# MATH G4307 FINAL EXAM
# FALL 2011

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V2: Problem 3(a) corrected.

**Instructions.** This exam is due (in my mailbox or e-mail box) by 5:00 p.m. on Monday, December 19. Please work on it by yourself. You are welcome to use Hatcher, but do not use other references (including the Internet). You are welcome to ask me or Kristen if you have questions or need a hint (but we may or may not answer). If you accidentally receive help from some other source, note that on your exam.

**Problems.** The ordering represents my impression of increasing difficulty.

(1) (Hatcher, 3.3.32, p. 260): Show that a compact manifold with boundary does not retract onto its boundary.

(2) For a topological space $X$ and basepoint $x_0 \in X$, $\pi_n(X, x_0)$ is the set of homotopy classes of maps from $S^n$ to $X$ sending the north pole $N \in S^n$ to $x_0 \in X$. The co-group structure on $S^n$ given by pinching a circle through $N$ (a map $S^n \to S^n \vee S^n$) makes $\pi_n(X, x_0)$ into a group.

(a) Show that for $n \geq 1$, the identity map $\mathbb{I}\colon S^n \to S^n$ generates a subgroup of $\pi_n(S^n)$ isomorphic to $\mathbb{Z}$. (Hint: this should be fairly easy using the tools we have learned.)

The Hurewicz theorem states that $\pi_n(S^n)$ is exactly this $\mathbb{Z}$; you may assume this for the rest of the problem. Also, let $[S^2] \in H^2(S^2)$ denote a generator (i.e., the dual of the fundamental class, or the Poincaré dual of the point class).

(b) Let $X$ be a 3-dimensional CW complex. Show that for any element $c \in H^2(X)$ there is a map $f\colon X \to S^2$ so that $f^*[S^2] = c$. (Hint: reduce to the case that $X$ has a unique 0-cell and no 1-cells, and use the Hurewicz theorem.)

(c) The following is false: For any connected 3-dimensional CW complex and any element $\alpha \in H_2(X)$ there is a map $f\colon S^2 \to X$ so that $\alpha = f_*[S^2]$ (where now $[S^2] \in H_2(S^2)$ is a generator). Give a counterexample, and prove it is a counterexample.

(d) The following is also false: for $X$ any 4-dimensional CW complex and $c \in H^2(X)$ there is a map $f\colon X \to S^2$ so that $f^*[S^2] = c$. Give a counterexample, and prove it is a counterexample.

(3) Let $S$ denote the 2-dimensional analogue of the Hawaiian earring,

$$E_\infty = \bigcup_{n=1}^{\infty} \{(x, y, z) \in \mathbb{R}^3 \mid (x - 1/n)^2 + y^2 + z^2 = 1/n^2\}.$$

Let $h\colon S^3 \to S^2$ denote the Hopf map (which is the attaching map for the 4-cell in $\mathbb{C}P^2$). There is a continuous map $\widetilde{h}\colon S^3 \to E_\infty$ so that the projection of $\widetilde{h}$ to any $S^2$ in $E_\infty$ is homotopic to the Hopf map. (You don't have to prove that, but it should be clear and easy to prove.)

Let $C_{\widetilde{h}}$ denote the mapping cone of $\widetilde{h}$.

(a) Prove that $H^2(C_{\widetilde{h}})$ contains an infinitely-generated subgroup $\mathbb{Z}\langle \xi_1, \xi_2, \dots \rangle$ and $H^4(C_{\widetilde{h}}) \cong \mathbb{Z}\langle \eta \rangle$ where

$$\xi_i \cup \xi_i = \eta \qquad \text{and} \qquad \xi_i \cup \xi_j = 0 \text{ if } i \neq j.$$

(b) Let $[S^3]$ denote the fundamental class of $S^3$. Suppose that $\widetilde{h}_*[S^3] = 0 \in H_3(E_\infty)$. Then there is a finite simplicial complex $X$ with boundary $S^3$ so that $\widetilde{h}$ extends to a map $k\colon X \to E_\infty$. (You don't have to prove that.) Let $Y = X \cup \mathbb{D}^4$, where the $\mathbb{D}^4$ is glued to $S^3$ in the obvious way. Then $k$ extends to a map $\ell\colon Y \to C_{\widetilde{h}}$, sending the $B^4$ to the cone $S^3 \times [0,1]/S^3 \times 1$. Prove that $\ell^*(\eta)$ is a nontrivial element of $H^4(Y)$.

(c) Prove that the (infinitely many) elements $\ell^*(\xi_i)$ are all linearly independent. (Hint: use naturality of the cup product.)

(d) Since $Y$ was a finite CW complex, $H^2(Y)$ is finitely generated. So, you have a contradiction. What (counter-intuitive) result that I mentioned several times in class have you proved?

(4) Suppose that $K_1$ and $K_2$ are embedded circles in $S^3$ (i.e., knots), and $K_1$ and $K_2$ are disjoint. We can define the *linking number* of $K_1$ and $K_2$ in two different ways:

(a) By Alexander duality, $H^1(S^3 \setminus K_1) \cong H_1(K_1) \cong \mathbb{Z}$. Let $\ell$ be a generator of $H^1(S^3 \setminus K_1)$. The knot $K_2$ gives a class $i_*[K_2] \in H_1(S^3 \setminus K_1)$, where $i\colon K_2 \to S^3$ denotes inclusion. The *linking number* of $K_1$ and $K_2$ is

$$lk(K_1, K_2) = c(i_*[K_2]).$$

(b) Let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be projection onto the $xy$-plane. Deforming $K_1$ and $K_2$ if necessary, we can arrange that $K_1, K_2 \subset \mathbb{R}^3 \subset S^3$, $\pi(K_1)$ is transverse to $\pi(K_2)$, and $\pi(K_1)$ and $\pi(K_2)$ meet only in double points. So, we can record $K_1$ and $K_2$ by a link diagram, as in Figure 1; the breaks indicate which strand is lower (has small $z$-coordinate). Orient $K_1$ and $K_2$. Then each crossing has one of the two forms shown in Figure 2. Given a crossing $c$, define $\epsilon(c)$ as in Figure 2. Then

$$lk'(K_1, K_2) = \sum_{\text{Crossings } c \text{ between } K_1 \text{ and } K_2} \epsilon(c).$$

(Note that this sum does *not* include places $K_1$ crosses over itself or $K_2$ crosses over itself.)

Prove that these definitions of linking number agree, at least up to a sign. (You do not have to prove that the second definition is well-defined, though you probably get that as a byproduct.)

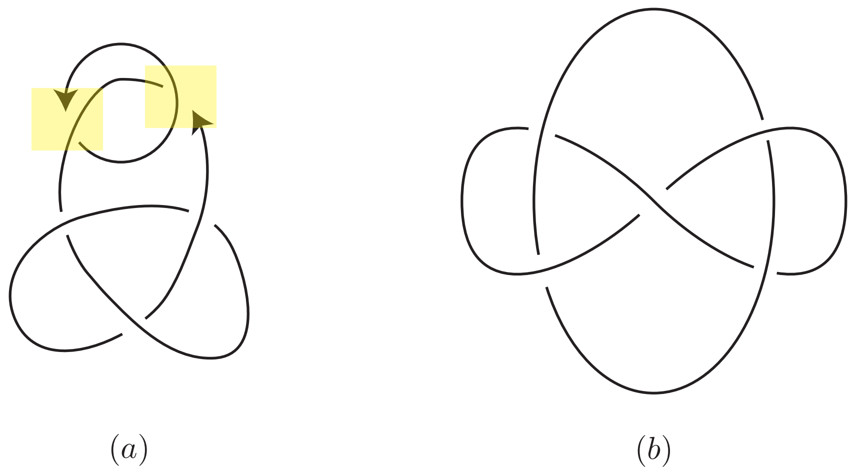


FIGURE 1. **Link diagrams.** (a) A link with linking number 1. The crossings contributing to the linking number (according to the second definition) are highlighted. (b) The Whitehead link, which has linking number 0.

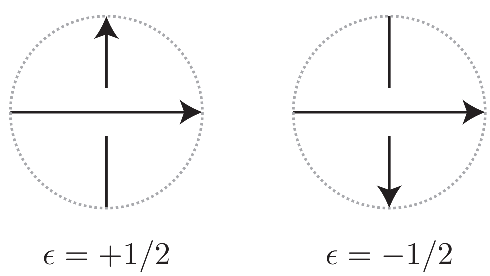


FIGURE 2. **Local contributions of crossings.** Only crossings in which the two strands are from different link components contribute to the linking number.

*Remark.* The example in Problem (3) is a special case of Barratt-Milnor, "An Example of Anomalous Singular Homology", Proceedings of the AMS 13 (2), 1962, 293–297. The simplified proof in this case was explained to me by Greg Brumfiel.

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