Homework 1
Due Tuesday July 12

Please show your work on the problems from the book. We can’t give partial credit if there’s no work.
From Stewart:
§5.3: 8, 13, 46, 76, 77 (hint: try differentiating)
§5.4: 40 (remember the definitions of sinh x and cosh x), 54
§5.5: 13, 21, 45, 89, 90 (hint: use the rule sin (a + b) = sin a cos b + sin b cos a to see that sin (π – u) = sin u)
Chapter 5 Review: 20, 38
§7.1: 24, 33, 48, 52, 67

Additional Problems:
1. I promised you a formula for
\[ \int p(t) e^t dt \]
where p(t) is a polynomial. Here it is. Let n be the degree of the polynomial p(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_0, that is, the highest power of t in the polynomial (here a_n ≠ 0). Then
\[ \int p(t) e^t dt = \left( \sum_{k=0}^{n} (-1)^k p^{(k)}(t) \right) e^t + C \]
We’re going to prove this.
(a). Consider the formula of 7.1.52,
\[ \int t^n e^t dt = t^n e^t - n \int t^{n-1} e^t \]
Use this to show
\[ \int t^n e^t dt = \left( \sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k} (t^n) \right) e^t + C \]
While I would prefer a completely rigorous proof, I am ok with you just writing in words how to deduce this formula from the one in 7.1.52. If rigor is your thing, use an induction argument.

(b). Why does this prove the formula for general polynomials? (hint: How do derivatives work with sums and multiplication by constants?)
(c). Write down and prove a similar formula for
\[ \int p(t) e^{at} dt \]
where a is a constant.

2. On the first day, I computed some Riemann sums to approximate the definite integral
\[ \int_0^1 x^2 dx \]
(a). What is this value?
Now that we know the answer, let’s prove this, but directly, without using the Fundamental Theorem of Calculus. Take the following partition \( \mathcal{P}_n \) of the interval \([0, 1] : 0 < \frac{1}{n} < \frac{2}{n} < ... < \frac{n-1}{n} < 1 \). That is, partition the unit interval into smaller intervals of size \( \frac{1}{n} \). That means when we write \( 0 = x_0 < x_1 < ... < x_n = 1 \), that \( x_k = \frac{k}{n} \).
(b). In each interval \([x_{k-1}, x_k]\), for what value of \(x_k^{\text{min}} \in [x_{k-1}, x_k]\) is \(f(x_k^{\text{min}}) = \min_{x \in [x_{k-1}, x_k]} f(x)\). That is, at what point in the interval \([x_{k-1}, x_k]\) does \(f(x)\) hit its minimum value. What about its maximum value? What are these maximum and minimum values in each interval \([x_{k-1}, x_k] = \left[\frac{k-1}{n}, \frac{k}{n}\right]\)? What is \(\Delta x_k = x_k - x_{k-1}\)?

(c). Write down the Riemann sums that I called \(S\) and \(\overline{S}\) in class, i.e.

\[
S = S(P_n) = \sum_{k=1}^{n} f(x_k^{\max}) \Delta x_k
\]

\[
\overline{S} = \overline{S}(P_n) = \sum_{k=1}^{n} f(x_k^{\min}) \Delta x_k
\]

for the values of \(f(x_k^{\max}), f(x_k^{\min}), \text{ and } \Delta x_k\) you computed in (b). Note that we are using the notation \(S(P_n)\) and \(\overline{S}(P_n)\) to emphasize the dependence the quantities I in class called \(S\) and \(\overline{S}\) have on the partition \(P_n\) we took.

(d). Why do we have the following inequalities? (hint: it’s a stupidly easy reason)

\[
\overline{S}(P_n) \leq \int_{0}^{1} x^2 \, dx
\]

\[
\int_{0}^{1} x^2 \, dx \leq S(P_n)
\]

(e). Use the formula

\[
\sum_{k=1}^{n} k^2 = \frac{1}{3} n (n + 1) \left( n + \frac{1}{2}\right)
\]

to write \(S(P_n)\) and \(\overline{S}(P_n)\) as functions of \(n\).

(f). Take the limit as \(n \to \infty\) of both, and note that they both tend towards the value in (a). This means, by the Squeeze Theorem that \(\int_{0}^{1} x^2 \, dx = \int_{0}^{1} x^2 \, dx\). So we have shown from scratch that \(x^2\) is integrable on \([0, 1]\) and that its integral is what the Fundamental Theorem says it is.

**Extra Credit:** Prove the formula \(\sum_{k=1}^{n} k^2 = \frac{1}{3} n (n + 1) \left( n + \frac{1}{2}\right)\). There are lots of nice ways to see this, so I won’t spoil it by giving hints.