MATH V1201 SECTIONS 002 & 003 HOMEWORK 3
DUE FEBRUARY 16, 2015

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1. Parametric equations for planes

(Since you type the answers in WebAssign in little boxes, you can forget to learn the forms that answers should take. These problems should all be really easy: I want to make sure you know how to write the answers correctly.)

(I.1) Find two vectors $\vec{v}$ and $\vec{w}$ parallel to the plane $P = \{(x, y, z) \mid x + 2y + 3z = 1\}$ and use $\vec{v}$ and $\vec{w}$ to give a parametric equation for P.

(I.2) The lines $\langle 1, 2, 3\rangle + t\langle 1, 1, 1\rangle$ and $\langle -1, 0, 2\rangle + t\langle 1, 1, 0\rangle$ intersect in one point, and hence lie in a plane P. Give a parametric equation for P.

(I.3) The lines $t\langle 1, 1, 1\rangle$ and $\langle 1, 2, 3\rangle + t\langle 1, 1, 1\rangle$ are parallel, and hence lie in a plane P. Give a parametric equation for P and also an implicit equation for P.

2. Conic sections are conic sections

(II.1) Any plane which does not go through the origin can be written in the form $ax+by+cz = 1$. I will be interested in *non-vertical* planes, i.e., planes which do not contain a vertical line ($\vec{k}$ is vertical). When is $ax + by + cz = 1$ non-vertical?

(II.2) Now, consider a non-vertical plane P given by $ax + by + cz = 1$ and the double cone C given by $z^2 = x^2 + y^2$. The intersection of P and C is a curve γ. Write an equation for the projection (shadow) of γ in the xy-plane. (Your answer should be an equation in x and y.)

(II.3) Your equation represents a conic section, right? For which a, b, c does it represent an ellipse? A hyperbola? A parabola?

Upshot: conic sections (cutting a cone by a plane) are conic sections (solutions to quadratic equations in x and y).

3. Some Stewart problems

(II.1) Stewart 12.3.53

(II.2) Stewart 12.3.63.

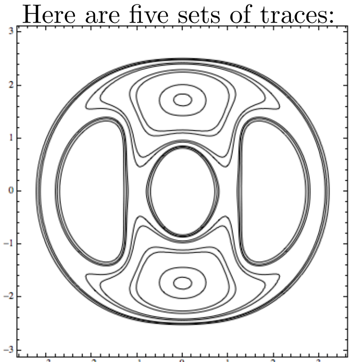
(II.3) Stewart 12.4.50.

(II.4) Stewart 12.4.51. Also, check this holds in a couple of examples.

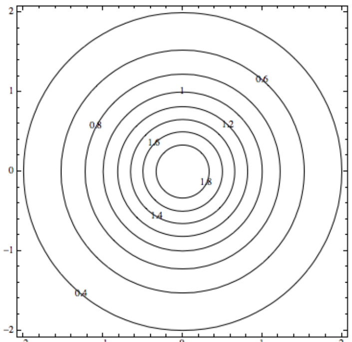
(II.5) Stewart 12.6.49.

4. Some surfaces and their level sets

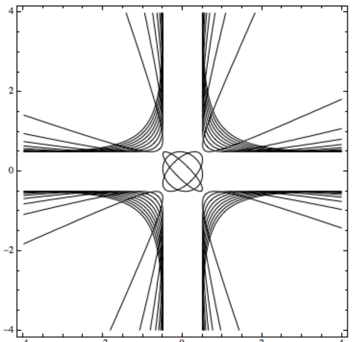
Here are five sets of traces:



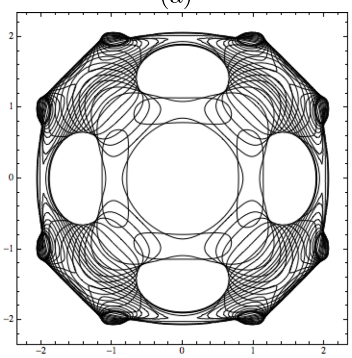
(a)



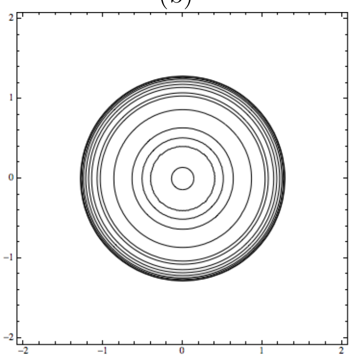
(b)



(c)

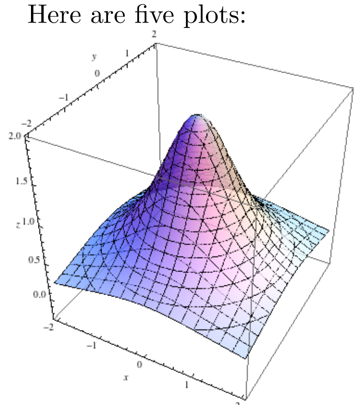


(d)

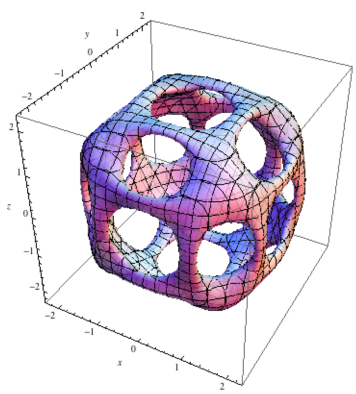


(e)

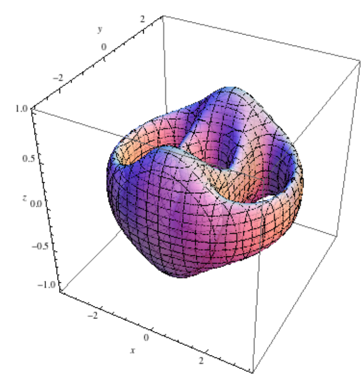
Here are five plots:



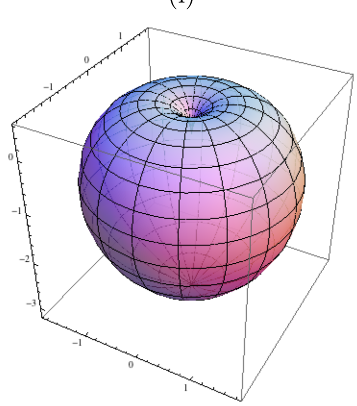
(i)



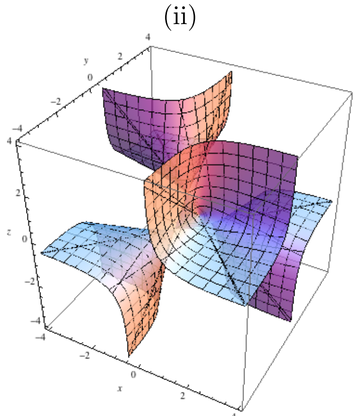
(ii)



(iii)



(iv)



(v)

For each of the following equations, say which set of traces and which plot it corresponds to. *Justify your answers.*

(IV.1) $z = 2/(r^2 + 1)$, in cylindrical coordinates. (Inspired by the Witch of Agnesi.)

(IV.2) $\rho = \phi$ in spherical coordinates.

(IV.3) $(\sqrt{(x-1)^2 + y^2} - 2)^2 + z^2) * ((\sqrt{(x+1)^2 + y^2} - 2)^2 + z^2) = .25$

(IV.4) $4(x^2 + y^2 + z^2) + 16xyz = 1$ (the "Cayley Cubic")

(IV.5) $((x^2 + y^2 - 4)^2 + (z^2 - 1)^2)((y^2 + z^2 - 4)^2 + (x^2 - 1)^2)((z^2 + x^2 - 4)^2 + (y^2 - 1)^2) = 4$

(The last three are inspired by `http://virtualmathmuseum.org/Surface/gallery_o.html`.)

Hints:

- Is the surface bounded or unbounded? How big can the different coordinates get?
- What does the trace $z = 0$ look like? Other traces you can compute?
- Intersections with the coordinate planes?
- Any obvious symmetry?

5. Mathematica

(1) In HW2, you learned how to use `ContourPlot3D` to plot planes. Plotting quadric surfaces works almost exactly the same way. Use Mathematica to plot the surface $x^2 + y^2 - z^2 = 5$.

(2) You can also use the Mathematica `ContourPlot` to plot traces. For example, to plot the traces $z = -5, -4, -3, \ldots, 0$ for $x^2 + y^2 - z^2 = 5$ you use the command:

```
ContourPlot[Sqrt[x^2 + y^2 - 5], {x, -5, 5}, {y, -5, 5}, ContourShading -> None,
ContourLabels -> All, ContourStyle -> Black, Contours -> {0, 1, 2, 3, 4, 5},
PlotPoints -> 40]
```

(Try it.)

(3) Use Mathematica to check your work on the "surfaces and their level sets" section. (You should notice something interesting if you play around with the Cayley cubic: is it really disconnected?) You might want to use `SphericalPlot3D` for the second equation, but see the warning below. The contours may be hard to plot; you don't have to plot them if you don't feel you need them.

(4) Just like one can have parametric equations for planes, one can have parametric equations for more general surfaces. Mathematica plots them with `ParametricPlot3D`. Just to test it, here's code to plot a Möbius band, borrowed from MathWorld:

```
ParametricPlot3D[{(1 + v*Cos[u/2])*Cos[u], (1 + v*Cos[u/2])*Sin[u],v*Sin[u/2]},
{u, 0, 2*Pi}, {v, -.25, .25}]
```

Try it.

(Some surfaces are much easier to define parametrically than implicitly, and some are much easier to define implicitly. I think the Möbius band is in the former category.)

(5) Use Mathematica to draw a picture illustrating Stewart 12.5.49 (Exercise (II.5)).

Warning! According to Wikipedia, mathematicians and physicists have different conventions for spherical coordinates. (Mathematicians are right.) Mathematica uses the *physics* convention for spherical coordinates, i.e., it exchanges θ and ϕ. Be careful if you use Mathematica's `SphericalPlot3D` feature not to get confused.

In case you're interested, here's how I plotted the traces of $4(x^2 + y^2 + z^2) + 16xyz = 1$. (Maybe there's an easier way to do this, but I don't know it; if you find one, please let me know.)

```
mythirdplot[z_] := ContourPlot[4*(x^2 + y^2 + z^2) + 16* x* y* z == 1, {x, -4, 4},
{y, -4, 4}, ContourShading -> None, ContourStyle -> Black]
Show[Map[mythirdplot, Range[-4, 4, .37]]]
```

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