MATH G4307 PROBLEM SET 2
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Exercises to turn in:

(E1) Hatcher 1.1.5 (p. 38). Be careful about basepoints.

(E2) Hatcher 1.1.7 (p. 38).

(E3) Hatcher 1.1.9 (p. 38).

(E4) Hatcher 1.1.18 (p. 39).

(E5) In class, we defined π_1 by using maps

$$\Delta\colon S^1 \to S^1 \vee S^1 \qquad r\colon S^1 \to S^1 \qquad \ell\colon S^1 \vee S^1 \to S^1.$$

We asserted that

$$\ell \circ (\mathbb{I} \otimes r) \circ \Delta = \iota \circ \epsilon = \ell \circ (r \otimes \mathbb{I}) \circ \Delta.$$

Prove this.

(E6) Using the previous problem, verify that every element of $\pi_1(X, x_0)$ has an inverse.

(E7) Let $H^1(X) = [X, S^1]$ denote the set of homotopy classes of continuous maps from X to S^1. (There are no basepoints around here.)
  (a) Recall that S^1 is a topological group. Use the group structure on S^1 to make $H^1(X)$ into a group. Note that this group is abelian for any X.
  (b) Compute $H^1(\{pt\})$. (This should be easy.)
  (c) Compute $H^1(S^1)$. (Use the fact that $\pi_1(S^1) \cong \mathbb{Z}$.)
  (d) Show that H^1 is *functorial* in the following sense: if $f\colon X \to Y$ is continuous then there is an induced map $f^*\colon H^1(Y) \to H^1(X)$. Moreover, if $g\colon Y \to Z$ then $(g \circ f)^* = f^* \circ g^*\colon H^1(Z) \to H^1(X)$. (This should be easy.)
  (e) Show that if $f \sim g$ then $f^* \sim g^*$. Conclude that if $X \simeq Y$ then $H^1(X) \cong H^1(Y)$. (This should be easy.)
  (f) Use H^1 to prove that there is no retraction $\mathbb{D}^2 \to S^1$, the key step in proving the Brouwer fixed point theorem.

Problems to think about but not turn in:

(P1) Here is an alternate approach to proving that homotopy equivalences induce isomorphisms on π_1. Suppose $f\colon (X, x_0) \to (Y, y_0)$ is a homotopy equivalence (not necessarily relative to x_0). Let $X' = X \amalg [0,1]/(0 \sim x_0)$ and $Y' = Y \amalg [0,1]/(0 \sim y_0)$. Let $x_1 = 1 \in X'$ and $y_1 = 1 \in Y'$. Observe that f extends in an obvious way to a map $f'\colon (X', x_1) \to (Y', y_1)$.
  (a) Show that (X', x_1) satisfies the homotopy extension property.
  (b) Show that X' deformation retracts to X.

(c) Show that (X', X) satisfies the homotopy extension property. (The previous part might help.) Conclude that f' is a homotopy equivalence.

(d) Use the previous parts, and Hatcher Proposition 0.19 to prove that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

(P2) Find other based spaces (X, x_0) so that $[(X, x_0), (Y, y_0)]$ is a group for any based space (Y, y_0).

(P3) If you haven't seen it before, read the definition of a *category* and a *functor*. Think about what categories and functors we have seen so far in this course.

(P4) Read through the remaining problems in this section, and do any that seem difficult or surprising.

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