

μ -CONSTANCY DOES NOT IMPLY CONSTANT BI-LIPSCHITZ TYPE

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ABSTRACT. We show that a family of isolated complex hypersurface singularities with constant Milnor number may fail, in the strongest sense, to have constant bi-Lipschitz type. Our example is the Briançon–Speder family $X_t := \{(x, y, z) \in \mathbb{C}^3 \mid x^5 + z^{15} + y^7 z + txy^6 = 0\}$ of normal complex surface germs; we show the germ $(X_0, 0)$ is not bi-Lipschitz homeomorphic with respect to the inner metric to the germ $(X_t, 0)$ for $t \neq 0$.

1. INTRODUCTION

Given a germ (X, p) of a point of a complex analytic set, a choice of generators x_1, \dots, x_N of its local ring gives an embedding of (X, p) into $(\mathbb{C}^N, 0)$. It then carries two induced metric space structures: the “outer metric” induced from distance in \mathbb{C}^N and the “inner metric” induced by arc-length of curves on X . In the Lipschitz category each of these metrics is independent of choice of embedding: different choices give metrics for which the identity map is a bi-Lipschitz homeomorphism. These metric structures have so far seen much more study in real algebraic geometry than in the complex algebraic world. The inner metric, which is given by a Riemannian metric off the singular set, is the one that interests us most here. It is determined by the outer metric, so germs that are distinguished by their inner metrics are certainly distinguished by their outer ones.

It is easy to see that two complex curve germs with the same number of components are bi-Lipschitz equivalent (inner metric). So for curve germs bi-Lipschitz geometry is equivalent to topology. This is even so for outer bi-Lipschitz geometry of plane curves: two germs of algebraic curves in \mathbb{C}^2 are bi-Lipschitz homeomorphic for the outer metric if and only if they are topologically equivalent as embedded germs (Fernandes [9])

We show here that for complex surface germs the picture is very different. Our main result (announced in [3]) is that in the Briançon–Speder family [5]

$$X_t := \{(x, y, z) \in \mathbb{C}^3 \mid x^5 + z^{15} + y^7 z + txy^6 = 0\}$$

of germs of algebraic complex surfaces in \mathbb{C}^3 , which are of constant embedded topological type, X_0 and X_t for $t \neq 0$ are not bi-Lipschitz homeomorphic, even for the inner metric. In particular, μ -constancy does not imply bi-Lipschitz equisingularity.

The criterion we use is the existence of so-called *separating sets*. A separating set of a complex surface germ (X, p) is a “thin” 3-dimensional subset Y through p which separates X into two “fat” subsets (precise definitions are below). The existence or non-existence of separating sets is a bi-Lipschitz invariant, and it turns out that the existence of a separating set is a fairly common thing in the geometry

Key words and phrases. bi-Lipschitz, complex surface singularity.

of complex algebraic surfaces. For example, A_k -singularities were shown to have separating sets for $k > 1$ and odd in [1] (Theorem 3.1 below is more general). We prove that in the Briançon-Speder family X_t has a separating set when $t \neq 0$ and does not have a separating set when $t = 0$.

Acknowledgements. The authors acknowledge research support under the grants: CNPq grant no 300985/93-2 (Lev Birbrair), CNPq grant no 300393/2005-9 (Alexandre Fernandes) and NSF grant no. DMS-0456227 (Walter Neumann). They are grateful for useful correspondence with Bruce Kleiner and Frank Morgan.

2. SEPARATING SETS

Let $V \subset \mathbb{R}^n$ be a k -dimensional rectifiable subset. Recall that the inferior and superior k -densities of V at the point $x_0 \in \mathbb{R}^n$ are defined by:

$$\underline{\Theta}^k(V, x_0) = \liminf_{\epsilon \rightarrow 0^+} \frac{\mathcal{H}^k(V \cap \epsilon B(x_0))}{\eta \epsilon^k}$$

and

$$\overline{\Theta}^k(V, x_0) = \limsup_{\epsilon \rightarrow 0^+} \frac{\mathcal{H}^k(V \cap \epsilon B(x_0))}{\eta \epsilon^k},$$

where $\epsilon B(x_0)$ is the n -dimensional ball of radius ϵ centered at x_0 , η is the volume of the k -dimensional unit ball and \mathcal{H}^k is k -dimensional Hausdorff measure in \mathbb{R}^n . If

$$\underline{\Theta}^k(V, x_0) = \theta = \overline{\Theta}^k(V, x_0),$$

then θ is called the k -dimensional density of V at x_0 (or simply k -density at x_0).

Remark 2.1. Recall [8] that if $V \subset \mathbb{R}^n$ is a semialgebraic subset, then the above two limits are equal and the k -density of V is well defined for any point of \mathbb{R}^n .

Definition 2.2. Let $X \subset \mathbb{R}^n$ be a k -dimensional semialgebraic set and let $x_0 \in X$ be a point such that the k -density of X at x_0 is positive. A $(k-1)$ -dimensional rectifiable subset $Y \subset X$ such that $x_0 \in Y$ is called a *separating set of X at x_0* if (see Fig. 1)

- for some small $\epsilon > 0$ the subset $(\epsilon B(x_0) \cap X) \setminus Y$ has at least two connected components A and B ,
- the $(k-1)$ -density of Y at x_0 is zero,
- the inferior k -densities of A and B at x_0 are nonzero.

Proposition 2.3 (Lipschitz invariance of separating sets). *Let X and Z be two real semialgebraic sets. If there exists a bi-Lipschitz homeomorphism of germs $F: (X, x_0) \rightarrow (Z, z_0)$ with respect to the inner metric, then X has a separating set at $x_0 \in X$ if and only if Z has a separating set at $z_0 \in Z$.*

Proof. The result would be immediate if separating sets were defined in terms of the inner metrics on X and Z . So we must show that separating sets can be defined this way.

Let $X \subset \mathbb{R}^n$ be a connected semialgebraic subset. Consider the set X equipped with the inner metric and with the Hausdorff measure \mathcal{H}_X^k associated to this metric. Let $V \subset X$ be a k -dimensional rectifiable subset. We define the inner inferior and superior densities of V at $x_0 \in X$ with respect to inner metric on X as follows:

$$\underline{\Theta}^k(X, V, x_0) = \liminf_{\epsilon \rightarrow 0^+} \frac{\mathcal{H}_X^k(V \cap \epsilon B_X(x_0))}{\eta \epsilon^k}$$

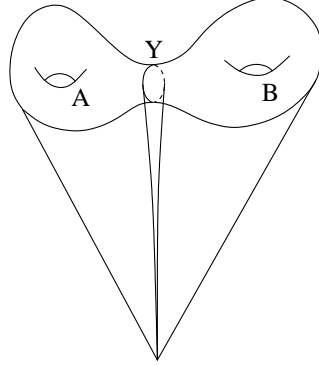


FIGURE 1. Separating set

and

$$\overline{\Theta}^k(X, V, x_0) = \lim_{\epsilon \rightarrow 0^+} \sup \frac{\mathcal{H}_X^k(V \cap \epsilon B_X(x_0))}{\eta \epsilon^k},$$

where $\epsilon B_X(x_0)$ denotes the closed ball in X (with respect to the inner metric) of radius ϵ centered at x_0 . The fact that separating sets can be defined using the inner metric now follows from the following proposition, completing the proof. \square

Proposition 2.4. *Let $X \subset \mathbb{R}^n$ be a semialgebraic connected subset. Let $V \subset X$ be a k -dimensional rectifiable subset and $x_0 \in X$. Then, there exist two positive constants κ_1 and κ_2 such that:*

$$\kappa_1 \underline{\Theta}^k(X, V, x_0) \leq \underline{\Theta}^k(V, x_0) \leq \kappa_2 \underline{\Theta}^k(X, V, x_0)$$

and

$$\kappa_1 \overline{\Theta}^k(X, V, x_0) \leq \overline{\Theta}^k(V, x_0) \leq \kappa_2 \overline{\Theta}^k(X, V, x_0).$$

Proof. This follows immediately from the Kurdyka’s “Pancake Theorem” ([11], [4]) which says that if $X \subset \mathbb{R}^n$ is a semialgebraic subset then there exists a finite semialgebraic partition $X = \bigcup_{i=1}^l X_i$ such that each X_i is a semialgebraic connected set whose inner metric and Euclidean metric are bi-Lipschitz equivalent. \square

The following proposition shows, among other things, that the germ of an isolated complex singularity which has a separating set cannot be *metrically conical*, i.e., bi-Lipschitz homeomorphic to the metric cone on its link.

Proposition 2.5. *Let (X, x_0) be a $(n + 1)$ -dimensional metric cone whose base is a compact connected Lipschitz manifold (possibly with boundary). Then, X does not have a separating set at x_0 .*

Proof. Let M be an n -dimensional compact connected Lipschitz manifold with boundary. For convenience of exposition we will suppose that M is a subset of the Euclidean sphere $S^{k-1} \in \mathbb{R}^k$ centered at 0 and with radius 1 and X the cone over M with vertex at the origin $0 \in \mathbb{R}^k$. Suppose that $Y \subset X$ is a separating set, so $X \setminus Y = A \cup B$ with A and B open in $X \setminus Y$; the n -density of Y at 0 is equal to zero and the inferior $(n + 1)$ -densities of A and B at 0 are unequal to zero. In particular, there exists $\xi > 0$ such that these inferior densities of A and B at 0 are bigger than ξ . For each $t > 0$, let $\rho_t: X \cap tD^k \rightarrow X$ be the map $\rho_t(x) = \frac{1}{t}x$, where tD^k is the

ball about $0 \in \mathbb{R}^k$ of radius t . Denote $Y_t = \rho_t(Y \cap tD^k)$, $A_t = \rho_t(A \cap tD^k)$ and $B_t = \rho_t(B \cap tD^k)$. Since the n -density of Y at 0 is equal to zero, we have:

$$\lim_{t \rightarrow 0^+} \mathcal{H}^n(Y_t) = 0.$$

Also, since the inferior densities of A and B at 0 are bigger than ξ , we have that $\mathcal{H}^{n+1}(A_t) > \xi$ and $\mathcal{H}^{n+1}(B_t) > \xi$ for all sufficiently small $t > 0$.

Let r be a radius such that $X \cap rD^k$ has volume $\leq \xi/2$ and denote by X' , A'_t , B'_t , Y'_t the result of removing from X , A_t , B_t , Y_t the intersection with the interior of the ball rB^k . Then X' is a Lipschitz $(n+1)$ -manifold (with boundary), A'_t and B'_t subsets of $(n+1)$ -measure $> \xi/2$ separated by Y'_t of arbitrarily small n -measure.

The following lemma then gives the contradiction to complete the proof. \square

Lemma 2.6. *Let X' be a $(n+1)$ -dimensional compact connected Lipschitz manifold with boundary. Then, for any $\xi > 0$ there exists $\epsilon > 0$ such that if $Y' \subset X'$ is a n -dimensional rectifiable subset with $\mathcal{H}^n(Y') < \epsilon$, then $X' \setminus Y'$ has a connected component A of $(n+1)$ -measure exceeding $\mathcal{H}^{n+1}(X') - \xi/2$ (so any remaining components have total measure $< \xi/2$).*

Proof. If X' is bi-Lipschitz homeomorphic to a ball then this follows from standard isoperimetric results: for a ball the isoperimetric problem is solved by spherical caps normal to the boundary (Burago and Maz'ja [7] p. 54, see also Hutchins [10]). Since the isoperimetric problem is generally formulated in terms of currents, one needs also that the mass of the current boundary of a region is less than or equal to the Hausdorff measure of the topological boundary ([8] 4.5.6 or [13] Section 12.2).

Let $\{T_i\}_{i=1}^m$ be a cover of X' by subsets which are bi-Lipschitz homeomorphic to balls and such that

$$T_i \cap T_j \neq \emptyset \Rightarrow \mathcal{H}^{n+1}(T_i \cap T_j) > 0.$$

Without loss of generality we may assume

$$\xi/m < \min\{\mathcal{H}^{n+1}(T_i \cap T_j) \mid T_i \cap T_j \neq \emptyset\}.$$

Since T_i is bi-Lipschitz homeomorphic to a ball there exists ϵ_i satisfying the conclusion of this lemma for ξ/m . Let $\epsilon = \min(\epsilon_1, \dots, \epsilon_m)$. So if $Y' \subset X'$ is an n -dimensional rectifiable subset such that $\mathcal{H}^n(Y') < \epsilon$, then for each i the largest component A_i of $T_i \setminus Y'$ has complement B_i of measure $< \xi/2m$.

We claim $\bigcup_{i=1}^m A_i$ is connected. It suffices to show that

$$T_i \cap T_j \neq \emptyset \Rightarrow A_i \cap A_j \neq \emptyset.$$

So suppose $T_i \cap T_j \neq \emptyset$. Then $B_i \cup B_j$ has measure less than ξ , which is less than $\mathcal{H}^n(T_i \cap T_j)$, so $T_i \cap T_j \not\subset B_i \cup B_j$. This is equivalent to $A_i \cap A_j \neq \emptyset$.

Thus there exists a connected component A of $X' \setminus Y'$ which contains $\bigcup_{i=1}^m A_i$. Its complement B is a subset of $\bigcup_{i=1}^m B_i$ and thus has measure less than $\xi/2$. \square

3. SEPARATING SETS IN NORMAL SURFACE SINGULARITIES

Theorem 3.1. *Let $X \subset \mathbb{C}^3$ be a weighted homogeneous algebraic surface with respect to the weights $w_1 \geq w_2 > w_3$ and with an isolated singularity at 0. If $(X \setminus \{0\}) \cap \{z = 0\}$ is not connected, then X has a separating set at 0.*

Example 3.2. This theorem applies to the Brieskorn singularity

$$X(p, q, r) := \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\}$$

if $p \leq q < r$ and $\gcd(p, q) > 1$. In particular it is not metrically conical. This was known for a different reason by [2]: a weighted homogeneous surface singularity (not necessarily hypersurface) whose two lowest weights are distinct is not metrically conical.

Proof of Theorem 3.1. Take ϵ small enough that the intersection $V \cap \epsilon S^5$ is transverse and gives the singularity link. Let $\tilde{A}, \tilde{B} \subset (V \cap \epsilon S^5) \cap \{z = 0\}$ be two semialgebraic closed subsets such that $\tilde{A} \cap \tilde{B} = \emptyset$. Let \tilde{M} be the conflict set of \tilde{A} and \tilde{B} on ϵS^5 , i.e.,

$$\tilde{M} = \{p \in \epsilon S^5 \mid d(p, \tilde{A}) = d(p, \tilde{B})\},$$

where $d(\cdot, \cdot)$ is the standard metric on ϵS^5 (euclidean metric in \mathbb{C}^3 gives the same set). Clearly, \tilde{M} is a compact semialgebraic subset and there exists $\delta > 0$ such that $d(\tilde{M}, \{z = 0\}) > \delta$. Let $M = \mathbb{C}^* \tilde{M} \cup \{0\}$ (the closure of the union of \mathbb{C}^* -orbits through \tilde{M}). Note that the \mathbb{C}^* -action restricts to a unitary action of S^1 , so the construction of \tilde{M} is invariant under the S^1 -action, so $M = \mathbb{R}^* \tilde{M}$, and is therefore 3-dimensional. It is semi-algebraic by the Tarski-Seidenberg theorem. We will use the weighted homogeneous property of M to show $\dim(T_0 M) \leq 2$, where $T_0 M$ denotes the tangent cone of M at 0, from which will follow that M has zero 3-density. In fact, we will show that $T_0 M \subset \{x = 0, y = 0\}$.

Let $T: \tilde{M} \times [0, +\infty) \rightarrow M$ be defined by:

$$T((x, y, z), t) = (t^{\frac{w_1}{w_3}} x, t^{\frac{w_2}{w_3}} y, tz).$$

Clearly, the restriction $T|_{\tilde{M} \times (0, +\infty)}: \tilde{M} \times (0, +\infty) \rightarrow M \setminus \{0\}$ is a bijective semi-algebraic map. Let $\gamma: [0, \epsilon) \rightarrow M$ be a semianalytic arc; $\gamma(0) = 0$ and $\gamma'(0) \neq 0$. We consider $\phi(s) = T^{-1}(\gamma(s))$ for all $s \neq 0$. Since ϕ is a semialgebraic map and M is compact, $\lim_{s \rightarrow 0} \phi(s)$ exists and belongs to $M \times \{0\}$. For the same reason, $\lim_{s \rightarrow 0} \phi'(s)$ also exists and is nonzero. Therefore, the arc ϕ can be extended to $\phi: [0, \epsilon) \rightarrow \tilde{M} \times [0, +\infty)$ such that $\phi(0) \in \tilde{M} \times \{0\}$ and $\phi'(0)$ exists and is nonzero. We can take the $[0, \infty)$ component of ϕ as parameter and write $\phi(t) = ((x(t), y(t), z(t), t)$. Then $\gamma(t) = (t^{w_1/w_3} x(t), t^{w_2/w_3} y(t), tz(t))$, so

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{\gamma(t)}{t} &= \left(\lim_{t \rightarrow 0} \frac{t^{\frac{w_1}{w_3}}}{t} x(t), \lim_{t \rightarrow 0} \frac{t^{\frac{w_2}{w_3}}}{t} y(t), \lim_{t \rightarrow 0} z(t) \right) \\ &= (0, 0, z(0)). \end{aligned}$$

This is a nonzero vector in the set $\{x = 0, y = 0\}$, so we obtain that

$$T_0 \subset \{x = 0, y = 0\}.$$

Since M is a 3-dimensional semialgebraic set and $\dim(T_0 M) \leq 2$, we obtain that the 3-dimensional density of M at 0 is equal to zero ([12]).

Now, we have the following decomposition:

$$V \setminus M = A \cup B,$$

where $\tilde{A} \subset A$, $\tilde{B} \subset B$, A and B are \mathbb{C}^* -invariant and $A \cap B = \emptyset$. Since A and B are semialgebraic sets, the 4-densities $\text{density}_4(A, 0)$ and $\text{density}_4(B, 0)$ are defined. We will show that these densities are nonzero. It is enough to prove that

$\dim_{\mathbb{R}}(T_0A) = 4$ and $\dim_{\mathbb{R}}(T_0B) = 4$. Let $\Gamma \subset A$ be a connected component of $A \cap \{z = 0\}$. Note that $\bar{\Gamma} = \Gamma \cup \{0\}$ is a complex algebraic curve. We will show that T_0A contains the set $\{(x, y, v) \mid (x, y, 0) \in \bar{\Gamma}, v \in \mathbb{C}\}$ if $w_1 = w_2$ (note that $\bar{\Gamma}$ is the line through $(x, y, 0)$ in this case) or either the y - z or the x - z plane if $w_1 < w_2$.

Given a smooth point $(x, y, 0) \in \Gamma$ and $v \in \mathbb{C}$, we may choose a smooth arc $\gamma: [0, \epsilon) \rightarrow A$ of the form $\gamma(t) = (\gamma_1(t), \gamma_2(t), t^m \gamma_3(t))$ with $(\gamma_1(0), \gamma_2(0)) = (x, y)$ and $\gamma_3(0) = v$. Then, using the \mathbb{R}^* -action, we transform this arc to the arc $\phi(t) = t^j \gamma(t)$ with j chosen so $jw_3 + m = jw_2$. Now $\phi(t) = (t^{jw_1} \gamma_1(t), t^{jw_2} \gamma_2(t), t^{jw_2} \gamma_3(t))$ is a path in A starting at the origin. Its tangent vector ρ at $t = 0$,

$$\rho = \lim_{t \rightarrow 0^+} \frac{\phi(t)}{t^{jw_2}},$$

is $\rho = (x, y, v)$ if $w_1 = w_2$ and $\rho = (0, y, v)$ if $w_1 > w_2$. If $w_1 > w_2$ and $y = 0$ then the same argument, but with j chosen with $jw_3 + m = jw_1$, gives $\rho = (x, 0, v)$. This proves our claim and completes the proof that T_0A has real dimension 4. The proof for T_0B is the same. \square

4. THE BRIANÇON-SPEDER EXAMPLE

For each $t \in \mathbb{C}$, let $X_t = \{(x, y, z) \in \mathbb{C}^3 \mid x^5 + z^{15} + y^7 z + txy^6 = 0\}$. This X_t is weighted homogeneous with respect to weights $(3, 2, 1)$ and has an isolated singularity at $0 \in \mathbb{C}^3$.

Theorem 4.1. *X_t has a separating set at 0 if $t \neq 0$ but does not have a separating set at 0 if $t = 0$.*

Proof. Note that for $t \neq 0$ Theorem 3.1 applies, so X_t has a separating set. So from now on we take $t = 0$. Denote $X := X_0$. In the following, for each sufficiently small $\epsilon > 0$, we use the notation

$$X^\epsilon = \{(x, y, z) \in X \mid \epsilon|y| \leq |z| \leq \frac{1}{\epsilon}|y|\}.$$

We need a lemma.

Lemma 4.2. *X^ϵ is metrically conical at the origin with connected link.*

Proof. Note that the lemma makes a statement about the germ of X^ϵ at the origin. We will restrict to the part of X^ϵ that lies in a suitable closed neighborhood of the origin.

Let $P: \mathbb{C}^3 \rightarrow \mathbb{C}^2$ be the orthogonal projection $P(x, y, z) = (y, z)$. The restriction P_X of P to X is a 5-fold cyclic branched covering map branched along $\{(y, z) \mid z^{15} + y^7 z = 0\}$. This is the union of the y -axis in \mathbb{C}^2 and the seven curves $y = \zeta z^2$ for ζ a 7-th root of unity. These seven curves are tangent to the z -axis.

Let

$$C^\epsilon = \{(y, z) \in \mathbb{C}^2 \mid \epsilon|y| \leq |z| \leq \frac{1}{\epsilon}|y|\}.$$

Notice that no part of the branch locus of P_X with $|z| < \epsilon$ is in C^ϵ . In particular, if D is a disk in \mathbb{C}^2 of radius $< \epsilon$ around 0, then the map P_X restricted to X^ϵ has no branching over this disk. We choose the radius of D to be $\epsilon/2$ and denote by Y the part of part of X^ϵ whose image lies inside this disk. Then Y is a covering of $C^\epsilon \cap D$, and to complete the proof of the lemma we must show it is a connected covering space and that the covering map is bi-Lipschitz.

Since it is a Galois covering with group $\mathbb{Z}/5$, to show it is a connected cover it suffices to show that there is a closed curve in $C^\epsilon \cap D$ which does not lift to a closed curve in Y . Choose a small constant $c \leq \epsilon/4$ and consider the curve $\gamma: [0, 1] \rightarrow C^\epsilon \cap D$ given by $\gamma(t) = (ce^{2\pi it}, c)$. A lift to Y has x -coordinate $(c^{15} + c^8 e^{14\pi it})^{1/5}$, which starts close to $c^{8/5}$ (at $t = 0$) and ends close to $c^{8/5} e^{(14/5)\pi i}$ (at $t = 1$), so it is not a closed curve.

To show that the covering map is bi-Lipschitz, we note that locally Y is the graph of the implicit function $(y, z) \mapsto x$ given by the equation $x^5 + z^{15} + y^7 z = 0$, so it suffices to show that the derivatives of this implicit function are bounded. Implicit differentiation gives

$$\frac{\partial x}{\partial y} = -\frac{7y^6 z}{5x^4}, \quad \frac{\partial x}{\partial z} = -\frac{15z^{14} + y^7}{5x^4}.$$

It is easy to see that there exists $\lambda > 0$ such that

$$|15z^{14} + y^7| \leq \lambda|z|^4, \quad |y^7| \leq \lambda|z^{14} + y^7| \quad \text{and} \quad |15z^{14} + y^7| \leq \lambda|z^{14} + y^7|,$$

for all $(y, z) \in C^\epsilon \cap D$. We then get

$$\left| \frac{\partial x}{\partial y} \right|^5 = \frac{7^5 |y^{30} z^5|}{5^5 |z^{14} + y^7|^4 |z|^4} \leq \frac{7^5}{5^5} \lambda^4 |y^2 z| < \frac{7^5 \lambda^4 \epsilon^3}{5^5 2^3}$$

and

$$\left| \frac{\partial x}{\partial z} \right|^5 = \frac{|15z^{14} + y^7|^5}{5^5 |z^{14} + y^7|^4 |z|^4} \leq \frac{\lambda^5}{5^5},$$

completing the proof. \square

We now complete the proof of Theorem 4.1. Let us suppose that X has a separating set. Let $A, B, Y \subset X$ be subsets satisfying:

- for some small $\epsilon > 0$ the subset $[\epsilon B(x_0) \cap X] \setminus Y$ is the union of relatively open subsets A and B ,
- the 3-dimensional density of Y at 0 is equal to zero,
- the 4-dimensional inferior densities of A and B at 0 are unequal to zero.

Set

$$N^\epsilon = \{(x, y, z) \in \mathbb{C}^3 \mid |z| \leq \epsilon|y| \text{ or } |y| \leq \epsilon|z|\}.$$

For each subset $H \subset \mathbb{C}^3$ we denote

$$H^\epsilon = H \cap [\mathbb{C}^3 \setminus N^\epsilon].$$

In this step, it is valuable to observe that there exists a positive constant K (independent of ϵ) such that

$$(1) \quad \mathcal{H}^4(X \cap N^\epsilon \cap B(0, r)) \leq K\epsilon r^4$$

for all $0 < r \leq 1$ (see, e.g., Comte-Yomdin [14], chapter 5). By definition, the 4-dimensional inferior density of A at 0 is equal to

$$\liminf_{r \rightarrow 0^+} \left(\frac{\mathcal{H}^4(A^\epsilon \cap B(0, r))}{r^4} + \frac{\mathcal{H}^4(A \cap N^\epsilon \cap B(0, r))}{r^4} \right)$$

Then, if $\epsilon > 0$ is sufficiently small, we can use inequality (1) in order to show that the 4-dimensional inferior density of A^ϵ is a positive number. In a similar way, we can show that if $\epsilon > 0$ is sufficiently small, then the 4-dimensional inferior density of B^ϵ at 0 is a positive number. These facts are enough to conclude that Y^ϵ is

a separating set of X^ϵ . But in view of Lemma 4.2 this contradicts Proposition 2.5. \square

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