MATH G4307 PROBLEM SET 8
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Exercises to turn in:

(E1) Recall that given chain complexes C_* and D_* we defined $\mathrm{Mor}(C_*, D_*)$ to be the chain complex with k^{th} chain group

$$\mathrm{Mor}_k(C_*, D_*) = \bigoplus_i \mathrm{Hom}_{\mathbb{Z}}(C_i, D_{i-k})$$

and $d_k\colon \mathrm{Mor}_k(C_*, D_*) \to \mathrm{Mor}_{k+1}(C_*, D_*)$ given by $d_k(f) = \partial_D \circ f + (-1)^{k-1} f \circ \partial_C$.

Prove: given $f \in \mathrm{Mor}_k(C_*, D_*)$ and $g \in \mathrm{Mor}_\ell(D_*, E_*)$,

$$d_{\mathrm{Mor}(C_*,E_*)}(g \circ f) = (d_{\mathrm{Mor}(D_*,E_*)} g) \circ f + (-1)^\ell g \circ (d_{\mathrm{Mor}(C_*,D_*)} f).$$

(Hint: this is very easy.)

(E2) Prove that $C^n(X, A; G) \cong \mathrm{Hom}_{\mathbb{Z}}(C_n(X, A), G)$, and that this isomorphism commutes with the coboundary maps $d_n\colon C^n(X, A; G) \to C^{n+1}(X, A; G)$.

(Hint: this is also easy.)

(E3) (a) Suppose

$$A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is an exact sequence of R-modules (e.g., abelian groups), and M is an R-module. Then

$$\mathrm{Hom}_R(A, M) \xleftarrow{i^T} \mathrm{Hom}_R(B, M) \xleftarrow{p^T} \mathrm{Hom}_R(C, M) \leftarrow 0$$

is exact.

(b) Suppose

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is a short exact sequence of R-modules. If C is free then

$$0 \leftarrow \mathrm{Hom}_R(A, M) \leftarrow \mathrm{Hom}_R(B, M) \leftarrow \mathrm{Hom}_R(C, M) \leftarrow 0$$

is exact. (Hint: show that the sequence $0 \to A \to B \to C \to 0$ splits.)

(E4) Hatcher 3.1.4 (p. 205). (Hint: this is also very easy.)

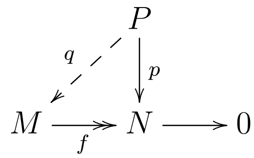
(E5) Hatcher 3.1.5 (p. 205).

(E6) Hatcher 3.1.6 (p. 205).

(E7) Hatcher 3.1.7 (p. 205).

Problems to think about but not turn in:

(P1) An R-module P is *projective* if given any modules M and N, a surjective map $f\colon M \to N$ and a map $p\colon P \to N$ there is a map $q\colon P \to M$ so that $p = f \circ q$, i.e.:



In Problem (E3), and in general when we're doing homological algebra in this course, the condition "free" could be replaced by "projective." Try solving Problem (E3) with "free" replaced by "projective"; it's easier, actually.

(P2) Change of coefficients. . .

(a) Suppose $f\colon G \to H$ is a map of abelian groups. Show: there is an induced map $f_*\colon H^n(X;G) \to H^n(X;H)$ for any topological space X.

(b) Suppose $0 \to G \to H \to K \to 0$ is a short exact sequence of abelian groups. Show that

$$0 \to H^n(X;G) \to H^n(X;H) \to H^n(X;K)$$

is exact. (In fact, you don't need that $H \to K$ is surjective, here.)

(c) In the situation of the previous part, find an example showing that

$$0 \to H^n(X;G) \to H^n(X;H) \to H^n(X;K) \to 0$$

need not be exact.

Remark. Though we wont focus much on change of coefficients in this class, the material is important. In particular, this is the basis for the definition of sheaf cohomology.

(P3) Read through the remaining problems in the section and do any that seem difficult, surprising or interesting.

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