You have 95 minutes to do the following six problems. Write solutions in the space provided. If you need more space, use the back of the page. No calculators or references may be used. Simplify your answers. Show your work. Good luck. (NB: This practice is a little harder than the real midterm, so if you can do it, you’ll be fine.)

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1. Evaluate the integrals
(a). [10 points]
\[ \int_{2}^{3} \frac{(\ln x)^2}{x^2} dx \]
Make the substitution \( u = \ln x \) and \( du = \frac{1}{x} \) \( dx \). Then our integral becomes, since \( \frac{1}{x} = e^{-u} \)

\[ \int_{2}^{3} \frac{(\ln x)^2}{x^2} dx = \int_{2}^{3} (\ln x)^2 \frac{1}{x} \frac{1}{x} dx = \int_{\ln 2}^{\ln 3} u^2 e^{-u} du \]

We integrate this by parts twice to see

\[ \int_{\ln 2}^{\ln 3} u^2 e^{-u} du = \left[ - (u^2 + 2u + 2) e^{-u} \right]_{\ln 2}^{\ln 3} \]

\[ = \frac{1}{2} ((\ln 2)^2 + 2 \ln 2 + 2) - \frac{1}{3} ((\ln 3)^2 + 2 \ln 3 + 2) \]

\[ = \frac{1}{2} (\ln 2)^2 + \ln 2 - \frac{1}{3} (\ln 3)^2 - \frac{2}{3} \ln 3 + \frac{1}{3} \]

(b). [5 points]

\[ \int \tan^4 \theta \sec^2 \theta d\theta \]

Make the substitution \( u = \tan \theta, \ du = \sec^2 \theta d\theta \). Then we have

\[ \int \tan^4 \theta \sec^2 \theta d\theta = \int u^4 du = \frac{u^5}{5} + C = \frac{1}{5} \tan^5 \theta + C \]

(c). [5 points]

\[ \int \frac{2t}{t^4 - 6t^2 + 13} dt \]

We complete the square. Thus

\[ t^4 - 6t^2 + 13 = t^4 - 6t^2 + 9 + 4 = (t^2 - 3)^2 + 4 \]

So

\[ \int \frac{2t}{t^4 - 6t^2 + 13} dt = \int \frac{2t}{(t^2 - 3)^2 + 4} dt \]

\[ = \frac{1}{2} \int \frac{t}{(t^2 - 3)^2 + 1} dt \]

taking \( u = (t^2 - 3)/2 \) and \( du = t dt \), we see

\[ \int \frac{2t}{t^4 - 6t^2 + 13} dt = \frac{1}{2} \int \frac{1}{u^2 + 1} du \]

\[ = \frac{1}{2} \tan^{-1} \left( \frac{t^2 - 3}{2} \right) + C \]
Compute the integral \( \int e^{\pi x} \sin 3x \cos 3x \, dx \) in two different ways.

(a). Use Euler’s identity. [10 points]

In both, we should first simplify by the identity \( 2 \sin \theta \cos \theta = \sin 2\theta \). This gives us

\[
\int e^{\pi x} \sin 3x \cos 3x \, dx = \frac{1}{2} \int e^{\pi x} \sin 6x \, dx
\]

\[
= \frac{1}{2} \int \text{Im} \left( e^{(\pi + 6i)x} \right) \, dx \\
= \frac{1}{2} \text{Im} \left( \int e^{(\pi + 6i)x} \, dx \right) \\
= \frac{1}{2} \text{Im} \left( \frac{1}{\pi + 6i} e^{(\pi + 6i)x} \right) + C \\
= \frac{1}{2 (\pi^2 + 36)} e^{\pi x} \text{Im} \left( (\pi - 6i) (\cos 6x + i \sin 6x) \right) \\
= \frac{1}{2 (\pi^2 + 36)} e^{\pi x} (\pi \sin 6x - 6 \cos 6x)
\]

(b). Use Integration by Parts. [10 points]

We compute

\[
\int e^{\pi x} \sin 6x \, dx = \frac{1}{\pi} e^{\pi x} \sin 6x - \frac{6}{\pi} \int e^{\pi x} \cos 6x \\
= \frac{1}{\pi} e^{\pi x} \sin 6x - \frac{6}{\pi} \left( \frac{1}{\pi} e^{\pi x} \cos 6x + \frac{6}{\pi} \int e^{\pi x} \sin 6x \right)
\]

so

\[
\left( 1 + \frac{36}{\pi^2} \right) \int e^{\pi x} \sin 6x \, dx = \frac{1}{\pi^2} e^{\pi x} (\pi \sin 6x - 6 \cos 6x)
\]

i.e.

\[
\int e^{\pi x} \sin 6x \, dx = \frac{1}{(\pi^2 + 36)} e^{\pi x} (\pi \sin 6x - 6 \cos 6x) + C
\]

so, since \( \int e^{\pi x} \sin 3x \cos 3x \, dx = \frac{1}{2} \int e^{\pi x} \sin 6x \, dx \), we find

\[
\int e^{\pi x} \sin 3x \cos 3x \, dx = \frac{1}{2 (\pi^2 + 36)} e^{\pi x} (\pi \sin 6x - 6 \cos 6x) + C
\]
as above.
3. [10 points] Compute

\[
\frac{d}{dx} \int_{1}^{x/\ln x} \ln t \, dt
\]

Let \( F(t) \) be an antiderivative for \( \ln t \). For example, \( F(t) = t \ln t - t \) will do. Then

\[
\int_{1}^{x/\ln x} \ln t \, dt = F\left(\frac{x}{\ln x}\right) - F(1),
\]

so

\[
\frac{d}{dx} \int_{1}^{x/\ln x} \ln t \, dt = \frac{d}{dx} \left( F\left(\frac{x}{\ln x}\right) - F(1) \right)
\]

\[
= F'\left(\frac{x}{\ln x}\right) \frac{d}{dx} \left( \frac{x}{\ln x} \right) = \ln \left( \frac{x}{\ln x} \right) \frac{\ln x - 1}{\ln^2 x}
\]
4. Derive the following identities using Euler’s identity $e^{ix} = \cos x + i \sin x$.

(a). [5 points]

\[ \cos 3x = \cos^3 x - 3 \cos x \sin^2 x \]

We know $\cos 3x = \Re \left( e^{3ix} \right) = \Re \left( (\cos x + i \sin x)^3 \right)$. So we expand

\[ (\cos x + i \sin x)^3 = \cos^3 x + 3 \cos x \sin^2 x + 3 \cos x (i \sin x)^2 + (i \sin x)^3 \]

\[ = (\cos^3 x - 3 \cos x \sin^2 x) + i (3 \cos^2 x \sin x + \sin^3 x) \]

taking real parts gives the answer.

(b). [10 points]

\[ \sin A \cos B = \frac{1}{2} [\sin (A - B) + \sin (A + B)] \]

We consider $\frac{1}{2} \left( e^{i(A-B)} + e^{i(A+B)} \right)$, since $\text{Im} \left( \frac{1}{2} \left( e^{i(A-B)} + e^{i(A+B)} \right) \right)$ is the left hand side. Well

\[ \frac{1}{2} \left( e^{i(A-B)} + e^{i(A+B)} \right) = e^{iA} \frac{1}{2} (e^{iB} + e^{-iB}) \]

Now, $\frac{1}{2} (e^{iB} + e^{-iB}) = \cos B$ by real-imaginary decomposition given in class. So

\[ \frac{1}{2} \left( e^{i(A-B)} + e^{i(A+B)} \right) = e^{iA} \cos B = \cos A \cos B + i \sin A \cos B \]

taking imaginary parts gives the answer.

(c). [5 points]

\[ \cot x = \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} \]

We know $\cot x = \frac{\cos x}{\sin x}$ but by the real-imaginary decomposition,

\[ \cos x = \frac{e^{ix} + e^{-ix}}{2} \]
\[ \sin x = \frac{e^{ix} - e^{-ix}}{2i} \]

dividing these expressions gives the result.
5. [15 points] A cylindrical water tank of radius $r$ is leaking water from the faucet. The rate of change of the total volume of water is

$$V'(t) = \begin{cases} \quad -V_0 e^{-rt} & \text{for } 0 \leq t < T \\ 0 & \text{for } t \geq T \end{cases}$$

That is, the rate decays exponentially until a time $T$, after which no more water leaks from the tank. Given that at time $t = 0$ the volume of water in the tank is $V_0$, find a formula for the total volume of water $V(t)$, valid for all $t \geq 0$.

We integrate to find an antiderivative, then shift by a constant to satisfy the initial condition $V(0) = V_0$. For $0 \leq t \leq T$, an antiderivative is given by

$$g(t) = \int_0^t -V_0 e^{-ru} du = \frac{V_0}{r} e^{-rt}$$

we want to add a constant $C$ so that $g(0) + C = V(0)$. Writing down this equation

$$\frac{V_0}{r} + C = V_0$$

so $C = V_0 - V_0/r$. So certainly for time $0 \leq t \leq T$, we have

$$V(t) = \frac{V_0}{r} e^{-rt} + V_0 - V_0/r$$

Now, for $t \geq T$, $V'(t) = 0$, so the water is neither leaking in or out from then on. So we have in total

$$V(t) = \begin{cases} \quad \frac{V_0}{r} e^{-rt} + V_0 - V_0/r & \text{for } 0 \leq t \leq T \\ \frac{V_0}{r} e^{-rT} + V_0 - V_0/r & \text{for } t \geq T \end{cases}$$
6. [15 points] Evaluate the integral
\[ \int \frac{x^4 - 3x^2 + 11}{(x - 1)(x^2 + 1)} \, dx \]

We do polynomial long division to write
\[ \frac{x^4 - 3x^2 + 11}{(x - 1)(x^2 + 1)} = x + 1 + \frac{-3x^2 + 12}{(x - 1)(x^2 + 1)} \]

This last term we do partial fractions on: we write
\[ \frac{-3x^2 + 12}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{ax + b}{x^2 + 1} \]
cross multiplying gives \( Ax^2 + A + ax^2 + bx - ax - b = -3x^2 + 12 \). So
\[
\begin{align*}
A + a &= -3 \\
A - b &= 12 \\
b - a &= 0
\end{align*}
\]
so \( 2A = 9 \), \( A = 9/2 \), \( a = b = -15/2 \), i.e.
\[
\frac{-3x^2 + 12}{(x - 1)(x^2 + 1)} = \frac{9}{2} \frac{1}{x - 1} - \frac{15}{2} \frac{x + 1}{x^2 + 1}
\]

Therefore, in whole, we have
\[
\frac{x^4 - 3x^2 + 11}{(x - 1)(x^2 + 1)} = x + 1 + \frac{-3x^2 + 12}{(x - 1)(x^2 + 1)}
= x + 1 + \frac{9}{2} \frac{1}{x - 1} - \frac{15}{2} \frac{x + 1}{x^2 + 1}
\]

Integrating gives us
\[
\int \frac{x^4 - 3x^2 + 11}{(x - 1)(x^2 + 1)} \, dx = \frac{x^2}{2} + x + \frac{9}{2} \ln |x - 1| - \frac{15}{4} \ln (x^2 + 1) - \frac{15}{2} \tan^{-1} x + C
\]