MATH W4051 PROBLEM SET 2
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Note: definition of the Zariski topology has been revised since this was first posted.

(1) Let $\mathcal{C}^\infty(\mathbb{R})$ denote the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that f is differentiable to all orders (i.e., $f^{(n)}$ exists for all $n \geq 0$). Notice that $\mathcal{C}^\infty(\mathbb{R})$ is a vector space in an obvious way.

We endow $\mathcal{C}^\infty(\mathbb{R})$ with two different metrics. Let[1]

$$d_0(f,g) = \sup\{|f(x) - g(x)| \mid x \in \mathbb{R}\}$$
$$d_1(f,g) = d_0(f,g) + \sup\{|f'(x) - g'(x)| \mid x \in \mathbb{R}\}$$

(d_0 is called the $\mathcal{C}^0$-metric and d_1 is called the $\mathcal{C}^1$-metric.)

(a) Convince yourself that d_0 and d_1 are, in fact, metrics. (You don't have to write anything for this part.)
(b) Is the topology induced by d_0 finer or coarser than the topology induced by d_1?
(c) Define a map $D\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ by $D(f)(x) = f'(x)$. Prove that D gives a continuous map $(\mathcal{C}^\infty, d_1) \to (\mathcal{C}^\infty, d_0)$.
(d) Prove that D does *not* give a continuous map $(\mathcal{C}^\infty, d_0) \to (\mathcal{C}^\infty, d_0)$. (If you haven't seen this before, this should surprise you: the map D is linear but not necessarily continuous!)

(2) (a) Let X be a set and $\mathcal{B}$ a sub-basis for a topology on X. Then the topology generated by $\mathcal{B}$ is the coarsest topology on X such that every set in $\mathcal{B}$ is open. Formulate precisely what this means.
(b) Prove it.
(c) For Y and Z topological spaces, the product topology on $Y \times Z$ is the finest topology on $Y \times Z$ such that for any topological space X and continuous maps $f\colon X \to Y$, $g\colon X \to Z$, $(f,g)\colon X \to Y \times Z$ is continuous. Prove this.
The product topology is also the coarsest topology so that the projections $\pi_Y\colon Y \times Z \to Y$ and $\pi_Z\colon Y \times Z \to Z$ are continuous. Prove this, too.
(d) Analogous statements hold for arbitrary (possibly infinite) products. Formulate and prove them.
(This problem should make you feel lucky that the product topology exists: it's the finest topology with one property you want, but the coarsest with another, so it's the only topology with both.)

(3) The Zariski topology on $\mathbb{C}^n$ is defined as follows: a subset $S \subset \mathbb{C}^n$ is closed iff there are is a set of polynomials $\{p_\alpha(z_1, \dots, z_n)\}$ so that

$$S = \{\vec{z} \in \mathbb{C}^n \mid p_\alpha(z_1, \dots, z_n) = 0 \text{for all } \alpha\}.$$

A set is defined to be open if its complement is closed.

[1] for a set S of real numbers, recall that $\sup(S)$ is the supremum or least upper bound of S, i.e., the smallest real number r such that for all $s \in S$, $s \leq r$.

(a) Verify that this defines a topology on $\mathbb{C}^n$. Is it coarser or finer than the usual one?
(b) Is the Zariski topology metrizable? Why or why not?
(c) For $n = 1$ the Zariski topology is the same as a topology defined in Munkres (or, briefly, in class). Which one?
(d) Show that the Zariski topology is the coarsest topology such that for any polynomial $p(z_1, \dots, z_n)$, the corresponding map $\mathbb{C}^n \to \mathbb{C}$ is continuous.
(e) Is the Zariski topology Hausdorff?
(f) Optional—uses some more abstract algebra: I originally wrote the problem with the sets $\{p_\alpha\}$ finite. Why is this equivalent to the current definition?

Remark. If $\mathbb{F}$ is any field (or even just a ring) then the same definition makes sense for $\mathbb{F}^n$. This allows one to use topology to study, say, algebraic sets over fields of characteristic p. (In practice if $\mathbb{F}$ is not algebraically closed this is not quite the topology one is looking for.) For this reason, the Zariski topology plays a central role in algebraic geometry.

(4) Munkres 17.13. (This is how the analogue of compactness in algebraic geometry is defined.)
(5) Munkres 17.14
(6) Munkres 18.6
(7) Munkres 19.6

Also, here's an *optional* problem:

- Find a subset S of $\mathbb{R}$ which becomes perfect after applying the Cantor derivative exactly n times.
- Find a subset S of $\mathbb{R}$ which becomes perfect after applying the Cantor derivative a countably infinite number of times (or more precisely, ω times), in the following sense:

  Let $S^{(n)}$ denote the result of applying the Cantor derivative n times to S. Let $S^{(\omega)} = \bigcap_{n=0}^{\infty} S^{(n)}$. Find a set such that no $S^{(n)}$ is perfect but $S^{(\omega)}$ is perfect.
- Find a subset S of $\mathbb{R}$ so that $S^{(\omega)}$ is not perfect but its Cantor derivative $S^{(\omega+1)}$ is.
- If you know about ordinals (or if you learn about them), prove:'

  Lemma 1. *For any countable ordinal o there is a set $S \subset \mathbb{R}$ so that $S^{(o)}$ is perfect but if $o' < o$ then $S^{(o')}$ is not perfect.*

  (The first step is defining $S^{(o)}$.)

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