# MATH W4051 PROBLEM SET 3
# DUE SEPTEMBER 23, 2008.

INSTRUCTOR: ROBERT LIPSHITZ

(1) Let $v, e, f \in \mathbb{N}$ with $v - e + f = 2$. Construct a CW complex homeomorphic to $S^2$ with exactly $v$ 0-cells, $e$ 1-cells and $f$ 2-cells.

(2) Let $X$ be a topological space, $A \subset X$. Define the *boundary of* $A$ to be $\partial A = A \setminus \mathrm{Int}(A)$.

Prove: if $A \subset X$ and $B \subset Y$ then $\partial(A \times B)$ in $X \times Y$ is $[(\partial A) \times B] \cup [A \times (\partial B)]$. (This fact partly explains using $\partial$ to denote the boundary.)

(3) Prove that any finite CW complex is locally connected, and also locally path connected.

(4) Quotients of spaces by group actions.[1] Let $G$ be a group, and $X$ a topological space. An *action of* $G$ *on* $X$ is a map $G \times X \to X$, which we will write $(g, x) \mapsto g \cdot x$, such that
- For any $x \in X$, $1_G \cdot x = x$.
- If $g, g' \in G$ and $x \in X$ then $(gg') \cdot x = g \cdot (g' \cdot x)$.
- For each $g \in G$, the map $X \to X$ defined by $x \mapsto g \cdot x$ is continuous.

Given an action of a group $G$ on a space $X$, we can define an equivalence relation $\sim$ by $x \sim y$ if there exists $g \in G$ so that $g \cdot x = y$. We let $X/G$ denote $X/\sim$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Prove that an action of $G$ on $X$ is a group homomorphism $G \to Homeo(X)$.

(c) $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(n, m) \cdot (x, y) = (x + n, y + m)$. What is $\mathbb{R}^2/\mathbb{Z}^2$? (Just identify it and draw a picture, don't try to prove it (whatever that would mean).)

(d) $\mathbb{Z}/3$ acts on $S^2$ by rotation by $2\pi/3$ around the $z$-axis. What is $S^2/(\mathbb{Z}/3)$? (Again, a correct statement and some pictures is enough.)

(e) Suppose that the space $X$ is a CW complex, and the action of $G$ respects the cell structure in the following way: for any $g \in G$, $g \neq 1_G$, and any cell $e$ in $X$, $g(e)$ is also a cell in $X$, and $g(e) \neq e$. Then $X/G$ inherits a cell structure from $X$. Explain how, briefly. Give an example illustrating this in the case of the action of $\mathbb{Z}/3$ on $S^2$ just described.

(f) The group $\mathbb{Z}/5$ acts on $S^3$ as follows. Write $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $\mathbb{Z}/5 = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$. Then define

$$\zeta^n(z, w) = (e^{2\pi i n/5} z, e^{6\pi i n/5} w).$$

The quotient $S^3/G$ is called the *lens space* $L(5, 3)$. Use the previous part to describe a CW complex structure on $L(5, 3)$. How many 0-cells, 1-cells, 2-cells and 3-cells does your CW complex have? (This takes some work; good luck.)

(5) Munkres 23.5

(6) Munkres 24.9

(7) Munkres 25.4

[1] A few of you haven't seen groups much yet. This would be a good time to start learning the definitions. The first four parts of this problem are pretty gentle.

**Optional** but recommended problems:

(1) Prove that $\mathbb{D}^m \times \mathbb{D}^n \cong \mathbb{D}^{m+n}$. (There are lots of possible approaches, but all of them take a little work. This is the sort of result I'm likely to call "obvious" in class, so it's good to be able to prove it.)

(2) Let $X_n$ be a topological space, $n = 1, \dots, \infty$, such that $X_{n-1} \subset X_n$ for all $n$. Let $X = \bigcup_{n=1}^{\infty} X_n$. Define a topology on $X$ by declaring $U \subset X$ to be open if $U \cap X_n$ is open for all $n$. (This is an instance of the *direct limit topology.*)
   (a) Let $Y$ be another topological space. A map $f\colon X \to Y$ is continuous if and only if the restriction of $f$ to each $X_n$ is continuous.
   (b) Use the direct limit topology to give a topology on infinite CW complexes with a finite number of cells in each dimension.
   (c) Let $X$ be a finite CW complex, and $Y$ an infinite one, given the topology from part 2b. Let $f\colon X \to Y$ be any continuous map. Then the image $f(X)$ of $X$ intersects only finitely many cells in $Y$. (You may want to postpone this for a week: you'll probably want to use compactness.)
   This result is somewhat counter-intuitive. Why?

(3) Munkres 25.9. (You'll want to read the supplementary exercises on pp. 145 and 146 first; this is interesting material that we don't have time for.)

*E-mail address*: `rl2327@columbia.edu`