. Then V_0 is the subset of V given by the quotient of the set $\bar V_0 = \{(u_1, u_2) \in \mathbb{C}^2 \mid u_1^a u_2^b = 1\}$ by the μ_n–action. Each fiber of the $\mathbb{C}^*$–action on $\mathbb{C}^2$ intersects $\bar V_0$ in exactly $a + b$ points, so the composition $\bar V_0 \to \mathbb{C}^2 - \{0\} \to P^1\mathbb{C}$ induces an $(a + b)$–fold covering $\bar V_0 \to P^1\mathbb{C} - \{0, \infty\}$. Note that $d = \gcd(q - 1, n)$ divides $a + b$ since $a + b = (qa + b) - (q - 1)a = nc - (q - 1)a$ for some c. Hence the subgroup $\mu_d = \mu_n \cap \mathbb{C}^*$ is in the covering transformation group of the above covering, so the covering $V_0 \to P_0$ obtained by quotienting by the μ_n–action has degree at most $(a + b)/d$. Restricting to a component V_0' of V_0 gives us possibly smaller degree. Since $(a + b)/d < n/d = n'$, the image of $\pi_1(V_0')$ in $\pi_1(P) = \mathbb{Z}/n'$ is non-trivial, completing the proof. ☐

Proof of Theorem 1. Assume $v_{r-1}/v_r > 1$. By Lemma 2.1 we can find a closed curve γ in V_0 which represents a non-trivial element of $\pi_1(V - \{p\})$. Suppose we have a bi-Lipschitz homeomorphism h from a neighborhood of p in V to a neighborhood in the cone CM. Using the $\mathbb{R}_+^*$–action on V, choose $\epsilon > 0$ small enough that $t\gamma$ is in the given neighborhood of p for $0 < t \le \epsilon$.

Consider the map H of $[0, 1] \times (0, \epsilon]$ to V given by $H(s, t) = t^{-v_r} h(t\gamma(s))$. Here $t\gamma(s)$ refers to the $\mathbb{R}_+^*$–action on V, and $t^{-v_r} h(v)$ refers to the $\mathbb{R}_+^*$–action on CM. Note that the coordinate z_r is constant equal to t^{v_r} on each $t\gamma$ and the other coordinates have been multiplied by at most $t^{v_{r-1}}$. Hence, for each t the curve $t\gamma$ is a closed curve of length of order bounded by $t^{v_{r-1}}$, so $h(t\gamma)$ has length of the same order, so $t^{-v_r} h(t\gamma)$ has length of order $t^{v_{r-1}-v_r}$. This length approaches zero as $t \to 0$, so H extends to a continuous map $H' \colon [0, 1] \times [0, \epsilon] \to V$ for which $H([0, 1] \times \{0\})$ is a point. Note that $t^{-v_r} h(t\gamma)$ is never closer to p than distance $1/K$, where K is the bi-Lipschitz constant of h, so the same is true for the image of H'. Thus H' is a null-homotopy of $\epsilon\gamma$ in $V - \{p\}$, contradicting the fact that γ was homotopically nontrivial. ☐

Proof of Theorem 2. Suppose (V, p) is a non-homogeneous cyclic quotient singularity, as in the proof of Lemma 2.1 and suppose Theorem 1 does not apply, so the two lowest weights are equal (in the notation of that proof this happens, for example, if $n = 4k$ and $q = 2k + 1$ for some $k > 1$: the generators of the ring of functions of lowest weight are $u_1 u_2^{2k-1}, u_1^3 u_2^{2k-3}, \dots$, of weight $2k$). Let $u_1^a u_2^b$ be the generator of lowest weight that has smallest u_1–exponent and choose this one to be the coordinate z_r in the notation of Lemma 2.1. Consider now the C^*–action induced by the action $t(u_1, u_2) = (t^\alpha u_1, t^\beta u_2)$ on $\mathbb{C}^2$ for some pairwise prime pair of positive integers

$\alpha > \beta$. With respect to this $\mathbb{C}^*$–action the weight $\alpha a' + \beta b'$ of any generator $u_1^{a'} u_2^{b'}$ with $a' > a$ will be greater that the weight $\alpha a + \beta b$ of z_r (since $a' + b' \geq a + b$, which implies $\alpha a' + \beta b' = \alpha a + \alpha(a' - a) + \beta b' > \alpha a + \beta(a' - a) + \beta b' \geq \alpha a + \beta b$). On the other hand, any generator $u_1^{a'} u_2^{b'}$ with $a' < a$ had $a' + b' > a + b$ by our choice of z_r, and if α/β is chosen close enough to 1 we will still have $\alpha a' + \beta b' > \alpha a + \beta b$, so it will still have larger weight than z_r. Thus z_r is then the unique generator of lowest weight, so we can carry out the proof of Theorem 1 using this $\mathbb{C}^*$–action to prove non-conicalness of the singularity. □

Proof Theorem 3. Let $h\colon (V, p) \to (W, q)$ be a K–bi-Lipschitz homeomorphism. Let us suppose that $\frac{v_{r-1}}{v_r} > \frac{w_1}{w_s}$. Let γ be a loop in V_0 representing a non-trivial element of $\pi_1(V - \{p\})$ (see Lemma 2.1). We choose ϵ as in the previous proof. For $t \in (0, \epsilon]$ consider the curve $t\gamma$, where $t\gamma$ refers to $\mathbb{R}_+^*$-action on V. Its length $l(t\gamma)$, considered as a function of t, has the order bounded by $t^{v_{r-1}}$. The distance of the curve $t\gamma$ from p is of order t^{v_r}. Since h is a bi-Lipschitz map, we obtain the same estimates for $h(t\gamma)$. Since the smallest weight for W is w_s, the curve $t^{-v_r/w_s} h(t\gamma)$ will be distance at least $1/K$ from p. Moreover its length will be of order at most $t^{-w_1 v_r/w_s} l(t\gamma)$ which is of order $t^{v_{r-1} - w_1 v_r/w_s}$. This approaches zero as $t \to 0$ so, as in the previous proof, we get a contradiction to the non-triviality of $[\gamma] \in \pi_1(V - \{p\}) = \pi_1(W - \{q\})$. By exchanging the roles of V and W we see that $\frac{w_{s-1}}{w_s} > \frac{v_1}{v_r}$ also leads to a contradiction. □

References

[1] Lev Birbrair and Alexandre Fernandes, Metric geometry of complex algebraic surfaces with isolated singularities, Preprint 2006.

[2] Lev Birbrair, Alexandre Fernandes, and Walter Neumann, Conical quasihomogeneous complex surface singularities, in preparation

[3] JP Brasselet, M Goresky, R MacPherson, Simplicial differential forms with poles. Amer. J. Math. **113** (1991), 1019–1052.

[4] Walter Neumann and Mark Jankins, *Lectures on Seifert manifolds.* Brandeis Lecture Notes, 2. Brandeis University, Waltham, MA, 1983.

[5] Walter D Neumann, Frank Raymond, Seifert manifolds, plumbing, -invariant and orientation reversing maps. *Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977)* Lecture Notes in Math., **664** (Springer, Berlin, 1978), 163–196

[6] Peter Scott, The geometries of 3-manifolds. Bull. London Math. Soc. **15** (1983), 401–487.

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