MATH W4051 PROBLEM SET 5
DUE OCTOBER 13, 2009.

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Essential problems (5-10 pts each):

(1) Munkres 26.4
(2) Munkres 28.6
(3) Munkres 29.1
(4) Munkres 30.3
(5) Munkres 31.1
(6) Munkres 31.2

Quasi-optional problems (2 pts each):

(6) Munkres 29.8
(7) Munkres 31.5
(8) In this problem, we'll exploit the version of the Baire category theorem that you proved last week in the homework, a little bit. (This is meant to give a bit of a flavor of how it's used, though our applications here are not so impressive.)

First, some terminology. Let $\mathbb{R}[x_1, \dots, x_n]$ denote the set (or ring) of polynomials in n variables, with real coefficients. Given a polynomial $p(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$, let $V(p) = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid p(a_1, \dots, a_n) = 0\}$. More generally, given a *set* of polynomials $S = \{p_i\} \subset \mathbb{R}[x_1, \dots, x_n]$ let

$$V(S) = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \forall i,\ p_i(a_1, \dots, a_n) = 0\} = \bigcap_{p_i \in S} V(p_i).$$

The sets $V(S)$ are called *real algebraic varieties*, or *algebraic subvarieties of* $\mathbb{R}^n$

As a degenerate case, $V(0) = \mathbb{R}^n$. If any of the p_i are nonzero then we call $V(\{p_1, \dots, p_k\})$ a proper algebraic subvariety of $\mathbb{R}^n$.

(a) Prove: let $p(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ be a nonzero polynomial. Then $V(p)$ is a closed set with empty interior.
Hint: the "closed" part should be easy. For the empty interior, think about the derivatives.
(b) Prove: $\mathbb{R}^n$ is not a countable union of proper algebraic subvarieties of $\mathbb{R}^n$.
(c) A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *real analytic* if for each $(x_1, \dots, x_n) \in \mathbb{R}^n$ is an open neighborhood $U \ni (x_1, \dots, x_n)$ so that the restriction of f to U, $f|_U$, is given by a convergent power series. That is, there are numbers $a_{i_1,\dots,i_n} \in \mathbb{R}$ such that for $(y_1, \dots, y_n) \in U$,

$$f(y_1, \dots, y_n) = \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n}(y_1 - x_1)^{i_1} \cdots (y_n - x_n)^{i_n}$$

(and the right-hand side is a convergent series).

A *real analytic subvariety of* $\mathbb{R}^n$ is the zero set of a real analytic function, i.e., a set of the form $V(f) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f(x_1, \dots, x_n) = 0\}$. It is a proper real analytic subvariety if f is not the zero function.

Prove: $\mathbb{R}^n$ is not a countable union of nontrivial real analytic varieties.

Hint: first show that for any f, $V(f)$ is a countable union of closed sets with empty interiors. Also, feel free to use facts from analysis like that real analytic functions are continuous, and in convergent power series you can move derivatives past $\sum$'s.

(d) The *Zariski topology* on $\mathbb{R}^n$ is the topology where the closed sets are exactly the real algebraic varieties. Prove that the Zariski topology is strictly coarser than the usual metric topology.

(e) Is the Zariski topology Hausdorff? T_1? Is $\mathbb{R}^n$ connected in the Zariski topology?

(f) We have seen the Zariski topology on $\mathbb{R}^1$ before, under a different name. What familiar topology is it?

Remark. Let F be any field. One can define the Zariski topology on F^n similarly to the way we did above. This allows one to use topology to study zero sets of polynomials over fields other than $\mathbb{R}$ (e.g., finite fields), and forms the basis for modern algebraic geometry.

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