MATH W4051 PROBLEM SET 13
DUE NEVER?

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The goal of this problem set is to prove the following:

Theorem 1. *S^2 is not contractible.*

Remark. I'm happy to solutions to any parts of this extremely optional problem set with any of you. Please do not try to turn it into Tom.

(1) Given a space X, the symmetric group S_n acts on $X^n = \overbrace{X \times X \times \cdots \times X}^{n \text{ copies}}$ by permuting the factors. That is, for $\sigma\colon \{1, \dots, n\} \to \{1, \dots, n\}$ a bijection, define
$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$
Define
  (a) Prove that the map X^n induced by σ is continuous.

  Define $\mathrm{Sym}^n(X) = X^n/S_n$. That is, a point in $\mathrm{Sym}^n(X)$ is an *unordered* n-tuple of points in X.
  (b) Prove that if $f\colon X \to Y$ is continuous then f induces a continuous map
  $$\mathrm{Sym}^n(f)\colon \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y).$$
  Show that Sym^n defines a functor from the category of topological spaces to itself, i.e., that $\mathrm{Sym}^n(Id) = Id$ and that $\mathrm{Sym}^n(f \circ g) = \mathrm{Sym}^n(f) \circ \mathrm{Sym}^n(g)$.
  (c) Prove that if $f \sim g$ then $\mathrm{Sym}^n(f) \sim \mathrm{Sym}^n(g)$. Conclude that if $X \simeq Y$ then $\mathrm{Sym}^n(X) \simeq \mathrm{Sym}^n(Y)$.
  (d) Prove that $\mathrm{Sym}^n(S^2) \cong \mathbb{C}P^n$, where $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus 0)/(p \sim \lambda p)$.
  (*Hint.* A point x in $\mathrm{Sym}^n(S^2)$ is an unordered k-tuple of points in $\mathbb{C} = S^2 \setminus \{pt\}$, for some $k < n$. Let $p_x(z) = a_0 + a_1 z + \cdots + a_k z^k$ be a degree k polynomial vanishing at these k points. Map x to $(a_0, a_1, \dots, a_k, 0, \dots, 0) \in (\mathbb{C}^{n+1} \setminus 0)/\sim = \mathbb{C}P^n$.)

(2) Let $X = X_1 \cup X_2 \cup \dots$ with $X_i \subset X_{i+1}$. Let $\mathcal{T}_i$ be a topology on X_i so that $\mathcal{T}_{i+1}|_{X_i} = \mathcal{T}_i$. Define a topology $\mathcal{T}$ on X by saying $U \subset X$ is open if for all i, $U \cap X_i \in \mathcal{T}_i$. The topology $\mathcal{T}$ is called the *weak topology*, the *union topology* or the *direct limit topology.*
  (a) Check that the weak topology $\mathcal{T}$ is, in fact, a topology.
  (b) Check that a map $f\colon (X, \mathcal{T}) \to Y$ is continuous if and only if $f|_{X_i}$ is continuous for all i. Check that the weak topology is the coarsest topology with this property.
  (c) Optional: formulate the direct limit topology in categorical terms, similar to the formulation of products or coproducts. (Which is it closer to?)

(3) Let $\mathbb{R}^\omega$ denote the set $\{(x_1, x_2, \dots)\}$ of infinite sequences of real numbers. Inside $\mathbb{R}^\omega$ we have $\mathbb{R}^1 = \{(x_1, 0, 0, \dots)$, $\mathbb{R}^2 = \{(x_1, x_2, 0, \dots)\}$ and so on. We also have
$$S^n = \{(x_1, x_2, \dots, x_{n+1}, 0, 0, \dots) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$
Let $S^\infty = \cup_{n=0}^\infty S^n$, and endow S^∞ with the weak topology induced by the S^n.

Prove that S^∞ is contractible. (Hint: it's easy to show any non-surjective map $S^\infty \to S^\infty$ is nullhomotopic. Show that Id is homotopic to the shift operator $s(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$.)

(4) Define $\mathrm{Sym}^\infty(X, x_0)$ as follows. Define the weakly-infinite permutation group S_∞ to be the set of bijective $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) = n$ for all but finitely many n. (In other words, $S_\infty = \bigcup_n S_n$.)

Now, consider the set $\prod^\infty(X, x_0)$ consisting of sequences of points in X where all but finitely many of the terms are x_0. That is, elements of $\prod^\infty(X, x_0)$ look like

$$(y_1, y_2, \dots, y_k, x_0, x_0, x_0, \dots).$$

Then S_∞ acts on $\prod^\infty(X, x_0)$ by permuting the coordinates. Define $\mathrm{Sym}^\infty(X, x_0)$ to be the quotient $\prod^\infty(X, x_0)/S_\infty$.

Note that there are inclusion maps $i_n\colon \mathrm{Sym}^n(X, x_0) \to \mathrm{Sym}^\infty(X, x_0)$. Moreover, any point $p \in \mathrm{Sym}^\infty(X, x_0)$ is in the image of i_n for large enough n. So, we can think of $\mathrm{Sym}^\infty(X, x_0) = \bigcup_n \mathrm{Sym}^n(X, x_0)$. Give $\mathrm{Sym}^n(X, x_0)$ the union topology induced from the $\mathrm{Sym}^n(X, x_0)$.

We take $(x_0, x_0, \dots)$ as the basepoint of $\mathrm{Sym}^\infty(X, x_0)$.

(a) Prove that Sym^∞ defines a functor $\mathcal{T}_* \to \mathcal{T}_*$.

(b) Prove that if $f, g\colon (X, x_0) \to (Y, y_0)$ are homotopic rel x_0 then

$$\mathrm{Sym}^\infty(f), \mathrm{Sym}^\infty(g)\colon \mathrm{Sym}^\infty(X, x_0) \to \mathrm{Sym}^\infty(Y, y_0)$$

are homotopic rel $\mathrm{Sym}^\infty(x_0)$. Conclude that if X deformation retracts to x_0 then $\mathrm{Sym}^\infty(X, x_0)$ deformation retracts to $\mathrm{Sym}^\infty(x_0)$.

(c) Prove that $\mathrm{Sym}^\infty(S^2)$ is a quotient space of S^∞.

(d) Let $\pi\colon S^\infty \to \mathrm{Sym}^\infty(S^2)$. Check that for any $p \in \mathrm{Sym}^\infty(S^2)$, $\pi^{-1}(p)$ is a circle.

(5) Given a based space (X, x_0), define the *based loop space of* (X, x_0) to be $\{f\colon ([0,1], \{0,1\}) \to (X, \{x_0\})\}$ with the compact-open topology.

(a) Check that Ω defines a functor $\mathcal{T}_* \to \mathcal{T}_*$.

(b) Prove that if $f, g\colon (X, x_0) \to (Y, y_0)$ are homotopic rel x_0 then $\Omega(f), \Omega(g)\colon \Omega(X, x_0) \to \Omega(Y, y_0)$ are homotopic rel $\Omega(x_0)$. Conclude that if X deformation retracts to x_0 then $\Omega(X, x_0)$ deformation retracts to $\Omega(x_0)$.

Remark. I will sometimes drop the basepoint from the notation $\Omega(X, x_0)$, writing simply $\Omega(X)$.

(6) Verify that $\Omega(S^1)$ has infinitely many connected components, each of which is contractible. (Hint: the winding number gives a continuous map $W\colon \Omega(S^1) \to \mathbb{Z}$. Using the fact that the universal cover of S^1 is contractible, show that $W^{-1}(n)$ is contractible for each n.)

(7) Prove that if S^2 is contractible then S^2 deformation retracts to a point. Conclude that S^2 is not contractible if and only if $\Omega(\mathrm{Sym}^\infty(S^2))$ is not contractible.

(8) Since S^1 is a group (as the unit complex numbers), $\Omega(S^1, 1)$ is naturally a group under pointwise multiplication. (Here, we use 1, the identity in S^1, as the basepoint.) Show that since S^1 acts continuously on S^∞ with quotient $\mathrm{Sym}^\infty(S^2)$, the group $\Omega(S^1)$ acts continuously on $\Omega(S^\infty)$ with quotient $\Omega(\mathrm{Sym}^\infty(S^2))$.

(9) Let $\Omega_0(S^1)$ denote the connected component of S^1 containing the identity. Show that $S^\infty/\Omega_0(S^1)$ is a nontrivial, connected covering space of $\Omega(\mathrm{Sym}^\infty(S^2))$.

(10) Conclude that S^2 is not contractible.

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