

LIPSCHITZ GEOMETRY DOES NOT DETERMINE EMBEDDED TOPOLOGICAL TYPE

WALTER D NEUMANN AND ANNE PICHON

Dedicated to José Seade for a great occasion. Happy birthday, Pepe!

ABSTRACT. We investigate the relationships between the Lipschitz outer geometry and the embedded topological type of a hypersurface germ in $(\mathbb{C}^n, 0)$. It is well known that the Lipschitz outer geometry of a complex plane curve germ determines and is determined by its embedded topological type. We prove that this does not remain true in higher dimensions. Namely, we give two normal hypersurface germs $(X_1, 0)$ and $(X_2, 0)$ in $(\mathbb{C}^3, 0)$ having the same outer Lipschitz geometry and different embedded topological types. Our pair consist of two superisolated singularities whose tangent cones form an Alexander-Zariski pair having only cusp-singularities. Our result is based on a description of the Lipschitz outer geometry of a superisolated singularity. We also prove that the Lipschitz inner geometry of a superisolated singularity is completely determined by its (non embedded) topological type, or equivalently by the combinatorial type of its tangent cone.

1. INTRODUCTION

A complex germ $(X, 0)$ has two natural metrics up to bilipschitz equivalence, the *outer metric* given by embedding $(X, 0)$ in some $(\mathbb{C}^n, 0)$ and taking distance in \mathbb{C}^n and the *inner metric* given by shortest distance along paths in X .

In this paper we investigate the relationships between the Lipschitz outer geometry and the embedded topological type of a hypersurface germ in $(\mathbb{C}^n, 0)$.

It is well known that the Lipschitz outer geometry of a complex plane curve germ determines and is determined by its embedded topological type ([12], see also [5] and [9, Theorem 1.1.]). We prove that this does not remain true in higher dimensions:

Theorem 1.1. *There exist two hypersurface germs in $(\mathbb{C}^3, 0)$ having same Lipschitz outer geometry and distinct embedded topological type.*

It is worth noting that for families of isolated hypersurfaces in \mathbb{C}^3 , the constancy of Lipschitz outer geometry implies constancy of embedded topological type. Indeed, Varchenko proved in [13] that a Zariski equisingular family of hypersurfaces in any dimension has constant embedded topological type and it is proved in [10] that for a family of hypersurface singularities $(X_t, 0) \subset (\mathbb{C}^3, 0)$, Zariski equisingularity is equivalent to constant Lipschitz outer geometry.

It should also be noted that the converse question, which consists of examining which part of the outer Lipschitz geometry of a hypersurface can be recovered from

1991 *Mathematics Subject Classification.* 14B05, 32S25, 32S05, 57M99.

Key words and phrases. complex surface singularity, bilipschitz, Lipschitz geometry, embedded topological type, superisolated.

its embedded topological type seems difficult. In particular the outer geometry of a normal complex surface singularity determines its multiplicity ([10, Theorem 1.2 (2)]) so this question somehow contains the Zariski multiplicity question.

In order to prove Theorem 1.1 we construct two germs of hypersurfaces in $(\mathbb{C}^3, 0)$ having the same Lipschitz outer geometry and different embedded topological types. They consist of a pair of superisolated singularities whose tangent cones form an Alexander-Zariski pair of projective plane curves.

A surface singularity $(X, 0)$ is *superisolated* (SIS for short) if it is given by an equation

$$f_d(x, y, z) + f_{d+1}(x, y, z) + f_{d+2}(x, y, z) + \cdots = 0,$$

where $d \geq 2$, f_k is a homogeneous polynomial of degree k and the projective curve $\{f_{d+1} = 0\} \subset \mathbb{P}^2$ contains no singular point of the projective curve $C = \{[x : y : z] : f_d(x, y, z) = 0\}$. In particular, the projectivized tangent cone C of $(X, 0)$ is reduced. In the sequel we will just consider SISs with equations

$$f_d(x, y, z) + f_{d+1}(x, y, z) = 0.$$

Definition 1.2 (Combinatorial type of a projective plane curve). The *combinatorial type* of a reduced projective plane curve $C \subset \mathbb{P}^2$ is the homeomorphism type of a tubular neighborhood of it in \mathbb{P}^2 (see, e.g., [3, Remark 3]; a more combinatorial version is also given there, which we describe in Section 3).

It is well known that the combinatorial type of the projectivized tangent cone of a SIS $(X, 0)$ determines the topology of $(X, 0)$. In fact, we will show:

Theorem 1.3. (i). *The Lipschitz inner geometry of a SIS determines and is determined by the combinatorial type of its projectivized tangent cone.*

(ii). *There exist SISs with the same combinatorial types of their projectivized tangent cones but different Lipschitz outer geometry.*

Acknowledgments. We are grateful to H el ene Maugendre for fruitful conversations and for communicating to us the equations of the tangent cones for the examples in the proof of Theorem 1.3 (ii). Walter Neumann was supported by NSF grant DMS-1206760. Anne Pichon was supported by ANR-12-JS01-0002-01 SUSI. We are also grateful for the hospitality and support of the following institutions: Columbia University, Institut de Math ematiques de Marseille, Aix Marseille Universit e and CIRM Luminy, Marseille.

2. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 will need Lemma 2.2 and Proposition 2.3 below, which will be proved in section 4. First a definition:

Definition 2.1. We say that two germs $(C_1, 0)$ and $(C_2, 0)$ of reduced irreducible plane curves are *weak RL-equivalent* if for $i = 1, 2$ there are holomorphic maps $h_i: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ with $(h_i^{-1}(0), 0) = (C_i, 0)$, a homeomorphism $\psi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$, a constant $K \geq 1$ and a neighborhood \mathcal{U} of the origin in \mathbb{C}^2 such that for all $a, a' \in \mathcal{U}$.

$$\begin{aligned} \frac{1}{K} \|h_2(\psi(a))(1, \psi(a)) - h_2(\psi(a'))(1, \psi(a'))\|_{\mathbb{C}^3} &\leq \|h_1(a)(1, a) - h_1(a')(1, a')\|_{\mathbb{C}^3} \\ &\leq K \|h_2(\psi(a))(1, \psi(a)) - h_2(\psi(a'))(1, \psi(a'))\|_{\mathbb{C}^3} \end{aligned}$$

Lemma 2.2. *Weak RL-equivalence of reduced irreducible plane curve germs $(C_1, 0)$ and $(C_2, 0)$ does not depend on the choice of their defining functions h_1 and h_2 . Moreover, it is implied by analytic equivalence of $(C_1, 0)$ and $(C_2, 0)$ in the sense of Zariski [14] (also called RL-equivalence or \mathcal{A} -equivalence).*

Proposition 2.3. *Let $(X, 0)$ be a SIS with equation $f_d + f_{d+1} = 0$. The Lipschitz outer geometry of $(X, 0)$ is determined by the combinatorial type of its projectivized tangent cone and by the weak RL-equivalence classes of corresponding singularities of the projectivized tangent cones.*

Proof of Theorem 1.1. Recall that a Zariski pair is a pair of projective curves $C_1, C_2 \subset \mathbb{P}^2$ with the same combinatorial type but such that (\mathbb{P}^2, C_1) is not homeomorphic to (\mathbb{P}^2, C_2) . The first example was discovered by Zariski: a pair of sextic curves C_1 and C_2 , each with six cusps, distinguished by the fact that C_1 has the cusps lying on a quadric and C_2 does not. He constructed those of type C_1 in [14] and conjectured type C_2 , confirming their existence eight years later in [15]. He distinguished their embedded topology by the fundamental groups of their complements, but they can also be distinguished by their Alexander polynomials (Libgober [7]) so they are called *Alexander-Zariski pairs*.

Let $(X_1, 0)$ and $(X_2, 0)$ be two SISs whose tangent cones are sextics of types C_1 and C_2 as above. According to [16], the analytic type of a cusp is uniquely determined, so its weak RL-equivalence class is determined (Lemma 2.2). Then by Proposition 2.3, $(X_1, 0)$ and $(X_2, 0)$ are outer Lipschitz equivalent.

On the other hand, Artal showed that $(X_1, 0)$ and $(X_2, 0)$ do not have the same embedded topological type. In fact, he shows ([1, Theorem 1.6 (ii)]) that a Zariski pair is distinguished by its Alexander polynomials if and only if the corresponding SISs are distinguished by the Jordan block decompositions of their homological monodromies. \square

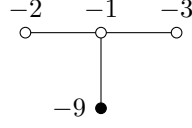
3. THE INNER GEOMETRY OF A SUPERISOLATED SINGULARITY

We first recall how the topological type of a SIS is determined by the combinatorial type of its projectivized tangent cone. We refer to [2] for details.

A SIS $(X, 0) \subset (\mathbb{C}^3, 0)$ is resolved by blowing up the origin of $(\mathbb{C}^3, 0)$. The exceptional divisor of this resolution of $(X, 0)$ is the projectivized tangent cone C of $(X, 0)$ and one obtains the minimal good resolution by blowing up the singularities of C which are not ordinary double points until one obtains a normal crossing divisor C' . Let Γ be the dual graph of this resolution. Following [4] we say \mathcal{L} -curve for a component of C' which is a component of C and \mathcal{L} -node any vertex of Γ representing an \mathcal{L} -curve.

One can also resolve the singularities of C as a projective plane curve to obtain the same graph Γ except that the self-intersection numbers of the \mathcal{L} -curves are different (in the example below the self-intersection number -9 becomes $+3$). The graph Γ with these data is equivalent to the combinatorial type of C .

Example 3.1. Consider the SIS $(X, 0) \subset (\mathbb{C}^3, 0)$ given by $F(x, y, z) = y^3 + xz^2 - x^4 = 0$. Blowing up the origin of \mathbb{C}^3 resolves the singularity: using the chart $(x, v, w) \mapsto (x, y, z) = (x, xv, xw)$, the equation of the resolved X^* is $v^3 + w^2 - x = 0$ and the exceptional curve has a cusp singularity $x = v^3 + w^2 = 0$. Blowing up further leads to the following dual graph Γ , the black vertex being the \mathcal{L} -node.



The self-intersection -9 of the \mathcal{L} -curve is computed as follows. Let E_1, \dots, E_4 be the components of the exceptional divisor indexed so that E_1 is the \mathcal{L} -curve and E_2, E_3 and E_4 correspond to the string of non \mathcal{L} -nodes indexed from left to right on the graph. Since the tangent cone is reduced with degree 3, the strict transform l_1^* of a generic linear form $l_1: (X, 0) \rightarrow (\mathbb{C}, 0)$ consists of three smooth curves transverse to E_1 . The total transform l_1 is given by the divisor:

$$(l_1) = E_1 + 3E_2 + 6E_3 + 2E_4 + l_1^*.$$

Since (l_1) is a principal divisor, we have $(l_1) \cdot E_1 = 0$, which leads to $E_1 \cdot E_1 = -9$.

Proof of Theorem 1.3 (i). Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a SIS with equation $f_d + f_{d+1} = 0$. We set $f = f_d$ and $g = -f_{d+1}$.

Let $\ell: \mathbb{C}^3 \rightarrow \mathbb{C}^2$ be a generic linear projection for $(X, 0)$, let Π be the polar curve of the restriction $\ell|_X$ and $\Delta = \ell(\Pi)$ its discriminant curve.

Let e be the blow-up of the origin of \mathbb{C}^3 and let p be a singular point of $e^{-1}(0) \cap X^*$. Without loss of generality, we can assume $\ell = (x, y)$. We can also choose our coordinates so that $p = (1, 0, 0)$ in the chart (x, v, w) given by $(x, v, w) \mapsto (x, y, z) = (x, xv, xw)$ in the blow-up e (so p corresponds to the x -axis in the tangent cone of X). Then X^* has equation

$$f(1, v, w) - xg(1, v, w) = 0$$

and $g(1, v, w)$ is a unit at p since $\{g = 0\} \cap \text{Sing}(f = 0) = \emptyset$ in \mathbb{P}^2 .

Let $e_0: Y \rightarrow \mathbb{C}^2$ be the blow-up of the origin of \mathbb{C}^2 . We consider e_0 in the chart $(x, v) \mapsto (x, y) = (x, xv)$, we set $q = (1, 0) \in Y$ in this chart, and we denote by $\tilde{\ell}: (X^*, p) \rightarrow (Y, q)$ the projection $(x, v, w) \mapsto (x, v)$. So we have the commutative diagram:

$$\begin{array}{ccc} (X^*, p) & \xrightarrow{e} & (X, 0) \\ \downarrow \tilde{\ell} & & \downarrow \ell \\ (Y, q) & \xrightarrow{e_0} & (\mathbb{C}^2, 0). \end{array}$$

Now $\Pi = X \cap \{f_z - g_z = 0\}$, so the strict transform Π^* of Π by e has equations:

$$f_w(1, v, w) - xg_w(1, v, w) = 0 \quad \text{and} \quad f(1, v, w) - xg(1, v, w) = 0,$$

which are also the equations of the polar curve of the projection $\tilde{\ell}: (X^*, p) \rightarrow (Y, q)$.

Since $g(1, v, w) \in \mathbb{C}\{v, w\}$ is a unit at p , the quotient $h(v, w) := \frac{f(1, v, w)}{g(1, v, w)}$ defines a holomorphic function germ $h: (\mathbb{C}_{(v, w)}^2, 0) \rightarrow (\mathbb{C}, 0)$. In terms of $h(v, w)$ the above equations for (Π^*, p) can be written:

$$h_w(v, w) = 0 \quad \text{and} \quad h(v, w) - x = 0.$$

Consider the isomorphism $proj: (X^*, p) \rightarrow (\mathbb{C}^2, 0)$ which is the restriction of the linear projection $(x, v, w) \mapsto (v, w)$. Then Π^* is the inverse image by $proj$ of the polar curve Π' of the morphism $\ell': (\mathbb{C}_{(v, w)}^2, 0) \rightarrow (\mathbb{C}_{(x, v)}^2, 0)$ defined by $(v, w) \mapsto (h(v, w), v)$, i.e., the relative polar curve of the map germ $(v, w) \mapsto h(v, w)$ for the generic projection $(v, w) \mapsto v$.

We set $\Delta' = \ell'(\Pi')$ and $q = (1, 0)$ in $\mathbb{C}_{(x,v)}^2$. We then have a commutative diagram:

$$\begin{array}{ccccc} (\mathbb{C}^2, \Pi', 0) & \xleftarrow{proj} & (X^*, \Pi^*, p) & \xrightarrow{e} & (X, \Pi, 0) \\ & \searrow \ell' & \downarrow \tilde{\ell} & & \downarrow \ell \\ & & (Y, \Delta', q) & \xrightarrow{e_0} & (\mathbb{C}^2, \Delta, 0) \end{array}$$

Let $(\Pi_0, 0)$ be the part of $(\Pi, 0)$ which is tangent to the x -axis (i.e., it corresponds to $p \in e^{-1}(0)$ in our chosen coordinates) and let $(\Delta_0, 0)$ be its image by ℓ . Let V be a cone around the x -axis in $(\mathbb{C}^3, 0)$. As in [4], consider a carousel decomposition of $(\ell(V), 0)$ with respect to the curve germ $(\Delta_0, 0)$ such that the Δ -wedges around Δ_0 are D-pieces. We then consider the geometric decomposition of $(V, 0)$ into A-, B- and D-pieces obtained by lifting by ℓ this decomposition. Lifting the carousel decomposition of $\ell(V)$ by e_0 we get a carousel decomposition of (Y, q) with respect to Δ' . On the other hand the lifting by e of the geometric decomposition of V is a geometric decomposition of (X^*, p) which coincides with the lifting by $\tilde{\ell}$ of the carousel decomposition of (Y, q) just defined.

By the Lê Swing Lemma [8, Lemma 2.4.7], the union of pieces beyond the first Puiseux exponents of the branches of Δ' at q lift to pieces in X^* which have trivial topology, i.e., their links are solid tori. Therefore these are absorbed by the amalgamation process consisting of amalgamating iteratively any D-piece which is not a conical piece with the neighbor piece using [4, Lemma 13.1].

Moreover, since Δ' is the strict transform of Δ by e_0 , the rate of each piece of the obtained decomposition of X^* equals $q + 1$, where q is the first Puiseux exponent of a branch of Δ' . Let Γ_p be the minimal resolution graph of the curve $h = 0$ at p . Let us call a *node* of Γ_p any vertex having at least three incident edges including the arrows representing the components of h and the root vertex of Γ_p if $h = 0$ has more than one line in its tangent cone. According to [8, Théorème C], the rate q equals the polar quotient

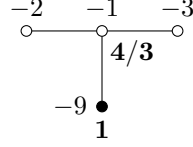
$$\frac{m_{E_i}(l)}{m_{E_i}(h)}$$

where v_i is the corresponding node in Γ_p and where $l: (\mathbb{C}_{v,w}^2, p) \rightarrow (\mathbb{C}, 0)$ is a generic linear form at p .

Now, set $\tilde{f}(v, w) = f(1, v, w)$. Since $g(1, v, w)$ is a unit at p , the curves $h = 0$ and $\tilde{f} = 0$ coincide, so $m_{E_i}(h) = m_{E_i}(\tilde{f})$. Since the strict transform of \tilde{f} coincides with the germ of \mathcal{L} -curves at p , Γ_p is a connected component of Γ minus its \mathcal{L} -nodes with free edges replaced by arrows. Therefore the rates $\frac{m_{E_i}(l)}{m_{E_i}(\tilde{f})}$, and then the inner rate of $(X, 0)$ are computed from Γ . \square

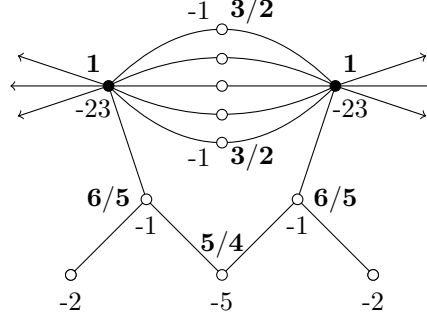
Example 3.2. Consider again the SIS $(X, 0)$ of Example 3.1 with equation $xz^2 + y^3 - x^4 = 0$. Its projectivized tangent cone $xz^2 + y^3 = 0$ has a unique singular point, and the corresponding graph Γ_p is the resolution graph of the cusp $w^2 + v^3 = 0$, i.e., the graph Γ of Example 3.1 with the \mathcal{L} -node replaced by an arrow. The multiplicity of \tilde{f} along the curve E_3 corresponding to the node of Γ_p equals 6 while that of a generic linear form $(v, w) \mapsto l(v, w)$ equals 2. We then obtain the polar quotient $\frac{m_{E_3}(l)}{m_{E_3}(\tilde{f})} = 1/3$, which gives inner rate $1/3 + 1 = 4/3$.

The Lipschitz inner geometry is then completely described (see [4, Section 15]) by the graph Γ completed by labeling its nodes by the inner rates of the corresponding geometric pieces:



Example 3.3. Consider the SIS $(X, 0)$ with equation $(zx^2 + y^3)(x^3 + zy^2) + z^7 = 0$, that we already considered in [4, Example 15.2] and in [10]. The tangent cone consists of two unicuspidal curves C and C' with 6 intersecting points p_1, \dots, p_6 , the germ $(C \cup C', p_1)$ consisting of two transversal cusps, and the remaining 5 points being ordinary double points of $C \cup C'$.

For each $i = 1, \dots, 6$, the tangent cone of $(C \cup C', p_i)$ has two tangent lines and the quotient $m_{E_{v_0}}(l)/m_{E_{v_0}}(\tilde{f})$ at the root vertex v_0 of Γ_{p_i} is then a polar quotient in the sense of [8]. The root vertex v_0 has valency 2 and it corresponds to a special annular piece in the sense of [4], with inner rate $m_{E_{v_0}}(l)/m_{E_{v_0}}(\tilde{f}) + 1$. For p_2, \dots, p_6 , we obtain inner rate $1/2 + 1 = 3/2$ for that special annular piece and for p_1 , we obtain $1/4 + 1 = 5/4$. The inner rates at the two other nodes of Γ_{p_1} both equal $2/10 + 1 = 6/5$. We have thus recovered the inner geometry:



This was also computed in [4] with the help of Maple, in terms of the carousel decomposition of the discriminant curve of a generic projection of $(X, 0)$.

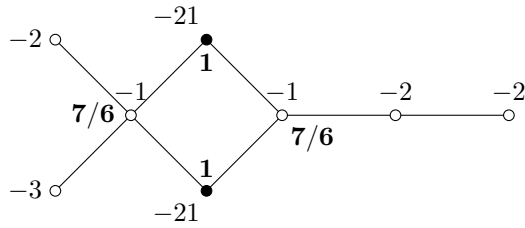
Proof of Theorem 1.3 (ii). Consider the two SISs $(X_1, 0)$ and $(X_2, 0)$ with equations respectively:

$$X_1 : F_1(x, y, z) = (y^3 - z^2x)(y^3 + z^2x) + (x + y + z)^7 = 0$$

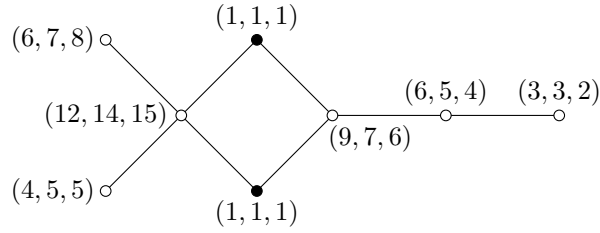
$$X_2 : F_2(x, y, z) = (y^3 - z^2x)(y^3 + 2z^2x) + (x + y + z)^7 = 0$$

We will prove that they have same inner geometry and different outer geometries.

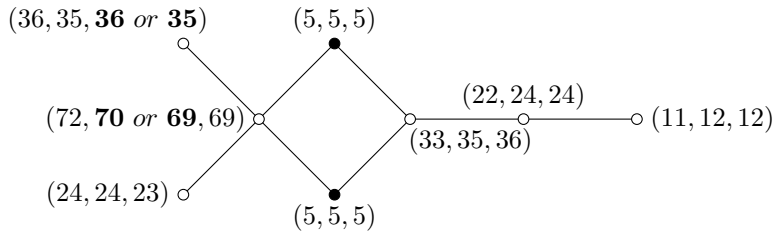
On one hand, the projectivized tangent cones of $(X_1, 0)$ and $(X_2, 0)$ have same combinatorial type, so $(X_1, 0)$ and $(X_2, 0)$ have same Lipschitz inner geometry (Theorem 1.3). The tangent cone consists of two unicuspidal components C and C' with two intersection points: one, p_1 , at the cusps, with maximal contact there, and one, p_2 , at smooth points of C and C' intersecting with contact 3 there. The inner geometry is given by the following graph. In particular, the inner rates at the two non \mathcal{L} -nodes are computed from the corresponding polar rates in the two graphs Γ_{p_1} and Γ_{p_2} . They both equal $1/6 + 1 = 7/6$.



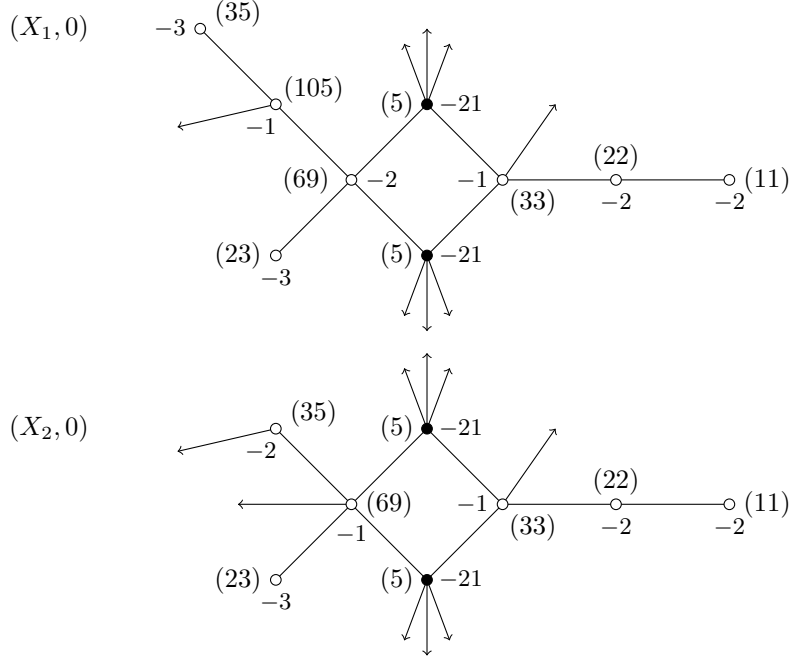
On the other hand, let us compute the multiplicities of the three functions x, y and z at each component of the exceptional locus. We obtain the following triples $(m_{E_j}(x), m_{E_j}(y), m_{E_j}(z))$ for both X_1 and X_2 :



We compute from this the partial derivatives $\frac{\partial F_i}{\partial x}$, $\frac{\partial F_i}{\partial y}$ and $\frac{\partial F_i}{\partial z}$ along the curves of the exceptional divisor. We obtain different values for two multiplicities (in bold) for $(X_1, 0)$ and $(X_2, 0)$, written in that order on the graph:



We compute from this the resolution graph of the family of polar curves $a \frac{\partial F_i}{\partial x} + b \frac{\partial F_i}{\partial y} + c \frac{\partial F_i}{\partial z} = 0$. In the X_1 case one has to blow up once more to resolve a basepoint. We then get the resolution graph of the polar curve of a generic plane projection of $(X_1, 0)$ resp. $(X_2, 0)$ (the arrows represent the strict transform, the numbers in parentheses are the multiplicities of the function $a \frac{\partial F_i}{\partial x} + b \frac{\partial F_i}{\partial y} + c \frac{\partial F_i}{\partial z}$ for generic a, b, c and the negative numbers are self-intersections):



The polar curves of $(X_1, 0)$ and $(X_2, 0)$ have different Lipschitz geometry since they don't even have the same number of components. Therefore, by [10, Theorem 1.2 (6)], $(X_1, 0)$ and $(X_2, 0)$ have different outer Lipschitz geometries. \square

4. THE OUTER GEOMETRY OF A SUPERISOLATED SINGULARITY

Proof of Lemma 2.2. We first re-formulate the definition of weak RL-equivalence. We will use coordinates (v, w) in \mathbb{C}^2 and (x, y, z) in \mathbb{C}^3 . We have functions $h_1(v, w)$ and $h_2(v, w)$ whose zero sets are the curves $(C_1, 0)$ and $(C_2, 0)$, a homeomorphism $\psi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ of germs, a constant $K \geq 1$ and a neighborhood \mathcal{U} of the origin in \mathbb{C}^2 such that for all $a, a' \in \mathcal{U}$.

$$\begin{aligned} \frac{1}{K} \|h_2(\psi(a))(1, \psi(a)) - h_2(\psi(a'))(1, \psi(a'))\|_{\mathbb{C}^3} &\leq \|h_1(a)(1, a) - h_1(a')(1, a')\|_{\mathbb{C}^3} \\ &\leq K \|h_2(\psi(a))(1, \psi(a)) - h_2(\psi(a'))(1, \psi(a'))\|_{\mathbb{C}^3} \end{aligned}$$

For $i = 1, 2$ we define $H_i: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ by

$$H_i(v, w) = h_i(v, w)(1, v, w)$$

and denote by $(S_i, 0)$ the image of H_i in $(\mathbb{C}^3, 0)$. Note that H_i maps $(C_i, 0)$ to 0 and is otherwise injective. We can thus complete the maps ψ , H_1 and H_2 to a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \xrightarrow{H_1} & (S_1, 0) \\ \downarrow \psi & & \downarrow \psi' \\ (\mathbb{C}^2, 0) & \xrightarrow{H_2} & (S_2, 0) \end{array} \quad (\star)$$

and ψ' is bijective. Weak RL-equivalence is now the statement that ψ' is bilipschitz for the outer geometry.

Now write $h_1 = Uh'_1$ and $H_1 = UH'_1$ where $U = U(v, w) \in \mathbb{C}\{v, w\}$ is a unit. Then we obtain a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \xrightarrow{H'_1} & (S'_1, 0) \\ \parallel & & \downarrow \eta \\ (\mathbb{C}^2, 0) & \xrightarrow{H_1} & (S_1, 0) \end{array}$$

where η is $(x, y, z) \mapsto U(\frac{y}{x}, \frac{z}{x})(x, y, z)$. The factor $U(\frac{y}{x}, \frac{z}{x}) = U(v, w)$ has the form $\alpha_0 + \sum_{i,j \geq 0} \alpha_{ij}v^i w^j$ with $\alpha_0 \neq 0$ so if the neighborhood \mathcal{U} is small then the factor is close to α_0 , so η is bilipschitz. Thus $\psi' \circ \eta: (S'_1, 0) \rightarrow (S_2, 0)$ is bilipschitz, so we have shown that modifying h_1 by a unit does not affect weak RL-equivalence. The same holds for h_2 , so weak RL-equivalence does not depend on the choice of defining functions for the curves $(C_1, 0)$ and $(C_2, 0)$.

It remains to show that analytic equivalence of $(C_1, 0)$ and $(C_2, 0)$ implies weak RL-equivalence. Analytic equivalence means that there exists a biholomorphic germ $\psi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ and a unit $U \in \mathbb{C}\{v, w\}$ such that $Uh_1 = h_2 \circ \psi$. We have already dealt with multiplication with a unit, so we will assume we have $h_1 = h_2 \circ \psi$. If ψ is a linear change of coordinates, then we get a diagram as in (\star) above, with ψ' given by the corresponding coordinate change in the y, z coordinates of \mathbb{C}^3 , so ψ' is bilipschitz and we have weak RL-equivalence. For general ψ the same is true up to higher order in v and w , so we still get weak RL-equivalence. \square

Proof of Proposition 2.3. Let $(X_1, 0)$ and $(X_2, 0)$ be two SISs with equations respectively

$$f_1(x, y, z) - g_1(x, y, z) = 0 \text{ and } f_2(x, y, z) - g_2(x, y, z) = 0,$$

where for $i = 1, 2$, f_i and g_i are homogeneous polynomials of degrees d and $d + 1$ respectively. We can assume that the projective line $x = 0$ does not contain any singular point of the projectivized tangent cones $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$. We assume also that C_1 and C_2 have the same combinatorial types and that corresponding singular points of C_1 and C_2 are weak RL-equivalent.

Since the tangent cone of a SIS $(X, 0)$ is reduced, the general hyperplane section of $(X, 0)$ consists of smooth transversal lines. Therefore, adapting the arguments of [11, Section 4] by taking simply a line as test curve, we obtain that the inner and outer metrics are Lipschitz equivalent inside the conical part of $(X, 0)$, i.e., outside cones around its exceptional lines. So we just have to control outer distance inside conical neighborhoods of the exceptional lines of $(X_1, 0)$ and $(X_2, 0)$ whose projective points are corresponding singular points of C_1 and C_2 .

Let $p_1 \in \text{Sing}(C_1)$ and $p_2 \in \text{Sing}(C_2)$ be two singular points in correspondence. After modifying $(X_1, 0)$ and $(X_2, 0)$ by analytic isomorphisms, we can assume that $p_i = (1, 0, 0)$ for $i = 1, 2$. We use again the notations of the proof of Theorem 1.3, and we work in the chart $(x, v, w) = (x, y/x, z/x)$ for the blow-up e .

Set $h_i(v, w) = f_i(1, v, w)/g_i(1, v, w)$. Then the germs (X_i^*, p_i) have equations $h_i(v, w) + x = 0$.

Since C_1 and C_2 are weak RL-equivalent and $h_i = 0$ is an equation of C_i , there exists a local homeomorphism $\psi: (\mathbb{C}^2_{(v,w)}, 0) \rightarrow (\mathbb{C}^2_{(v,w)}, 0)$, a constant $K \geq 1$ and a

neighborhood U of the origin in \mathbb{C}^2 such that for all $(v, w), (v', w') \in U$.

$$\begin{aligned} \frac{1}{K} \|h_2(\psi(v, w))(1, \psi(v, w)) - h_2(\psi(v', w'))(1, \psi(v', w'))\|_{\mathbb{C}^3} &\leq \\ \|h_1(v, w)(1, v, w) - h_1(v', w')(1, v', w')\|_{\mathbb{C}^3} &\leq \quad (*) \\ K \|h_2(\psi(v, w))(1, \psi(v, w)) - h_2(\psi(v', w'))(1, \psi(v', w'))\|_{\mathbb{C}^3} & \end{aligned}$$

Locally,

$$X_1^* = \{x = h_1(v, w)\} \quad \text{and} \quad X_2^* = \{x = h_2(\psi(v, w))\}.$$

As in the proof of Theorem 1.3 we consider the isomorphisms $\text{proj}_i: (X_i^*, p_i) \rightarrow (\mathbb{C}^2, 0)$ for $i = 1, 2$, the restrictions of the linear projections $(x, v, w) \mapsto (v, w)$. The composition $\text{proj}_2^{-1} \circ \psi \circ \text{proj}_1$ gives a local homeomorphism $\psi': (W_1, p_1) \rightarrow (W_2, p_2)$, where W_i is an open neighborhood of p_i in X_i^* . Then, ψ' induces a local homeomorphism $\psi'': e(W_1) \rightarrow e(W_2)$ such that $\psi'' \circ e = e \circ \psi'$. Notice that each $e(W_i)$ contains the intersection of X_i with a cone in $(\mathbb{C}^3, 0)$ around the exceptional line represented by p_i .

Consider a pair of points $q = (x, xv, xw)$ and $q' = (x', x'v', x'w')$ in $e(W_1)$. By definition of ψ'' , we have

$$\|q - q'\| = \|h_1(v, w)(1, v, w) - h_1(v', w')(1, v', w')\|_{\mathbb{C}^3},$$

$$\|\psi''(q) - \psi''(q')\| = \|h_2(\psi(v, w))(1, \psi(v, w)) - h_2(\psi(v', w'))(1, \psi(v', w'))\|_{\mathbb{C}^3}.$$

Then (*) implies that the ratio $\frac{\|\psi''(q) - \psi''(q')\|}{\|q - q'\|}$ is bounded above and below in a neighborhood of the origin.

Now let \widetilde{W}_i be the union of the W_i 's and let $\psi': \widetilde{W}_1 \rightarrow \widetilde{W}_2$ be the homeomorphism whose restriction to each W_1 is the local ψ' . Then $\psi'': e(\widetilde{W}_1) \rightarrow e(\widetilde{W}_2)$ is the outer bilipschitz homeomorphism induced by ψ' and we must extend ψ'' over all of X_1 .

Let B be a Milnor ball for X_1 and X_2 around 0. We set $\widetilde{Y}_i = (e^{-1}(B \cap X_i) \setminus \widetilde{W}_i)$. For $i = 1, 2$ we can adjust \widetilde{W}_i so that \widetilde{Y}_i is a D^2 -bundle over the exceptional divisor C_i minus its intersection with \widetilde{W}_i , i.e., over $\widetilde{C}_i := C_i \setminus \widetilde{W}_i$, and whose fibers are curvettes of C_i . We want to extend $\psi'': e(\widetilde{W}_1) \rightarrow e(\widetilde{W}_2)$ to a bilipschitz map over the conical regions $e(\widetilde{Y}_1)$ and $e(\widetilde{Y}_2)$. For this it suffices to extend ψ' by a bundle isomorphism $\widetilde{Y}_1 \rightarrow \widetilde{Y}_2$, since the resulting $e(\widetilde{Y}_1) \rightarrow e(\widetilde{Y}_2)$ is then bilipschitz.

$(X_1, 0)$ and $(X_2, 0)$ are inner bilipschitz equivalent by Theorem 1.3), so by [4, 1.9 (2)] the image by ψ'' of the foliation of $e(\widetilde{W}_1)$ by Milnor fibers of a generic linear form ℓ_1 has the homotopy class of the corresponding foliation by fibers of ℓ_2 in $e(\widetilde{W}_2)$. Since the projectivized tangent cones C_1 and C_2 are reduced, a fiber of $\ell_i \circ e$ intersects each D^2 -fiber over $\partial\widetilde{C}_i$ in one point. This gives a trivialization of the D^2 -bundle over each $\partial\widetilde{C}_i$ and therefore determines a relative Chern class for each component of the bundle \widetilde{Y}_i over \widetilde{C}_i . The map ψ' restricted to the bundle over $\partial\widetilde{C}_1$ extends to bundle isomorphisms between the components of \widetilde{Y}_1 and \widetilde{Y}_2 if and only if their relative Chern classes agree. But for $i = 1, 2$ these relative Chern classes are given by the negative of the number of intersection points of ℓ_i^* with each component of C_i (i.e., the degrees of these components of C_i), and these degrees agree since C_1 and C_2 are combinatorially equivalent.

We have now constructed a map $\psi'': (X_1, 0) \rightarrow (X_2, 0)$ which is outer bilipschitz if we restrict to distance between pairs of points x, y which are either both in a single component of $e(\widetilde{W}_1)$ or both in the conical region $e(\widetilde{Y}_1)$. Let $N\widetilde{Y}_i$ be a larger

version of the bundle \widetilde{Y}_i , so $e(N\widetilde{Y}_i)$ is a conical neighborhood of $e(\widetilde{Y}_i)$. We still have an outer bilipschitz constant for ψ'' for any x and y which are both in a single component of $e(\widetilde{W}_1)$ or both in the conical region $e(N\widetilde{Y}_1)$. Otherwise, either one of x, y is in $e(\widetilde{W}_1) \setminus e(N\widetilde{Y}_1)$ and the other in $e(\widetilde{Y}_1)$ or x and y are in different components of $e(\widetilde{W}_1)$. The ratio of inner to outer distance is clearly bounded for such point pairs, so since ψ'' is inner bilipschitz, it is outer bilipschitz. \square

REFERENCES

- [1] Enrique Artal Bartolo, Forme de Jordan de la monodromie des singularités superisolées de surfaces, *Mem. Amer. Math. Soc.* **109** (1994), no. 525.
- [2] Enrique Artal Bartolo, Ignacio Luengo and Alejandro Melle Hernández, Superisolated surface singularities, *Singularities and computer algebra*, 1339, London Math. Soc. Lecture Note Ser., 324, Cambridge Univ. Press, Cambridge, 2006.
- [3] Enrique Artal Bartolo, José Ignacio Cogolludo and Hiro-o Tokunaga, A survey on Zariski pairs, *Algebraic geometry in East Asia—Hanoi 2005*, 1–100, Adv. Stud. Pure Math. **50**, Math. Soc. Japan, Tokyo, 2008.
- [4] Lev Birbrair, Walter D Neumann and Anne Pichon, The thick-thin decomposition and the bilipschitz classification of normal surface singularities, *Acta Math.* **212** (2014), 199–256.
- [5] Alexandre Fernandes, Topological equivalence of complex curves and bi-Lipschitz maps, *Michigan Math. J.* **51** (2003), 593–606.
- [6] Abramo Hefez and Marcelo Escudeiro Hernandes, The analytic classification of plane branches. *Bull. Lond. Math. Soc.* **43** (2011), no. 2, 289–298.
- [7] A Libgober, Alexander polynomials of plane algebraic curves and cyclic multiple planes, *Duke Math. J.* **49** (1982), 833–851.
- [8] Lê Dũng Tráng, Françoise Michel and Claude Weber, Courbes polaires et topologie des courbes planes, *Ann. Sci. École Norm. Sup.* **24** (1991), 141–169.
- [9] Walter D Neumann and Anne Pichon, Lipschitz geometry of complex curves, *Journal of Singularities* volume **10** (2014), 225–234.
- [10] Walter D Neumann and Anne Pichon, Lipschitz geometry of complex surfaces: analytic invariants and equisingularity (2014), arXiv:1211.4897v2.
- [11] Walter D Neumann, Helge Møller Pedersen and Anne Pichon, Minimal surface singularities are Lipschitz normally embedded (2015), 24 pages, arXiv:1503.03301.
- [12] Frédéric Pham and Bernard Teissier, Fractions Lipschitziennes d’une algèbre analytique complexe et saturation de Zariski. *Prépublications Ecole Polytechnique* No. M17.0669 (1969). Available at <http://hal.archives-ouvertes.fr/hal-00384928/fr/>
- [13] A. N. Varchenko, *Algebro-Geometrical Equisingularity and Local Topological Classification of Smooth Mappings*, Proc. Internat. Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 1, pp. 427–431. Canad. Math. Congress, Montreal, Que., 1975.
- [14] Oscar Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, *Amer. J. Math.* **51** (1929), 305–328.
- [15] Oscar Zariski, The topological discriminant genus p , *Amer. J. Math.* **59** (1937), 335–358.
- [16] Oscar Zariski, *The moduli problem for plane branches*, with an appendix by Bernard Teissier, Amer. Math. Soc., University Lecture Series 39 (2006) [translated from original French edition of 1973].

DEPARTMENT OF MATHEMATICS, BARNARD COLLEGE, COLUMBIA UNIVERSITY, 2009 BROADWAY
MC4424, NEW YORK, NY 10027, USA

E-mail address: `neumann@math.columbia.edu`

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MAR-
SEILLE, FRANCE

E-mail address: `anne.pichon@univ-amu.fr`