MATH V2020 PROBLEM SET 9
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Revised. to make problem 3 a bit shorter.

(1) Prove that for an $n \times n$ matrix A, $e^{\mathrm{tr}(A)} = \det(e^A)$. (Hint: prove it first if A is in JNF. Then deduce the general result.)

(2) Consider the space $\mathcal{C}^\infty[-1,1]$ of smooth functions on $[-1,1]$, with inner product $\langle f, g\rangle = \int_{-1}^1 f(x)g(x)dx$.

(a) Prove that if $n \neq m \in \mathbb{N}$ then $\sin(\pi n x)$ is orthogonal to $\sin(\pi m x)$. Prove that $\sin(\pi n x)$ has length 1. Prove that for any n, m, $\sin(\pi n x)$ is orthogonal to $\cos(\pi m x)$. Prove that all of these functions are orthogonal to $f(x) = \frac{1}{\sqrt{2}}$.

Remark. Of course, it's also true that if $n \neq m \in \mathbb{N}$ then $\cos(\pi n x)$ is orthogonal to $\cos(\pi m x)$, and that $\cos(\pi m x)$ has length one.

Hint. There are some useful trig identities on the next page.

(b) Let $g(x) = |x|$.[1] Let

$$\begin{aligned} a_n &= \langle g(x), \sin(\pi n x)\rangle \\ b_n &= \langle g(x), \cos(\pi n x)\rangle \\ c &= \langle g(x), 1/\sqrt{2}\rangle. \end{aligned}$$

Compute a_n, b_n and c. *Hint.* One of the three integrals is obviously zero. For the others, you'll have to split it up into $\int_{-1}^0 + \int_0^1$ and use integration by parts. The answer is different for n odd and n even.

(c) The following theorem follows from general results in Fourier analysis:

Theorem 1. *For any* $x \in [-1,1]$,

$$g(x) = c/\sqrt{2} + \sum_{n=1}^\infty a_n \sin(\pi n x) + \sum_{n=1}^\infty b_n \cos(\pi n x).$$

What theorem for inner product spaces is this analogous to?

(d) What does the theorem say when you plug in $x = 0$?

(e) Use the previous part to prove that $\sum_{n \text{ odd}} 1/n^2 = \pi^2/8$.

(f) Use the previous part to prove that $\sum_{n=1}^\infty 1/n^2 = \pi^2/6$. (Hint: break the sum into the sum over n even and n odd. The sum over n even is equal to $1/4$ times the sum over all n.)

(g) *Optional* but encouraged: use a computer to plot the sum of the first three terms in the Fourier series for $|x|$, and the first six terms. Describe what you see. (For example, does the series seem to be converging faster to $|x|$ in some places than others?)

(3) Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{C}^\infty([0,1])$ by $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

[1]See remark on next page.

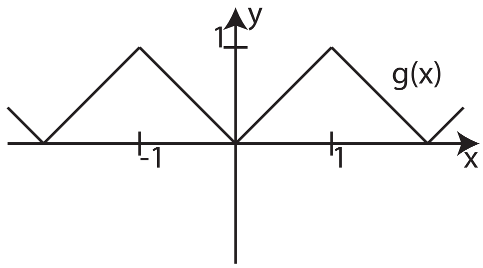

FIGURE 1. The saw-toothed function obtained by periodizing g.

(a) Compute $\langle 1, \sin(x)\rangle$ and $\langle x, \sin(x)\rangle$.
(b) Find the degree one polynomial $p(x)$ which approximates $\sin(x)$ as well as possible on $[0,1]$, in the sense that $\int_0^1 (p(x) - \sin(x))^2 dx$ is as small as possible.
(Suggestion: Project $\sin(x)$ onto the first two of the basis you found in Problem 9 last week. Use your answers from part (3a) and bilinearity of the inner product to avoid computing any new integrals in this part.)
(c) Optional: find the degree 2 polynomial which best approximates $\sin(x)$ in the same sense.

(4) Let
$$A = \begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}.$$
Find an orthogonal matrix P so that $P^{-1}AP$ is diagonal.

(5) Let
$$A = \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix}.$$
Find a unitary matrix U so that $U^{-1}AU$ is diagonal.

Useful trig identities. Recall that $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$. So,
$$\begin{aligned} \sin(\pi nx + \pi mx) &= \sin(\pi nx)\cos(\pi mx) + \sin(\pi mx)\cos(\pi nx) \\ \sin(\pi nx - \pi mx) &= \sin(\pi nx)\cos(\pi mx) - \sin(\pi mx)\cos(\pi nx) \\ \sin(\pi nx)\cos(\pi mx) &= \frac{1}{2}\left(\sin(\pi nx + \pi mx) + \sin(\pi nx - \pi mx)\right). \end{aligned}$$
Similarly, $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ so
$$\begin{aligned} \cos(pinx + \pi mx) &= \cos(\pi nx)\cos(\pi mx) - \sin(\pi nx)\sin(\pi mx) \\ \cos(pinx - \pi mx) &= \cos(\pi nx)\cos(\pi mx) + \sin(\pi nx)\sin(\pi mx) \\ \cos(\pi nx)\cos(\pi mx) &= \frac{1}{2}\left(\cos(pinx + \pi mx) + \cos(pinx - \pi mx)\right) \\ \sin(\pi nx)\sin(\pi mx) &= \frac{1}{2}\left(\cos(pinx - \pi mx) - \cos(pinx + \pi mx)\right). \end{aligned}$$

Remark. Regarding problem 2: from the perspective of Fourier series, we should really view the function g as a periodic function on $\mathbb{R}$ of period 2. That is, g is a saw-tooth function, as shown in Figure 1

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