MATH W4051 PROBLEM SET 1
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Note: there are a lot of problems this week, but most of their solutions are fairly short. Please time how long it takes you to do the problem set: I'll ask you about this so I can adjust the length of future problem sets accordingly.

(1) Use the ϵ-δ definition of continuity to prove that the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is continuous. (If you find this hard, do several more, like $f(x) = x^3\colon \mathbb{R} \to \mathbb{R}$, $f(x, y) = x^2 + y^2\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x) = 1/x\colon (0, 1) \to (1, \infty)$, etc., until they become fairly easy.)

(2) Given metric spaces (X, d_X) and (Y, d_Y), we define the *product metric* $d_{X\times Y}$ on $X \times Y$ by setting

$$d_{X\times Y}((x, y), (x', y')) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}.$$

(a) Prove that $(X \times Y, d_{X\times Y})$ is, in fact, a metric space.

(b) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces, and $f\colon X \to Y$, $g\colon X \to Z$ continuous maps. Prove that the map $(f, g)\colon X \to Y \times Z$ is continuous (where $Y \times Z$ is given the product metric).

(3) Let X be a space, and d, d' metrics on X. The metrics d and d' are called *equivalent* if there are constants $c, C > 0$ such that for any $p, q \in X$, $cd(p, q) \leq d'(p, q) \leq Cd(p, q)$.

(a) Show that equivalence of metrics is, in fact, an equivalence relation on the set of metrics. (See Munkres, §3 for the definition of an equivalence relation.)

(b) Given metric spaces (X, d_X) and (Y, d_Y), define the *box metric* d_{box} on $X \times Y$ by

$$d_{box}((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}.$$

Is the product metric equivalent to the box metric? If so, prove it; if not, give a counter example.

(c) Using the definition of continuity for maps of metric spaces, prove that equivalent metrics induce the same continuous functions. That is, let d_X, d'_X be equivalent metrics on X and d_Y, d'_Y equivalent metrics on Y. Then a function $f\colon X \to Y$ is continuous with respect to d_X and d_Y if and only if f is continuous with respect to d'_X and d'_Y.

(d) Prove that equivalent metrics induce the same topology.

(e) Part (3d) implies part (3c); say why in one sentence.

(f) Some inequivalent metrics induce the same topology. Give an example of this.

(4) Given metric spaces (X, d_X) and (Y, d_Y), a map $f\colon X \to Y$ is called an *isometry* if

- f is surjective and
- for all $p, q \in X$, $d_Y(f(p), f(q)) = d_X(p, q)$.

Spaces X and Y are called isometric if there exists an isometry $f\colon X \to Y$.

(a) Show that if X and Y are isometric then X and Y are homeomorphic.

(b) For (X, d_X) a metric space, show that the set of isometries $X \to X$ forms a group (under composition of maps). (This should be easy.) We'll denote this group by $\mathrm{Isom}(X, d_X)$.

(c) Let X be an equilateral triangle in $\mathbb{R}^2$, given the metric induced from the standard metric on $\mathbb{R}^2$. Which group is $\mathrm{Isom}(X, d_X)$? (You don't have to prove your answer.)

(d) Say a sentence of two comparing the size of $\mathrm{Isom}(X, d_X)$ and $\mathrm{Homeo}(X, d_X)$.

(5) Homeomorphisms preserve topological properties. As an example of this, prove:

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be homeomorphic topological spaces. Then $(X,\mathcal{U})$ is metrizable if and only if $(Y,\mathcal{V})$ is metrizable.

(6) Let (X, d) be a metric space. A function $f\colon X \to \mathbb{R}$ is called *uniformly continuous* if for any $\epsilon > 0$ there exists $\delta > 0$ so that if $d(p, q) < \delta$ then $|f(p) - f(q)| < \epsilon$. (The difference from just plain continuity is that δ is not allowed to depend on p.)

(a) Uniform continuity of a function is not a purely topological property. Formulate precisely what this means. (Hint: compare with problem (5). Take your time: you'll want to re-word or re-think your statement several times before it's just right.)

(b) Prove it.

(7) Do Munkres problem 13.8.

(8) (Countability) Read Munkres Section 7. Do problems 7.1, 7.4.

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