MATH G4307 PROBLEM SET 9
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Exercises to turn in:

(E1) Hatcher Exercise 3.1.11 (p. 205)

(E2) Hatcher Exercise 3.1.2 (p. 204).

(E3) Hatcher Exercise 3.1.3 (p. 204).

(E4) Compute:

  (a) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/6)$.
  (b) $\mathrm{Ext}^n_R(R, M)$ for any ring R and R-module M.
  (c) $\mathrm{Ext}^n_{\mathbb{Q}[t]}(\mathbb{Q}[t]/(t^2+1), \mathbb{Q}[t])$ for each n.
  (d) $\mathrm{Ext}^n_{\mathbb{Q}[t]}(\mathbb{Q}[t]/(t^2+1), \mathbb{Q}[t]/(t^2+1))$ for each n.
  (e) $\mathrm{Ext}^n_{\mathbb{Q}[t]}(\mathbb{Q}[t]/(t^2+1), \mathbb{Q}[t]/(t))$ for each n.

(E5) Universal coefficients for homology.

  (a) Given chain complexes C_* and D_* over a ring R, define $C_* \otimes_R D_*$ to be the chain complex with
$$(C_* \otimes_R D_*)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j.$$
  Define $\partial_n \colon (C_* \otimes_R D_*)_n \to (C_* \otimes_R D_*)_{n-1}$ by
$$\partial(x \otimes y) = (\partial_C x) \otimes y + (-1)^{|x|} x \otimes (\partial_D y),$$
  and extending linearly. (Here, $|x|$ denotes the grading of x.)
  Verify that $(C_* \otimes D_*, \partial)$ is a chain complex.
  (b) Show that if $f \colon C_* \to C'_*$ then f induces a chain map $(f \otimes \mathbb{I}) \colon C_* \otimes_R D_* \to C'_* \otimes_R D_*$. Show that if f is homotopic to g then $(f \otimes \mathbb{I})$ is homotopic to $(g \otimes \mathbb{I})$. Show that if C_* is homotopy equivalent to C'_* then $C_* \otimes_R D_*$ is homotopy equivalent to $C'_* \otimes D_*$.
  (c) Define $\mathrm{Tor}^R_n(C_*, D_*)$ as follows. Let $f \colon P_* \to C_*$ be a projective resolution. Then $\mathrm{Tor}^R_n(C_*, D_*) = H_n(P_* \otimes D_*)$.
  Show that Tor^R_n is well-defined up to isomorphism, and that if C_* is quasi-isomorphic to C'_* then $\mathrm{Tor}^R_n(C_*, D_*) \cong \mathrm{Tor}^R_n(C'_*, D_*)$. (Hint: this should be very little work, using the previous part and what we proved in class.)
  (d) Let G and H be abelian groups. Show that $\mathrm{Tor}^R_n(G, H) = 0$ if $n > 1$.
  (e) Given a space X and abelian group G, define $C_n(X; G) = C_n(X) \otimes_{\mathbb{Z}} G$. Let $H_n(X; G) = H_*(C_n(X; G), \partial)$. Prove that
$$H_n(X; G) \cong (H_n(X; \mathbb{Z}) \otimes_{\mathbb{Z}} G) \oplus \mathrm{Tor}^{\mathbb{Z}}_1(H_{n-1}(X), \mathbb{Z}). \tag{0.1}$$

Remark. The isomorphism in Equation (0.1) is not natural, but there is a natural short exact sequence
$$0 \to H_n(X; \mathbb{Z}) \otimes_{\mathbb{Z}} G \to H_n(X; G) \to \mathrm{Tor}^{\mathbb{Z}}_1(H_{n-1}(X), \mathbb{Z}) \to 0.$$

(E6) Compute:
  (a) $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/4,\mathbb{Z}/6)$.
  (b) $\mathrm{Tor}_n^R(R,M)$ for any ring R and R-module M.
  (c) $\mathrm{Tor}_n^{\mathbb{Q}[t]}(\mathbb{Q}[t]/(t^2+1),\mathbb{Q}[t])$ for each n.
  (d) $\mathrm{Tor}_n^{\mathbb{Q}[t]}(\mathbb{Q}[t]/(t^2+1),\mathbb{Q}[t]/(t^2+1))$ for each n.
  (e) $\mathrm{Tor}_n^{\mathbb{Q}[t]}(\mathbb{Q}[t]/(t^2+1),\mathbb{Q}[t]/(t))$ for each n.

(E7) Hatcher Exercise 3.A.1.
(E8) Hatcher Exercise 3.A.2.
(E9) Hatcher Exercise 3.A.3.

Problems to think about but not turn in:

(P1) Let $R = \mathbb{Z}[x]/(x^2-1) = \mathbb{Z}[\mathbb{Z}/2]$ and let

$$C_* = \mathbb{Z}[x] \xleftarrow{1-x} \mathbb{Z}[x] \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

$$D_* = \mathbb{Z}[x]/(1-x) \xleftarrow{0} \mathbb{Z}[x]/(1-x) \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots .$$

Show that C_* and D_* have the same homology but C_* is not quasi-isomorphic to D_*. (Hint: compute $\mathrm{Tor}(C_*,\mathbb{Z}[x]/(x-1))$ and $\mathrm{Tor}(D_*,\mathbb{Z}[x]/(x-1))$, say.)

(P2) Suppose X and Y are CW complexes and $f\colon X\to Y$ is a cellular map.
  (a) The mapping cylinder $\mathrm{Cyl}(f)$ of f inherits a CW complex structure, with one cell for each cell of Y and two cells for each cell of X; explain how. What is the cellular chain complex for $\mathrm{Cyl}(f)$, in terms of $C_*^{cell}(X)$, $C_*^{cell}(Y)$ and f_*?
  (b) Inspired by the previous part, suppose C_* and D_* are chain complexes and $f\colon C_*\to D_*$ is a chain map. Define a new chain complex E_* and maps $g\colon C_*\to E_*$, $h\colon E_*\to D_*$ so that:
    - Each $g_n\colon C_n\to E_n$ is an inclusion, and $0\to C_n\to E_n\to E_n/C_n$ splits.
    - The map h is a homotopy equivalence.
    - The following diagram commutes:

$$C_n \xrightarrow{g} E_n \xrightarrow{h} D_n, \qquad h\circ g = f.$$

  (Note that we used a similar lemma in class. This construction is called the *algebraic mapping cone* of f.)
  (c) The mapping cone $\mathrm{Cone}(f)$ inherits a CW complex structure, with one cell for each cell of X and one cell for each cell of Y; explain how. What is the cellular chain complex for $\mathrm{Cone}(f)$, in terms of $C_*^{cell}(X)$, $C_*^{cell}(Y)$ and f_*?
  (d) Inspired by the previous part, suppose C_* and D_* are chain complexes and $f\colon C_*\to D_*$ is a chain map. Define a new chain complex E_* and chain maps $g\colon D_*\to E_*$, $h_n\colon C_{n+1}\to E_n$ (that is, h shifts degree by 1) so that

$$0\to D_n\to E_n\to C_{n+1}\to 0$$

  is exact.

(This construction is called the *algebraic mapping cylinder* of f.)

(P3) Exercise (E1) shows that the universal coefficient theorem is not natural (in X). Where did we lose naturality in the proof from class?

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