MATH W4051 PROBLEM SET 11
DUE DECEMBER 15, 2009.

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(1) Let $p\colon \tilde{X} \to X$ be a covering space. Prove: if $\tilde{X}$ is compact then for any $x \in X$, $p^{-1}(x)$ is finite.

(2) Draw the universal covers of the following spaces:
   (a) The union of a sphere and a diameter of the sphere.
   (b) A sphere and a circle glued together at one point.

(3) Find a 2-fold cover of the Möbius strip by an orientable surface. Find a 2-fold cover of the Klein bottle by an orientable surface. (I know, we haven't defined orientable.) To what subgroups of π_1 do these covers correspond?

(4) Prove that any map $\mathbb{R}P^2 \to S^1$ is nullhomotopic. (Hint: what is the induced map on π_1?)

Optional:

(1) Recall from problem set 3: The group $\mathbb{Z}/5$ acts on S^3 as follows. Write $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $\mathbb{Z}/5 = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$. Then define
$$\zeta^n(z, w) = (e^{2\pi i n/5} z, e^{6\pi i n/5} w).$$
The quotient S^3/G is called the *lens space* $L(5, 3)$.
   (a) Prove that S^3 is the universal cover of $L(5, 3)$. (The work here is proving that the map $S^3 \to L(5, 3)$ is a covering map.)
   (b) Use covering space theory to compute $\pi_1(L(5, 3))$.
   (c) Generalize this to define lens spaces $L(p, q)$ for any relatively prime p and q. Use covering spaces to compute $\pi_1(L(p, q))$.

(2) (If you've taken a course in differential geometry. . .) Prove that if M is a non-orientable smooth manifold then M has a 2-fold cover by an orientable manifold. (If you want a hint, ask.)

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