SLICE DIAGRAMS AND OPERATOR INVARIANTS: OVERVIEW

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Abstract. Sliced diagrams of tangles provide rigorous constructions of operator invariants of tangles, allowing us to reconstruct the Jones polynomial in terms of these operator invariants. In this paper, we offer a brief introduction to the operator invariants of tangles via sliced diagrams and conclude by sketching their importance in terms of quantum invariants and TQFTs.

1. Introduction

The discovery of a new knot invariant by Jones in the 1980s [Jon85], [Jon87] triggered discoveries of infinitely many new knot invariants derived from interactions with various other fields (the theory of operator algebras, the Yang-Baxter equation in statistical mechanics, representations of quantum groups, etc.). Low-dimensional topology had met up with mathematical physics, yielding new invariants of knots and even 3-manifolds, quantum invariants. In particular, there was Witten's proposal that the partition function of quantum field theory would provide a topological invariant of the 3-manifold.

This brief expository paper written for *W4052: Knot Theory* serves as a primer to begin understanding this collaboration in greater detail. We will begin discussing the properties of tangles, then continue describing operator invariants of unoriented tangles and oriented tangles; in particular, we will construct the Jones polynomial with invariants of elementary tangle diagrams.

2. Tangles

Let's begin with the definition of a tangle. In link theory, a tangle is defined as a compact 1-manifold properly embedded in $\mathbb{R}^2 \times [0,1]$ such that the boundary of the embedded 1-manifold is a set of distinct points in $\{0\} \times \mathbb{R} \times \{0,1\}$ An oriented tangle is a tangle with each component oriented, and a framed tangle is a tangle with a framing of each component: imagine a tangle with a normal vector field defined as a tangle ribbon such that one edge of the ribbon is the tangle itself whereas the other one is obtained from the first by a small translation along the vector field. Isotopy classes of tangles with framings are called framed tangles or ribbons. See [Kas94], Chapter X.

Simply put, a tangle diagram is a diagram of a tangle in $\mathbb{R} \times [0,1]$. Like link diagrams (or rather, in particular link diagrams), tangle diagrams consist of crossings, vertical paths, and critical points. For such diagrams, we desire an isotopy preserving crossings and critical points, which we call Turaev moves, which are intuitively analogous to Redeimeister moves among link diagrams and Markov moves among braids.

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3. Sliced diagrams

Take Figure 2, which we'll call a sliced tangle diagram D. Notice this is a diagram which has horizontal lines such that there is exactly one of the elementary tangle diagrams between each pair of adjacent horizontal lines. Each domain between slices in a sliced diagram can be presented as a tensor product of a number of vertical lines and either a crossing or a critical point. Taking V as a vector space over $\mathbb{C}$, we associate V to each point of the section of the diagram by a horizontal line, and take the tensor product of the vector spaces along said line. Associate linear maps between the tensor products, as shown in Figure 1.

For any sliced tangle diagram D with i upper ends and j lower ends, we associate a bracket [D]$\in$ Hom (V$^{\otimes i}$, V$^{\otimes j}$), which is the composition of the tensor products of the linear maps shown.

Figure 1: The linear maps associated to the elementary tangle diagrams

$$\begin{pmatrix} V \\ \uparrow id_V \\ V \end{pmatrix}\begin{pmatrix} V\otimes V \\ \uparrow R \\ V\otimes V \end{pmatrix}\ \begin{pmatrix} V\otimes V \\ \uparrow R^{-1} \\ V\otimes V \end{pmatrix}\begin{pmatrix} \mathbb{C} \\ \uparrow n \\ V\otimes V \end{pmatrix}\ \begin{pmatrix} V\otimes V \\ \uparrow u \\ \mathbb{C} \end{pmatrix}$$

Explicitly, these moves for the identity on vertical lines, the invertible endomorphism of crossings, and bilinear forms for critical points are:

$$V \xrightarrow{id_V} V,\ V\otimes\, V \xrightarrow{R} V\otimes\, V,\ V\otimes\, V \xrightarrow{R^{-1}} V\otimes\, V,\ V\otimes\, V \xrightarrow{n} \mathbb{C},\, \mathbb{C} \xrightarrow{u} V\otimes\, V$$

In order for [D] to be invariant under a certain Tuarev move, maps n and u must satisfy the condition $(n \otimes id_V)(id_V \otimes u) = id_V = (id_V \otimes n)(u \otimes id_V)$

If we obtain the following relations for R of $V\otimes\, V$ and n on $V\otimes\, V$ satisfying

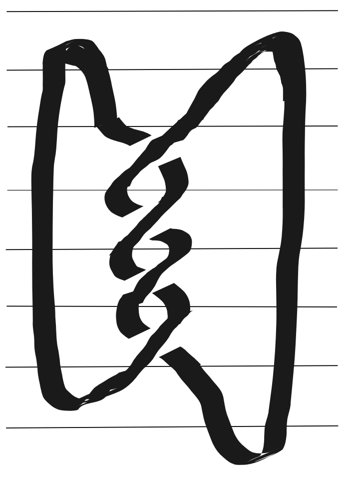
(1) $(id_V \otimes n)(R^{\pm 1} \otimes id_V) = (n \otimes id_V)(id_V \otimes R^{\mp 1})$

(2) $n \cdot R = n$

(3) $(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R)$

then the bracket [D] associated with R and n is an isotopy invariant for a tangle T (of which D is the sliced diagram). This invariant [T] is called the operator invariant, an isotopy invariant of tangles as an operator between vector spaces.

If T is a framed tangle, then (2) is rather $n \cdot R = c \cdot R$, with some non-zero scalar c. See [Oht02], Chapter 3, and [Saw94] for a more thorough treatment.



$$
\begin{array}{l}
\mathbb{C} \\
\uparrow n \otimes n \\
V \otimes V \otimes V \otimes V \\
\uparrow id_V \otimes id_V \otimes id_V \otimes id_V \\
V \otimes V \otimes V \otimes V \\
\uparrow id_V \otimes R \otimes id_V \\
V \otimes V \otimes V \otimes V \\
\uparrow id_V \otimes R \otimes id_V \\
V \otimes V \otimes V \otimes V \\
\uparrow id_V \otimes R \otimes id_V \\
V \otimes V \otimes V \otimes V \\
\uparrow id_V \otimes R \otimes id_V \\
V \otimes V \otimes V \otimes V \\
\uparrow u \otimes id_V \otimes id_V \\
V \otimes V \\
\uparrow u \\
\mathbb{C}
\end{array}
$$

Figure 2: A sliced link diagram of the FigureEightKnot and its operator invariant

4. Kauffman Bracket

With this operator invariant, we are able to reconstruct the Kauffman bracket!

Let V be a 2-dimensional vector space with basis $\{e_0, e_1\}$. With respect to $\{e_0 \otimes e_0, e_0 \otimes e_1, e_0 \otimes e_1, e_1 \otimes e_1\}$ of $V \otimes V$, take

$$R = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & A^{-1} & A - A^{-3} & 0 \\ 0 & 0 & 0 & A \end{pmatrix} \in End(V \otimes V)$$

$$n = \begin{pmatrix} 0 & A & -A^{1} & 0 \end{pmatrix} \in Hom(\mathbb{C}, V \otimes V)$$

One may check this satisfies (1) and (3).

(2) is verified as $n \cdot R = \begin{pmatrix} 0 & -A^{-2} & A^{-4} & 0 \end{pmatrix} = -A^{-3} \cdot n$ where c is $-A^{-3}$.

We may verify that these maps satisfy the relations of the Kauffman bracket. This operator invariant of some link [L] is equal to the Kauffman bracket.

5. Oriented sliced diagrams

We now will construct operator invariants of oriented tangles, which may be used to reconstruct the Jones polynomial [Jon85],[Jon87] and the Alexander polynomial. Following a similar approach to operator invariants of unoriented tangles, we show the linear maps associated to oriented elementary tangles.

We'll have to take into account the new structure with the following notation: We remind ourselves V^* is a dual vector space of V. Also, for an endomorphism in $End(V_1 \otimes V_2)$, we define the $trace_i$ as:

$$trace_1 = End(V_1 \otimes V_2) = V_2^* \otimes V_1^* \otimes V_1 \otimes V_2 \xrightarrow{contraction} V_2^* \otimes V_2 = End(V_2)$$
$$trace_2 = End(V_1 \otimes V_2) = V_2^* \otimes V_1^* \otimes V_1 \otimes V_2 \xrightarrow{contraction} V_1^* \otimes V_1 = End(V_1)$$

where the contractions are $V_i^* \otimes V_i \to \mathbb{C}$

For an endomorphism A, we have

$$A \in End(V_1 \otimes V_2) = (V_1 \otimes V_2)^* \otimes (V_1 \otimes V_2) = V_2^* \otimes V_1^* \otimes V_1 \otimes V_2$$

and denote $A^{\rightarrow}$, $A^{\leftarrow}$ the linear maps obtained from A by cyclic permutations of the entries of the previous tensor product such that

$A^{\rightarrow} \in Hom((V_2 \otimes V_2^*, V_1^* \otimes V_1) = (V_2 \otimes V_2^*)^* \otimes (V_1^* \otimes V_1) = V_2 \otimes V_2^* \otimes V_1^* \otimes V_1$
$A^{\leftarrow} \in Hom((V_1^* \otimes V_1, V_2 \otimes V_2^*) = (V_1^* \otimes V_1)^* \otimes (V_2 \otimes V_2^*) = V_1^* \otimes V_1 \otimes V_2 \otimes V_2^*$

Take an oriented tangle T: if invertible endomorphisms $R \in End(V \otimes V)$ and $h \in End(V)$ satisfy
(1) $R \cdot (h \otimes h) = (h \otimes h) \cdot R$
(2) $trace_2((id_V \otimes h) \cdot R^{\pm 1}) = id_V$
(3) $(R^{-1})^{\leftarrow} \cdot ((id_V \otimes h) \cdot R \cdot (h^{-1} \otimes id_V))^{\rightarrow} = id_V \otimes id_{V^*}$
(4) $(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R)$
then the bracket [D] is an isotopy invariant of T, denoted by [T].

6. Jones polynomial

Our operator invariant for oriented tangles is able to redefine the Jones polynomial (and the Alexander polynomial!). With some complex parameter t, the former may be calculated with

$$R = \begin{pmatrix} t^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & t^{-\frac{1}{2}} - t^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & t^{-\frac{1}{2}} \end{pmatrix} \in End(V \otimes V)$$

$$h = \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & t^{-\frac{1}{2}} \end{pmatrix} \in Hom(\mathbb{C}, V \otimes V)$$

Industrious readers may check R and h both satisfies conditions (1)-(4) and is equal to $V_L(t)$.

7. Further Perspective

A Hopf algebra is a structure that is a bialgebra (i.e. simultaneoulsy a unital associative algebra and counital coassociative coalgebra) equipped with an antiautomorphim (the antipode) (See[Abe80] for rigorous details). A generalization of Hopf algebras is the concept of quasi-Hopf algebras, whereby co-associativity of the co-algebra structure is not assumed.

(A quasi-triangular Hopf algebra is a Hopf algebras with an invertible element $R \in$ A $\otimes$ A such that

$$(P \circ \Delta)(x) = R\Delta(x)R^{-1},\ (\Delta \circ id)(R) = R_{13}R_{23},\ (id \circ \Delta)(R) = R_{13}R_{12},$$

where map P is the permutation P (x $\otimes$ y)=y$\otimes$x and R is an invertible R matrix. Said algebra has the property of the quantized Yang-Baxter equation. [Saw96] provides details and motivation).

A ribbon Hopf algebra is a quasi-triangular Hopf algebra with invertible element v, the ribbon element. We can consequently introduce invariants of links in ribbon Hopf algebras called the universal A invariant, denoted by $Q^{A;*}(L)$. $Q^{A;*}(L)$ induces an operator invariant of tangles. Taking the quantum group $U_q(sl_2)$ as a ribbon Hopf algebra, one can construct a functor from a topological category to some category of representations, which results in the Jones polynomial [See (Tin10]; [Kas94], Chapters VII, XX for details).

We are able to relate operator invariants to invariants derived from ribbon Hopf algebras, i.e. for a ribbon Hopf algebra (A, R, v), we introduce the universal A invariant of a link L, denoted by $Q^{A;*}(L)$. Values for $Q^{A;*}$ are given using the same oriented tangle diagrams offered previously. Just like before, for a diagram D of a framed link, we take a decomposition of D into elementary tangle diagrams and glue the values of the invariant to obtain a gradhical construction of $Q^{A;*}(D)$. (Note: $Q^{A;*}$ gives an isotopy invariant of framed links via Turaev moves.) For much more background, see [Oht02], [Tin10], [CP94].

An an example of a ribbon Hopf algebra, take quantum group $U_q(\mathfrak{sl}_2)$, an infinite dimensional algebra related to the Lie algebra $\mathfrak{sl}_2$ of 2×2 matrices with trace zero. We can derive an operator invariant from a representation V_n of $U_q(\mathfrak{sl}_2)$ which defined the quantum $(\mathfrak{sl}, V_n)$ invariant of oriented frame links. This we can use to reconstruct the Jones polynomial. (See [Tin10]; [Oht02], Chapter 4)

If we let V be the standard representation of $\mathfrak{sl}_2$, we calculate

$$R = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & q^{-\frac{1}{4}} & q^{\frac{1}{4}} - q^{-\frac{3}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix} \text{ and } h = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}$$

The operator invariant obtained with these matrices R and h (usually the method shown before) is the quantum $(\mathfrak{sl}_2, V)$ invariant $Q^{\mathfrak{sl}_2;V}(L_0)$ with framed link L, which satisfies the skein relation:

$$q^{\frac{1}{4}} Q^{\mathfrak{sl}_2;V}(L_+) - q^{-\frac{1}{4}} Q^{\mathfrak{sl}_2;V}(L_-) = (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) Q^{\mathfrak{sl}_2;V}(L_0)$$

This motivates the theorem that, where $\#L$ denotes the number of components of L and *f(L)* denotes the sum of the framings of the components of L,

$$Q^{\mathfrak{sl}_2;V}(L) = (-1)^{\#L + f(L)} < L >$$

If we take $A = q^{\frac{1}{4}}$, the theorem gives a reconstruction of the quantum $(\mathfrak{sl}_2, V)$ invariant by the Kauffman bracket. This is naturally only a sketch as to how the Kauffman bracket is in fact equal to quantum $(\mathfrak{sl}_2, V)$ invariant; one may regard the Jones polynomial as the simplest quantum invariant. (Following [Tin10]; [Oht02], Chapters 4-5. Refer for rigorous details and calculations.)

Reshetikhin and Turaev used quantum group techniques to construct a rigorous topological invariant corresponding to Chern-Simons theory with gauge group $SU(2)$ and showed that it satisfied a weaker version of Atiyahs axioms for TQFTs. [RT90], [RT91]. Manageable formulas for 3-manifold invariants soon transpired [KB91].

8. Relation of TQFTs to sliced diagrams

Let's interpret the operations of our sliced diagrams in terms of a monoid, a set M with an associative binary operation with a neutral element. (See [Kock04]). Indeed, if we allow ourselves to view our sliced diagram as a symmetric monoidal functor from the 2-framed cobordism category to the category *Vect*, we'll discover the functor the satisfies conditions of a TQFT (See [Saw99]), which in general (using [Ati90]) is formulated as a rule $\mathcal{A}$ which to each closed oriented manifold Σ associates a vector space $\Sigma\mathcal{A}$, and to each oriented manifold whose boundary is Σ associates a vector in $\Sigma\mathcal{A}$. That is, our "objects" are 0-dimensional submanifolds of $\mathbb{R}^2$, with morphisms 1-dimensional submanifolds with boundary of $\mathbb{R}^2 \times [0, 1]$. (This is a morphism from its intersection with $\mathbb{R}^2 \times 0$ to its intersection with $\mathbb{R}^2 \times 1$.) These morphisms are exactly the tangles of our sliced diagram.

In physical terms, look at the slice diagram as 2-dimensional space. Consider time increasing upwards vertically and the intersections of the link with some plane at some fixed time to be "states". "Time slices" are represented by the horizontal lines; as it sweeps upwards, particles are born, annihilated, etc.

(See [Ati90]; Chapter 2, [Wit89] for further details on how the Jones Polynomial fits into the axiomatic approach to TQFTs).

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