MATH W4051 PROBLEM SET 8 PART 2 OF 2
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Don't forget to do part 1, which was already posted!

(1) Munkres 68.2.
(2) Munkres 69.1.
(3) Munkres 69.3.
(4) Munkres 69.4.
(5) Let $G = \langle g_1, g_2, \dots, g_k \mid r_1, r_2, \dots, r_l \rangle$ be a group given in terms of generators and relations. Write $r_i = g_{i,1}^{n_{i,1}} g_{i,2}^{n_{i,2}} \dots g_{i,j_i}^{n_{i,j_i}}$.

Let H be any group, and $h_1, \dots, h_k \in H$. Then there is a group homomorphism $f\colon G \to H$ such that $f(g_i) = h_i$ $(i = 1, \dots, k)$ if and only if, for all

$$h_{i,1}^{n_{i,1}} h_{i,2}^{n_{i,2}} \dots h_{i,j_i}^{n_{i,j_i}} = 1_H$$

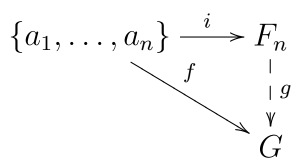
for $i = 1, \dots, l$.

Prove this. (Hint: one direction is easy. For the other, you'll use the definition of G as a quotient group of a free group, and probably the property of free groups in Optional Problem (6), below.)

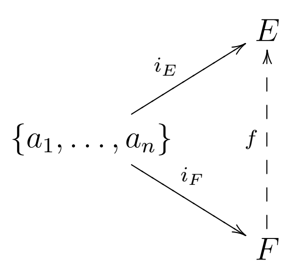
Optional:

(6) Here's another abstract description of free groups. The free group F_n on n symbols $a_1, \dots, a_n$ is characterized as follows: there is a map *of sets* $i\colon \{a_1, \dots, a_n\} \to F_n$, and for any group G and map *of sets* $f\colon \{a_1, \dots, a_n\} \to G$ there is a unique map $g\colon F_n \to G$ such that $f = g \circ i$.

In terms of diagrams:



(a) Prove that this property characterizes F_n up to unique isomorphism. That is, given any two groups E and F and maps $i_E\colon \{a_1, \dots, a_n\} \to E$ and $i_F\colon \{a_1, \dots, a_n\} \to F$ satisfying the condition given above there is a unique isomorphism $f\colon E \to F$ so that the following diagram commutes:



(b) Explain briefly that $\mathbb{Z}$ has this property for $n = 1$, so $\mathbb{Z} \cong F_1$.

(c) Explain why F_2, as defined in class, has this property for $n = 2$.

(7) Let $GL_n(\mathbb{R})$ denote the set of invertible $n \times n$ matrices, which we topologize as a subspace of $\mathbb{R}^{n^2}$. Let $O_n(\mathbb{R})$ denote the subgroup of $GL_n(\mathbb{R})$ of $n \times n$ orthogonal matrices (i.e., matrices P so that $P^T P = I$), topologized as a subspace.
 (a) Prove that $GL_n(\mathbb{R})$ deformation retracts to $O_n(\mathbb{R})$. (Hint: one way to do this is by doing the Gram-Schmidt process gradually to the columns.)
 (b) Prove that $O_n(\mathbb{R})$ has two connected components. (Hint: to see it has at least two, consider the determinant. To see it has at most two, use the spectral theorem. For the latter, you could restrict to the case $n = 3$ if you prefer.)

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