CALCULUS III: HOMEWORK 4

This assignment is due on Thursday, July 31th.

There is an additional component to the homework which you must access through WebAssign. If you are auditing the class and so have not purchased WebAssign, but would like additional problems to help with following the material, please contact me by email.

(1) Do the following exercises from the book: 14.2 Exercise 38, 14.4 Exercise 40, 14.4 Exercise 46.

(2) We have seen before the functions of trace and determinant on matrices. Recall that the trace takes as input a matrix $X = [x_{ij}]$ and outputs the sum of its diagonal elements $\text{Tr}(X) = \sum_{i=1}^{n} x_{ii}$: that is, it is a function

$$\text{Tr} : \mathbb{R}^{n^2} \rightarrow \mathbb{R}$$

$$X \mapsto \text{Tr}(X)$$

(The $\mathbb{R}^{n^2}$ is because we are thinking of a function on $n \times n$ matrices as a function on its $n^2$ entries.) Similarly, the determinant, as we described it in class, is a function

$$\det : \mathbb{R}^{n^2} \rightarrow \mathbb{R}$$

$$X \mapsto \det(X)$$

(a) (2 × 2 matrices) Show that, if we write

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

then

$$d \det = x_{22}dx_{11} - x_{21}dx_{12} - x_{12}dx_{21} + x_{11}dx_{22}$$

so in particular if we evaluate at $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we find that

$$d \det |_{I_2} = dx_{11} + dx_{22}$$

(b) (3 × 3 matrices) Let

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

find a formula for $d \det$s above, and use that to conclude that, evaluating at

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$d \det |_{I_3} = dx_{11} + dx_{22} + dx_{33}$$

(c) ($n \times n$ matrices) (Extra credit.) Show, by induction on $n$, that

$$d \det |_{I_n} = dx_{11} + dx_{22} + ... + dx_{nn}$$

so that, in general “the derivative of the determinant at the identity is the trace.”