



Kadomtsev–Petviashvili Turning Points and CKP Hierarchy

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Abstract: A characterization of the Kadomtsev–Petviashvili hierarchy of type C (CKP) in terms of the KP tau-function is given. Namely, we prove that the CKP hierarchy can be identified with the restriction of odd times flows of the KP hierarchy on the locus of turning points of the second flow. The notion of CKP tau-function is clarified and connected with the KP tau function. Algebraic–geometrical solutions and in particular elliptic solutions are discussed in detail. A new identity for theta-functions of curves with holomorphic involution having fixed points is obtained.

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1. Introduction

The Kadomtsev–Petviashvili (KP) hierarchy is one of the most fundamental in the modern theory of integrable systems. It has at least three well-known definitions/representations. In its original, the so-called Zakharov–Shabat form [1], it is an infinite system of equations on an infinite number of variables which are the coefficients of monic ordinary linear differential operators

$$B_k = \partial_x^k + \sum_{i=0}^{k-2} u_{k,i}(x, \mathbf{t}) \partial_x^i \tag{1.1}$$

depending on x and an infinite set of “times” $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$. The equations of the hierarchy are equivalent to the operator equations

$$\partial_{t_l} B_k - \partial_{t_k} B_l + [B_k, B_l] = 0, \quad \text{for all pairs } k, l. \tag{1.2}$$

For each pair (k, l) the operator equation (1.2) is equivalent to a system of partial differential equations on the coefficients of the operators B_k, B_l . The system is well-defined in the sense that the number of equations is equal to the number of unknown functions. For example, for the case $k = 2, l = 3$ in which $B_2 = \partial_x^2 + 2u$ and $B_3 = \partial_x^3 + 3u\partial_x + w$ Eq. (1.2) is equivalent to a system of two equations for u and w . After eliminating w from this system, and after the change of the notation for independent variables $t_2 = y, t_3 = t$, the remaining equation for u becomes the original KP equation

$$3u_{yy} = (4u_t - 12uu_x - u_{xxx})_x. \tag{1.3}$$

The second form of the KP hierarchy (which is often called the Sato form) was introduced in [2] as a system of commuting flows on the space of sequences $(u_1(x), u_2(x), \dots)$ of functions of one variable x , which can be identified with the space of pseudo-differential operators of the form

$$\mathcal{L} = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \dots \tag{1.4}$$

The flows are defined by the Lax equations

$$\partial_{t_k} \mathcal{L} = [B_k, \mathcal{L}], \quad B_k = (\mathcal{L}^k)_+, \quad k = 1, 2, 3, \dots \tag{1.5}$$

where $(\dots)_+$ stands for the differential part of a pseudo-differential operator. The statement that Eq. (1.2) follow from Eq. (1.5) is easy. The inverse statement is true up to a triangular change of time variables [3].

The third form of the KP hierarchy is an infinite system of equations for *one* function $\tau^{\text{KP}}(x, \mathbf{t})$ of an *infinite* number of variables generated by the Hirota bilinear equation [4,5]

$$\oint_{\mathcal{C}_\infty} \tau^{\text{KP}}(x, \mathbf{t} - [z^{-1}]) \tau^{\text{KP}}(x, \mathbf{t}' + [z^{-1}]) \exp\left(\sum_{k \geq 1} (t_k - t'_k) z^k\right) dz = 0, \tag{1.6}$$

which should be valid for all sets of times \mathbf{t}, \mathbf{t}' . Here and below C_∞ is a big circle around the infinity $z = \infty$ and $\mathbf{t} \pm [z^{-1}]$ denotes the following special shift of time variables:

$$\mathbf{t} \pm [z^{-1}] := \left\{ t_1 \pm \frac{1}{z}, t_2 \pm \frac{1}{2z^2}, t_3 \pm \frac{1}{3z^3}, \dots \right\}. \tag{1.7}$$

Note that Eq. (1.6) is invariant under the transformation

$$\tau^{\text{KP}}(x, \mathbf{t}) \longrightarrow \exp\left(\gamma_0 + \gamma_1 x + \sum_{j \geq 1} \gamma_j t_j\right) \tau^{\text{KP}}(x, \mathbf{t}) \tag{1.8}$$

with constant γ_j . The tau-functions which differ by multiplication by exponent of a linear function of times are called equivalent.

In [6] an infinite integrable hierarchy of partial differential equations with $\text{Sp}(\infty)$ symmetry was introduced and called the Kadomtsev–Petviashvili hierarchy of type C (CKP). It is a hierarchy of commuting flows that are the restriction of the flows of the KP hierarchy corresponding to “odd” times $\mathbf{t}_o = \{t_1, t_3, t_5, \dots\}$ onto the space of *anti self-adjoint* pseudo-differential operators \mathcal{L}_{CKP} of the form (1.4), i.e. such that

$$\mathcal{L}_{\text{CKP}}^\dagger = -\mathcal{L}_{\text{CKP}}, \tag{1.9}$$

where \dagger means the formal adjoint defined by the rule $(f(x) \circ \partial_x^m)^\dagger = (-\partial_x)^m \circ f(x)$. The CKP hierarchy was revisited in [7–10].

The first goal of this work is to characterize the CKP hierarchy in terms of the KP tau-function. More precisely, each solution of the CKP hierarchy has a unique extension to the solution of the full KP hierarchy via the flows (1.5) with even k (which obviously do not preserve constraint (1.9)). In what follows we will refer to the corresponding solution to the KP hierarchy as *KP extension of the solution to the CKP hierarchy*. In Sect. 2 we prove that the KP tau-function is the tau-function of such a solution if the equation

$$\partial_{t_2} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e=0} = 0 \tag{1.10}$$

holds for all \mathbf{t}_o , where all “even” times $\mathbf{t}_e = (t_2, t_4, \dots)$ are set equal to zero. Conversely, in the equivalence class of any KP tau-function which is the tau-function of the KP extension of a solution to the CKP hierarchy there exists one which satisfies the condition (1.10). We note that this condition selects “turning points” of the KP hierarchy in the sense that if x_i are zeros of the tau-function $\tau^{\text{KP}}(x, \mathbf{t})$, i.e., $\tau^{\text{KP}}(x_i, \mathbf{t}) = 0$, then $\partial_{t_2} x_i(t_1, t_3, \dots) = 0$ for all t_1, t_3, t_5, \dots

We also prove the existence of the tau-function $\tau(x, \mathbf{t}_o)$ for the CKP hierarchy which is a function of “odd” times \mathbf{t}_o only and prove that it is the square root of τ^{KP} satisfying the condition (1.10):

$$\tau(x, t_1, t_3, t_5, \dots) = \sqrt{\tau^{\text{KP}}(x, t_1, 0, t_3, 0, t_5, 0, \dots)}. \tag{1.11}$$

In the first part of Sect. 3 we present in detail the algebraic–geometrical construction of quasi-periodic solutions to the CKP hierarchy briefly outlined in [6]. We start from the general scheme proposed in [11, 12]. The specialization for the CKP hierarchy is a certain reduction of this general scheme. The data defining the algebraic–geometrical solutions of the CKP hierarchy are the following: a smooth algebraic curve Γ of genus g

with a holomorphic involution having at least one fixed point $P_\infty \in \Gamma$, a local parameter in a neighbourhood of P_∞ which is *odd* with respect to the involution and a generic *admissible* divisor of degree g . The locus of the admissible divisors in the Jacobian is a translate of the Prym variety of Γ . In the second part of Sect. 3 we prove a new identity for the Riemann theta-function of a curve with involution having at least one fixed point (Theorem 3.4). The identity is an algebraic–geometrical incarnation of the relations between KP- and CKP-tau-functions discussed in Sect. 2.

In Sect. 4 we study solutions to the C -version of the KP equation which are double-periodic (elliptic) in the variable $x = t_1$ and their trigonometric and rational degenerations.

In the seminal paper [13] the motion of poles of singular solutions to the Korteweg–de Vries and Boussinesq equations was considered. It was discovered that the poles move as particles of the many-body Calogero–Moser system [14–16] with some additional restrictions in the phase space. In [17, 18] it was shown that in the case of the KP equation this correspondence becomes an isomorphism: the dynamics of poles of rational solutions to the KP equation is given by equations of motion for the Calogero–Moser system with pairwise interaction potential $1/(x_i - x_j)^2$. This remarkable connection was further generalized to elliptic solutions in [19]: poles x_i of the elliptic solutions as functions of $t_2 = y$ move according to the equations of motion

$$\partial_y^2 x_i = 4 \sum_{k \neq i} \wp'(x_i - x_k) \tag{1.12}$$

of Calogero–Moser particles with the elliptic interaction potential $\wp(x_i - x_j)$ (\wp is the Weierstrass \wp -function). Moreover, in [19] it was shown that the origin of Eq. (1.12) is related to a more fundamental problem: when a linear equation with elliptic coefficients has *double-Bloch* solutions (i.e. solution which are sections of a line bundle over the elliptic curve, see [20]). Recently, the method proposed in [19] was applied to the theory of elliptic solutions of the BKP equation [21, 22].

Along the same line of arguments we derive in Sect. 4 the equations of motion for poles of elliptic solutions to the CKP equation:

$$\dot{x}_i = 3 \sum_{k \neq i}^n \wp(x_i - x_k) - 6c, \tag{1.13}$$

where c is a constant and dot means the t_3 -derivative. In contrast to the KP and BKP cases, where the equations of motion are of the second order (see [19, 21, 22]) Eq. (1.13) are of the first order. As follows from the comparison of the CKP and KP hierarchies in Sect. 2 Eq. (1.13) coincide with the restriction of the Calogero–Moser flow corresponding to the higher Hamiltonian H_3 to the manifold of *turning points* in the $2n$ -dimensional phase space (p_i, x_i) , i.e. the n -dimensional submanifold $p_i = \partial_y x_i = 0$ for all $i = 1, \dots, n$.

Remark. The notion of the turning points of the elliptic Calogero–Moser system and the study of the corresponding spectral curves in the forthcoming paper [23] was motivated by the problem of construction of explicit solutions to the two-dimensional $O(2m + 1)$ sigma-model.

2. The CKP Hierarchy

2.1. *The \mathcal{L} -operator and the dressing operator.* The set of independent variables (“times”) in the CKP hierarchy is $\mathbf{t}_0 = \{t_1, t_3, t_5, \dots\}$. Like in the BKP case, they are indexed by positive odd numbers. It is convenient to set $t_1 = x + \text{const}$, so that the vector fields ∂_{t_1} and ∂_x are identical: $\partial_{t_1} = \partial_x$. The hierarchy is defined on the space of pseudo-differential operators \mathcal{L}_{CKP} of the form

$$\mathcal{L}_{\text{CKP}} = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \dots \tag{2.1}$$

subject to the constraint

$$\mathcal{L}_{\text{CKP}}^\dagger = -\mathcal{L}_{\text{CKP}}, \tag{2.2}$$

The coefficients u_j of \mathcal{L}_{CKP} depend on x and on all the times. It is convenient to introduce the wave operator (or dressing operator)

$$W = 1 + \xi_1 \partial_x^{-1} + \xi_2 \partial_x^{-2} + \dots \tag{2.3}$$

such that

$$\mathcal{L}_{\text{CKP}} = W \partial_x W^{-1}. \tag{2.4}$$

The wave operator is unique up to multiplication from the right by a pseudo-differential operator with x -independent coefficients.

The constraint (2.2) implies that $W^\dagger W$ commutes with ∂_x , i.e., it is a pseudo-differential operator with x -independent coefficients. We fix the above mentioned ambiguity in the definition of the wave operator by imposing the equation $W^\dagger W = 1$, i.e.

$$W^\dagger = W^{-1}. \tag{2.5}$$

The hierarchy of flows is defined by the Lax equations

$$\partial_{t_k} \mathcal{L}_{\text{CKP}} = [B_k, \mathcal{L}_{\text{CKP}}], \quad B_k = \left(\mathcal{L}_{\text{CKP}}^k \right)_+, \quad k = 1, 3, 5, \dots, \tag{2.6}$$

which obviously preserve the constraint (2.5) since $B_k^\dagger = -B_k$ (for odd k).

The zero curvature (Zakharov–Shabat) equations

$$\partial_{t_l} B_k - \partial_{t_k} B_l + [B_k, B_l] = 0, \quad k, l \text{ odd} \tag{2.7}$$

is an easy corollary of (2.6). They are equivalent to the statement that the flows (2.6) commute with each other.

The first equation of the CKP hierarchy is the equation $\partial_{t_3} B_5 - \partial_{t_5} B_3 + [B_5, B_3] = 0$ with

$$B_3 = \partial_x^3 + 6u \partial_x + 3u', \quad u' \equiv \partial_x u, \quad u = \frac{1}{2} u_1, \tag{2.8}$$

$$B_5 = \partial_x^5 + 10u \partial_x^3 + 15u' \partial_x^2 + v \partial_x + \frac{1}{2} (v' - 5u'''). \tag{2.9}$$

Straightforward calculations give the following system of equations for the unknown functions u, v :

$$\begin{cases} 10 \partial_{t_3} u = 3v' - 35u''' - 120uu' \\ 6 \partial_{t_5} u - \partial_{t_3} v = \frac{57}{2} u'''' + 150uu''' + 180u'u'' - \frac{5}{2} v''' + 6vu' - 6uv'. \end{cases} \tag{2.10}$$

Note that the variable v can be excluded by passing to the unknown function U such that $U' = u$.

2.2. *The wave function and the tau-function.* The Lax equations (2.6) are compatibility conditions of the auxiliary linear problems

$$\partial_{t_k} \Psi = B_k \Psi, \quad \mathcal{L}_{\text{CKP}} \Psi = z \Psi \tag{2.11}$$

for the formal wave function

$$\Psi = \Psi(\mathbf{t}_0, z) = W e^{xz + \zeta(\mathbf{t}_0, z)}, \tag{2.12}$$

where z is the spectral parameter and

$$\zeta(\mathbf{t}_0, z) = \sum_{k \geq 1, \text{ odd}} t_k z^k. \tag{2.13}$$

Note that the operator ∂_x^{-1} acts to the exponential function as $\partial_x^{-1} e^{xz} = z^{-1} e^{xz}$. Therefore, from (2.3), (2.12), it follows that the wave function has the following expansion as $z \rightarrow \infty$:

$$\Psi(x, \mathbf{t}_0, z) = e^{xz + \zeta(\mathbf{t}_0, z)} \left(1 + \sum_{k \geq 1} \xi_k z^{-k} \right). \tag{2.14}$$

Proposition 2.1 [6]. *The wave function Ψ satisfies the bilinear relation*

$$\oint_{C_\infty} \Psi(x, \mathbf{t}_0, z) \Psi(x, \mathbf{t}'_0, -z) dz = 0 \tag{2.15}$$

for all $\mathbf{t}_0, \mathbf{t}'_0$.

For completeness, we give a sketch of the proof here. By virtue of differential equations (2.11), the bilinear relation is equivalent to vanishing of the coefficients

$$b_m = \frac{1}{2\pi i} \partial_{x'}^m \oint_{C_\infty} \Psi(x, \mathbf{t}_0, z) \Psi(x', \mathbf{t}_0, -z) dz \Big|_{x'=x} \quad \text{for all } m \geq 0.$$

We have:

$$\begin{aligned} b_m &= \frac{1}{2\pi i} \oint_{C_\infty} \left(\sum_{k \geq 0} \xi_k(x) z^{-k} \right) \partial_{x'}^m \left(\sum_{l \geq 0} \xi_l(x') (-z)^{-l} e^{(x-x')z} \right) dz \Big|_{x'=x} \\ &= \frac{1}{2\pi i} \oint_{C_\infty} \left(\sum_{k \geq 0} \xi_k z^{-k} \right) (\partial_x - z)^m \left(\sum_{l \geq 0} \xi_l (-z)^{-l} \right) dz \\ &= \sum_{j+k+l=m+1} (-1)^{m+j+l} \binom{m}{j} \xi_k \partial_x^j \xi_l. \end{aligned}$$

The last expression is the coefficient at $(-1)^m \partial_x^{-m-1}$ in the operator $W W^\dagger$:

$$W W^\dagger = 1 + \sum_{m \geq 0} (-1)^m b_m \partial_x^{-m-1}.$$

Since $W W^\dagger = 1$ (see (2.5)), we get that $b_m = 0$ for all $m \geq 0$.

Theorem 2.1 *There exists a function $\tau = \tau(x, \mathbf{t}_0)$ such that*

$$\Psi(x, \mathbf{t}_0, z) = (2z)^{-1/2} \sqrt{\partial_x \psi^2(x, \mathbf{t}_0, z)}, \tag{2.16}$$

where

$$\psi(x, \mathbf{t}_0, z) = e^{xz+\zeta(\mathbf{t}_0,z)} \frac{\tau(x, \mathbf{t}_0 - 2[z^{-1}]_0)}{\tau(x, \mathbf{t}_0)} \tag{2.17}$$

and we use the notation

$$\mathbf{t}_0 + j[z^{-1}]_0 := \left\{ t_1 + \frac{j}{z}, t_3 + \frac{j}{3z^3}, t_5 + \frac{j}{5z^5}, \dots \right\}, \quad j \in \mathbb{Z}. \tag{2.18}$$

Definition 2.1 The function $\tau = \tau(x, \mathbf{t}_0)$ is called the tau-function of the CKP hierarchy.

Proof of Theorem 2.1. Representing the right hand side of (2.16) in explicit form, we see that we should prove the formula

$$\Psi = e^{xz+\zeta(\mathbf{t}_0,z)} G(x, \mathbf{t}_0, z) \frac{\tau(x, \mathbf{t}_0 - 2[z^{-1}]_0)}{\tau(x, \mathbf{t}_0)}, \tag{2.19}$$

where

$$G(x, \mathbf{t}, z) = \left(1 + z^{-1} \partial_{t_1} \log \frac{\tau(x, \mathbf{t}_0 - 2[z^{-1}]_0)}{\tau(x, \mathbf{t}_0)} \right)^{1/2}. \tag{2.20}$$

The proof is based on the bilinear relation (2.15). Let us represent the wave function in the form

$$\Psi(x, \mathbf{t}_0, z) = e^{xz+\zeta(\mathbf{t}_0,z)} w(x, \mathbf{t}_0, z)$$

and set $\mathbf{t}'_0 = \mathbf{t}_0 - 2[a^{-1}]_0$ in the bilinear relation. We have $e^{\zeta(2[a^{-1}]_0,z)} = \frac{a+z}{a-z}$. Then the residue calculus yields

$$w(\mathbf{t}_0, a)w(\mathbf{t}_0 - 2[a^{-1}]_0, -a) = f(\mathbf{t}_0, a), \tag{2.21}$$

where

$$f(\mathbf{t}_0, z) = 1 + \frac{1}{2z} \left(\xi_1(\mathbf{t}_0) - \xi_1(\mathbf{t}_0 - 2[z^{-1}]_0) \right) \tag{2.22}$$

and we do not indicate the dependence on x for brevity. Next, we set $\mathbf{t}'_0 = \mathbf{t}_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0$ in the bilinear relation and the residue calculus yields

$$\begin{aligned} & \frac{a+b}{a-b} \left(aw(\mathbf{t}_0, a)w(\mathbf{t}_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0, -a) \right. \\ & \quad \left. - bw(\mathbf{t}_0, b)w(\mathbf{t}_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0, -b) \right) \\ & = a + b + \frac{1}{2} \left(\xi_1(\mathbf{t}_0) - \xi_1(\mathbf{t}_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0) \right). \end{aligned} \tag{2.23}$$

Using the relation (2.21), we can represent this equation in the form

$$\begin{aligned} & \frac{1}{a-b} \left(af(\mathbf{t}_0 - 2[b^{-1}]_0, a) \frac{w(\mathbf{t}_0, a)}{w(\mathbf{t}_0 - 2[b^{-1}]_0, a)} \right. \\ & \quad \left. - bf(\mathbf{t}_0 - 2[a^{-1}]_0, b) \frac{w(\mathbf{t}_0, b)}{w(\mathbf{t}_0 - 2[a^{-1}]_0, b)} \right) \\ & = 1 + \frac{\xi_1(\mathbf{t}_0) - \xi_1(\mathbf{t}_0 - 2[a^{-1}]_0 - 2[b^{-1}]_0)}{2(a+b)}. \end{aligned} \tag{2.24}$$

Shifting here $\mathbf{t}_0 \rightarrow \mathbf{t}_0 + 2[b^{-1}]_0$, changing the sign of b (i.e. changing $b \rightarrow -b$) and using (2.21) in the second term in the left hand side after that, we arrive at the equation

$$\begin{aligned} & \frac{1}{a+b} \left(af(\mathbf{t}_0, a) \frac{w(\mathbf{t}_0 - 2[b^{-1}]_0, a)}{w(\mathbf{t}_0, a)} - bf(\mathbf{t}_0, b) \frac{w(\mathbf{t}_0 - 2[a^{-1}]_0, b)}{w(\mathbf{t}_0, b)} \right) \\ & = 1 + \frac{\xi_1(\mathbf{t}_0 - 2[b^{-1}]_0) - \xi_1(\mathbf{t}_0 - 2[a^{-1}]_0)}{2(a-b)}. \end{aligned} \tag{2.25}$$

Together Eqs. (2.24), (2.25) form the system of equations

$$\begin{cases} \frac{1}{a-b} \left(af(\mathbf{t}_0 - 2[b^{-1}]_0, a) X_a^{-1} - bf(\mathbf{t}_0 - 2[a^{-1}]_0, b) X_b^{-1} \right) \\ \quad = \frac{af(\mathbf{t}_0, a) + bf(\mathbf{t}_0 - 2[a^{-1}]_0, b)}{a+b} \\ \frac{1}{a+b} \left(af(\mathbf{t}_0, a) X_a - bf(\mathbf{t}_0, b) X_b \right) = \frac{af(\mathbf{t}_0, a) - bf(\mathbf{t}_0, b)}{a-b} \end{cases} \tag{2.26}$$

for the “unknowns”

$$X_a = \frac{w(\mathbf{t}_0 - 2[b^{-1}]_0, a)}{w(\mathbf{t}_0, a)}, \quad X_b = \frac{w(\mathbf{t}_0 - 2[a^{-1}]_0, b)}{w(\mathbf{t}_0, b)}. \tag{2.27}$$

Multiplying the two Eq. (2.26), one obtains, using the identity

$$af(\mathbf{t}_0, a) - af(\mathbf{t}_0 - 2[b^{-1}]_0, a) - bf(\mathbf{t}_0, b) + bf(\mathbf{t}_0 - 2[a^{-1}]_0, b) = 0, \tag{2.28}$$

the following simple relation:

$$\frac{w(\mathbf{t}_0, a)w(\mathbf{t}_0 - 2[a^{-1}]_0, b)}{w(\mathbf{t}_0, b)w(\mathbf{t}_0 - 2[b^{-1}]_0, a)} = \left(\frac{f(\mathbf{t}_0, a)f(\mathbf{t}_0 - 2[a^{-1}]_0, b)}{f(\mathbf{t}_0, b)f(\mathbf{t}_0 - 2[b^{-1}]_0, a)} \right)^{1/2}. \tag{2.29}$$

Therefore, introducing $w_0(\mathbf{t}_0, z) = w(\mathbf{t}_0, z)f^{-1/2}(\mathbf{t}_0, z)$, we get

$$\frac{w_0(\mathbf{t}_0, a)w_0(\mathbf{t}_0 - 2[a^{-1}]_0, b)}{w_0(\mathbf{t}_0, b)w_0(\mathbf{t}_0 - 2[b^{-1}]_0, a)} = 1. \tag{2.30}$$

Our goal is to prove that there exists a function $\tau(\mathbf{t}_0)$ such that

$$w_0(\mathbf{t}_0, z) = \frac{\tau(\mathbf{t}_0 - 2[z^{-1}]_0)}{\tau(\mathbf{t}_0)}. \tag{2.31}$$

For that it is enough to show that there is a function τ such that the equation

$$\hat{D}(\log w_0(\mathbf{t}_0, z) + \log \tau(\mathbf{t}_0)) = 0 \tag{2.32}$$

with

$$\hat{D} := \partial_z - 2 \sum_{m \geq 1, \text{ odd}} z^{-m-1} \partial_{t_m} \tag{2.33}$$

holds.

Indeed, integrating equation $\hat{D}F = 0$ along its characteristics we get that a function $F(\mathbf{t}_0, z)$ is in the kernel of the differential operator \hat{D} if and only if it is of the form

$$F(\mathbf{t}_0, z) = f(\mathbf{t}_0 - 2[z^{-1}]_0)$$

for some function $f(\mathbf{t}_0)$. For F as in (2.32) the initial condition $\log w_0(\mathbf{t}_0, \infty) = 0$ allows one to identify the corresponding function f with τ .

Equation (2.32) is equivalent to the equations

$$Y_n := \operatorname{res}_{z=\infty} \left[z^n \hat{D} \log w_0 \right] = 2 \frac{\partial \log \tau}{\partial t_n}.$$

Therefore, to complete the proof of the existence of the tau-function it remains only to show that $\partial_{t_n} Y_m(\mathbf{t}_0) = \partial_{t_m} Y_n(\mathbf{t}_0)$.

Changing $a \rightarrow z, b \rightarrow \zeta$ in (2.30), and applying the operator \hat{D} to logarithm of this equality, we get

$$\hat{D} \left(\log w_0(\mathbf{t}_0, z) - \log w_0(\mathbf{t}_0 - 2[\zeta^{-1}]_0, z) + \log w_0(\mathbf{t}_0, \zeta) \right) = 0,$$

or

$$Y_n(\mathbf{t}_0) - Y_n(\mathbf{t}_0 - 2[\zeta^{-1}]_0) = -2\partial_{t_n} \log w_0(\mathbf{t}_0, \zeta). \tag{2.34}$$

Denote $F_{mn} = \partial_{t_m} Y_n - \partial_{t_n} Y_m$. Then, from (2.34) it follows that the equation

$$F_{mn}(\mathbf{t}_0) = F_{mn}(\mathbf{t}_0 - 2[\zeta^{-1}]_0) \tag{2.35}$$

holds identically in ζ . Expanding the right hand side in a power series,

$$F_{mn}(\mathbf{t}_0 - 2[\zeta^{-1}]_0) = F_{mn}(\mathbf{t}_0) - 2\zeta^{-1} \partial_{t_1} F_{mn}(\mathbf{t}_0) - \frac{2}{3} \zeta^{-3} (\partial_{t_3} F_{mn}(\mathbf{t}_0) + 2\partial_{t_1}^3 F_{mn}(\mathbf{t}_0)) + \dots,$$

we see from the ζ^{-1} -term that F_{mn} does not depend on t_1 . Then from the ζ^{-3} -term we conclude that it does not depend on t_3 and so on, so it does not depend on t_k for all (odd) k : $F_{mn} = 2a_{mn}$, where a_{mn} are some constants such that $a_{mn} = -a_{nm}$. Therefore, we can write

$$Y_n = \sum_m a_{mn} t_m + \partial_{t_n} h,$$

where $h = h(\mathbf{t}_0)$ is some function. Then from (2.34) we have

$$-2\partial_{t_n} \log w_0(\mathbf{t}_0, z) = \partial_{t_n} (h(\mathbf{t}_0) - h(\mathbf{t}_0 - 2[z^{-1}]_0)) + 2 \sum_{m \text{ odd}} \frac{a_{mn}}{m} z^{-m},$$

or, after integration,

$$\log w_0(\mathbf{t}_0, z) = \frac{1}{2} h(\mathbf{t}_0 - 2[z^{-1}]_0) - \frac{1}{2} h(\mathbf{t}_0) - \sum_{m \text{ odd}} \frac{a_{mn}}{m} z^{-m} t_n + \varphi(z),$$

where $\varphi(z)$ is a function of z only. Substituting this into logarithm of (2.30), we conclude that $a_{mn} = 0$.

Now, writing $w(\mathbf{t}_0, z) = f^{1/2}(\mathbf{t}_0, z)w_0(\mathbf{t}_0, z)$ and noting that $f(\mathbf{t}_0, z) = 1 + O(z^{-2})$, we see that

$$\xi_1(\mathbf{t}_0) = -2\partial_{t_1} \log \tau(\mathbf{t}_0) \tag{2.36}$$

and we arrive at (2.19) with $G = f^{1/2}$. □

Remark. The proof given above is rather involved. It is instructive to obtain (2.19) up to a common x -independent factor in the following easy way [7,8]. Let us apply ∂_{t_1} to (2.15) and set $\mathbf{t}'_0 = \mathbf{t}_0 - 2[a^{-1}]_0$. The residue calculus yields

$$\begin{aligned} &2a^2 \left(1 - w(\mathbf{t}_0, a)w(\mathbf{t}_0 - 2[a^{-1}]_0, -a) \right) - 2aw'(\mathbf{t}_0, a)w(\mathbf{t}_0 - 2[a^{-1}]_0, -a) \\ &+ 2a \left(\xi_1(\mathbf{t}_0) - \xi_1(\mathbf{t}_0 - 2[a^{-1}]_0) \right) + \xi_2(\mathbf{t}_0 - 2[a^{-1}]_0) + \xi_2(\mathbf{t}_0) + \xi'_1(\mathbf{t}_0) \\ &- \xi_1(\mathbf{t}_0)\xi_1(\mathbf{t}_0 - 2[a^{-1}]_0) = 0, \end{aligned} \tag{2.37}$$

where prime means the x -derivative and we again do not indicate the dependence on x explicitly. Letting $a \rightarrow \infty$, we get the relation

$$2\xi_2(\mathbf{t}_0) = \xi_1^2(\mathbf{t}_0) - \xi'_1(\mathbf{t}_0) \tag{2.38}$$

which also directly follows from $WW^\dagger = 1$. Plugging it back to (2.37), we can rewrite Eq. (2.37) in the form

$$w'(\mathbf{t}_0, a)w(\mathbf{t}_0 - 2[a^{-1}]_0, -a) = af(\mathbf{t}_0, a)(f(\mathbf{t}_0, a) - 1) + \frac{1}{2}f'(\mathbf{t}_0, a). \tag{2.39}$$

Using (2.21), we conclude that

$$\begin{aligned} \partial_x \log w(\mathbf{t}_0, a) &= a(f(\mathbf{t}_0, a) - 1) + \frac{1}{2} \partial_x \log f(\mathbf{t}_0, a) \\ &= \frac{1}{2} \left(\xi_1(\mathbf{t}_0) - \xi_1(\mathbf{t}_0 - 2[a^{-1}]_0) \right) + \frac{1}{2} \partial_x \log f(\mathbf{t}_0, a). \end{aligned} \tag{2.40}$$

Now, setting $\xi_1(x, \mathbf{t}_0) = -2\partial_x \log \tau(x, \mathbf{t}_0)$ and integrating, we arrive at (2.19) with $G = f^{1/2}$ up to a common multiplier which does not depend on x .

Remark. Substitution of (2.14) into (2.11) with $k = 3$ gives that the function u in (2.8) equals

$$u = -\frac{1}{2} \xi'_1 = \partial_x^2 \log \tau \tag{2.41}$$

2.3. *CKP hierarchy versus KP hierarchy.* The goal of this section is to prove that the CKP hierarchy can be identified with the restriction of *odd-times* flows of the KP hierarchy onto the locus of *turning points of even-times* flows.

Recall that wave function Ψ^{KP} and the adjoint wave function $\Psi^{\dagger\text{KP}}$ of the KP hierarchy are expressed through the tau-function τ^{KP} as

$$\Psi^{\text{KP}}(x, \mathbf{t}; z) = \exp\left(xz + \sum_{k \geq 1} t_k z^k\right) \frac{\tau^{\text{KP}}(x, \mathbf{t} - [z^{-1}])}{\tau^{\text{KP}}(x, \mathbf{t})}, \tag{2.42}$$

$$\Psi^{\dagger\text{KP}}(x, \mathbf{t}; z) = \exp\left(-xz - \sum_{k \geq 1} t_k z^k\right) \frac{\tau^{\text{KP}}(x, \mathbf{t} + [z^{-1}])}{\tau^{\text{KP}}(x, \mathbf{t})}. \tag{2.43}$$

where the notation (1.7) for the special shift of times is used. The origin of these expressions is the bilinear relation [4]

$$\oint_{C_\infty} \Psi^{\text{KP}}(x, \mathbf{t}, z) \Psi^{\dagger\text{KP}}(x, \mathbf{t}', z) dz = 0 \tag{2.44}$$

equivalent to (1.6).

A direct consequence of the bilinear relation (2.44) with the wave functions given by (2.42), (2.43) is the Hirota–Miwa equation for the tau-function of the KP hierarchy

$$\begin{aligned} &(z_1 - z_2)(z_3 - z_4) \tau^{\text{KP}}(\mathbf{t} - [z_1^{-1}] - [z_2^{-1}]) \tau^{\text{KP}}(\mathbf{t} - [z_3^{-1}] - [z_4^{-1}]) \\ &+ (z_2 - z_3)(z_1 - z_4) \tau^{\text{KP}}(\mathbf{t} - [z_2^{-1}] - [z_3^{-1}]) \tau^{\text{KP}}(\mathbf{t} - [z_1^{-1}] - [z_4^{-1}]) \\ &+ (z_3 - z_1)(z_2 - z_4) \tau^{\text{KP}}(\mathbf{t} - [z_1^{-1}] - [z_3^{-1}]) \tau^{\text{KP}}(\mathbf{t} - [z_2^{-1}] - [z_4^{-1}]) = 0. \end{aligned} \tag{2.45}$$

It is a generating equation for the differential equations of the hierarchy. The differential equations are obtained by expanding it in negative powers of z_1, z_2, z_3, z_4 . In the limit $z_4 \rightarrow \infty, z_3 \rightarrow \infty$ Eq. (2.45) becomes

$$\begin{aligned} &\partial_x \log \frac{\tau^{\text{KP}}(x, \mathbf{t} + [z_1^{-1}] - [z_2^{-1}])}{\tau^{\text{KP}}(x, \mathbf{t})} \\ &= (z_2 - z_1) \left(\frac{\tau^{\text{KP}}(x, \mathbf{t} + [z_1^{-1}]) \tau^{\text{KP}}(x, \mathbf{t} - [z_2^{-1}])}{\tau^{\text{KP}}(x, \mathbf{t}) \tau^{\text{KP}}(x, \mathbf{t} + [z_1^{-1}] - [z_2^{-1}])} - 1 \right). \end{aligned} \tag{2.46}$$

We will need a particular case of (2.46) at $z_2 = -z_1 = z$ which we write in the form

$$\begin{aligned} &\frac{1}{2z} \partial_x \left(e^{2xz} \frac{\tau^{\text{KP}}(x, \mathbf{t} + [-z^{-1}] - [z^{-1}])}{\tau^{\text{KP}}(x, \mathbf{t})} \right) \\ &= e^{2xz} \frac{\tau^{\text{KP}}(x, \mathbf{t} + [-z^{-1}]) \tau^{\text{KP}}(x, \mathbf{t} - [z^{-1}])}{(\tau^{\text{KP}}(x, \mathbf{t}))^2}. \end{aligned} \tag{2.47}$$

The following theorem gives an expression for the CKP tau-functions in terms of the KP tau-functions satisfying the “turning points” constraint (1.10).

Theorem 2.2 *The KP tau-function $\tau^{\text{KP}}(x, \mathbf{t})$ is the KP extension of a solution of the CKP hierarchy if the equation*

$$\partial_{t_2} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e=0} = 0 \tag{2.48}$$

holds for all t_1, t_3, t_5, \dots when “even” times $\mathbf{t}_e = \{t_2, t_4, t_6, \dots\}$ are set equal to zero. Conversely, in the equivalence class of any KP tau-function corresponding to KP extension of a solution to the CKP hierarchy there is one which satisfies (2.48). Moreover, the CKP tau-function defined in Theorem 2.1 is equal to

$$\tau(x, \mathbf{t}_0) = \sqrt{\tau^{\text{KP}}(x, t_1, 0, t_3, 0, \dots)}. \tag{2.49}$$

Proof. We first prove the “only if” part which is easier. Comparing (2.44) and (2.15), we see that the wave function for any solution of the CKP hierarchy and the wave function for its KP extension are related as

$$\begin{aligned} \Psi(x, \mathbf{t}_0, z) &= e^{\chi(z)} \Psi^{\text{KP}}(x, t_1, 0, t_3, 0, \dots, z), \\ \Psi(x, \mathbf{t}_0, -z) &= e^{-\chi(z)} \Psi^{\dagger \text{KP}}(x, t_1, 0, t_3, 0, \dots, z) \end{aligned}$$

with some function $\chi(z)$ depending only on z such that $\chi(\infty) = 0$ (this function may appear if one substitutes the tau-function by an equivalent one). Therefore, we have

$$\Psi^{\dagger \text{KP}}(x, t_1, 0, t_3, 0, \dots, z) = e^{2\chi_e(z)} \Psi^{\text{KP}}(x, t_1, 0, t_3, 0, \dots, -z), \tag{2.50}$$

where $\chi_e(z) = \frac{1}{2}(\chi(z) + \chi(-z))$ is the even part of the function $\chi(z)$. From (2.42), (2.43) and (2.50) it follows that the KP tau-function corresponds to the KP extension of a solution to the CKP hierarchy if and only if the equation

$$\begin{aligned} \tau^{\text{KP}}\left(x, t_1 + z^{-1}, \frac{1}{2} z^{-2}, t_3 + \frac{1}{3} z^{-3}, \frac{1}{4} z^{-4}, \dots\right) \\ = e^{2\chi_e(z)} \tau^{\text{KP}}\left(x, t_1 + z^{-1}, -\frac{1}{2} z^{-2}, t_3 + \frac{1}{3} z^{-3}, -\frac{1}{4} z^{-4}, \dots\right) \end{aligned} \tag{2.51}$$

holds identically for all $z, x, t_1, t_3, t_5, \dots$. Shifting the odd times, we can rewrite this condition as

$$\begin{aligned} \log \tau^{\text{KP}}\left(x, t_1, \frac{1}{2} z^{-2}, t_3, \frac{1}{4} z^{-4}, \dots\right) \\ - \log \tau^{\text{KP}}\left(x, t_1, -\frac{1}{2} z^{-2}, t_3, -\frac{1}{4} z^{-4}, \dots\right) = 2\chi_e(z). \end{aligned} \tag{2.52}$$

Comparing the coefficients at z^{-2} of the expansions of the left and right hand sides of (2.52) and passing to an equivalent tau-function if necessary, we get (2.48), i.e. the “only if” part of the theorem statement is proven. \square

We begin the proof of the “if part” by the following lemma.

Lemma 2.1 *On solutions of the KP hierarchy equation (2.48) implies that all derivatives of odd degree higher then 1 with respect to various even times are equal to zero for all x, \mathbf{t}_0 , i.e.*

$$\partial_{t_{2k_1}} \partial_{t_{2k_2}} \dots \partial_{t_{2k_{2m+1}}} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e=0} = 0 \tag{2.53}$$

for all $k_1, k_2, \dots, k_{2m+1} \geq 1, m \geq 1$. Besides, first order derivatives with respect to even times satisfy

$$\partial_x \partial_{t_4} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e=0} = \partial_x \partial_{t_6} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e=0} = \dots = 0. \tag{2.54}$$

The proof is given in ‘‘Appendix A’’. From Eq. (2.54) we see that

$$\tilde{\chi}_{2k}(t_3, t_5, \dots) := \partial_{t_{2k}} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e=0}, \quad k > 1$$

does not depend on x . Equation (2.48) means that $\tilde{\chi}_2 = 0$. Next, from (2.53) we conclude that

$$\log \tau^{\text{KP}}(x, t_1, t_2, t_3, t_4, \dots) - \log \tau^{\text{KP}}(x, t_1, -t_2, t_3, -t_4, \dots) = 2 \sum_{k \geq 2} \tilde{\chi}_{2k}(t_3, t_5, \dots) t_{2k}$$

is a linear function of \mathbf{t}_e . Therefore, we can write

$$\tau^{\text{KP}}\left(x, t_1, \frac{1}{2} z^{-2}, t_3, \frac{1}{4} z^{-4}, \dots\right) = e^{2\tilde{\chi}_e(t_3, t_5, \dots; z)} \tau^{\text{KP}}\left(x, t_1, -\frac{1}{2} z^{-2}, t_3, -\frac{1}{4} z^{-4}, \dots\right), \tag{2.55}$$

where $\tilde{\chi}_e(t_3, t_5, \dots; z)$ is a function of the times t_3, t_5, \dots and an even function of z . In its turn, (2.55) implies

$$\Psi^{\dagger \text{KP}}(x, \mathbf{t}, z) = C(t_3 + \frac{1}{3} z^{-3}, t_5 + \frac{1}{5} z^{-5}, \dots; z) \Psi^{\text{KP}}(x, \mathbf{t}, -z), \tag{2.56}$$

where $C(t_3, t_5, \dots; z) = e^{\tilde{\chi}_e(t_3, t_5, \dots; z)}$ and we use the short-hand notation

$$\mathbf{t} = \{t_1, 0, t_3, 0, \dots\}.$$

(In this notation Eq. (2.51) takes the form $\tau^{\text{KP}}(x, \mathbf{t}+[z^{-1}]) = e^{2\chi_e(z)} \tau^{\text{KP}}(x, \mathbf{t}-[-z^{-1}])$.)

The adjoint wave function $\Psi^{\dagger \text{KP}}$ satisfies the adjoint linear equation (see the independent proof in the next section), which restricted to the locus \mathbf{t} where $B_k^\dagger = -B_k$ for odd k coincides with the linear equation for Ψ^{KP} , so we simultaneously have

$$\begin{aligned} \partial_{t_k} \Psi^{\dagger \text{KP}}(x, \mathbf{t}, z) &= B_k \Psi^{\dagger \text{KP}}(x, \mathbf{t}, z), \\ \partial_{t_k} \Psi^{\text{KP}}(x, \mathbf{t}, z) &= B_k \Psi^{\text{KP}}(x, \mathbf{t}, z). \end{aligned} \tag{2.57}$$

for odd k . Substituting (2.56) into the first of these equations, we get, after the change $z \rightarrow -z$,

$$\partial_{t_k} \Psi^{\text{KP}}(x, \mathbf{t}, z) + \partial_{t_k} \log C\left(t_3 - \frac{1}{3} z^{-3}, t_5 - \frac{1}{5} z^{-5}, \dots; z\right) = B_k \Psi^{\text{KP}}(x, \mathbf{t}, z),$$

and from the second equation in (2.57) we conclude that

$$\partial_{t_k} \log C\left(t_3 - \frac{1}{3} z^{-3}, t_5 - \frac{1}{5} z^{-5}, \dots; z\right) = 0,$$

i.e. $C(t_3, t_5, \dots; z)$ is an even function of z which does not depend on the times. (This function can be eliminated in (2.55) by passing to an equivalent tau-function.) Therefore, the Eq. (2.51) which guarantees that τ^{KP} is the KP extension of a solution to the CKP hierarchy is proved.

Remark. Passing to an equivalent tau-function using the transformation (1.8), one obtains the condition $\partial_{t_2} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e} = \gamma_2$ instead of (2.48). Conversely, if $\partial_{t_2} \log \tau^{\text{KP}} \Big|_{\mathbf{t}_e} = \gamma_2$ with some nonzero γ_2 , it is possible to pass to an equivalent tau-function satisfying (2.48) by a transformation of the form (1.8).

In order to prove that $\tau = \sqrt{\tau^{\text{KP}}}$ ([9]) we compare two expressions for the wave function Ψ of the CKP hierarchy. The first one is in terms of the KP tau-function (satisfying (2.48)),

$$\Psi^{\text{KP}} = e^{xz+\zeta(\mathbf{t}_0,z)} \frac{\tau^{\text{KP}}(x, t_1 - z^{-1}, -\frac{1}{2}z^{-2}, t_3 - \frac{1}{3}z^{-3}, -\frac{1}{4}z^{-4}, \dots)}{\tau^{\text{KP}}(t_1, 0, t_3, 0, \dots)}, \tag{2.58}$$

and the second one (2.19) is in terms of the CKP tau-function τ . Recall that

$$\Psi = z^{-1/2} \sqrt{\partial_x \log \psi} \cdot \psi = (2z)^{-1/2} \sqrt{\partial_x \psi^2} \tag{2.59}$$

(see (2.16)), where

$$\psi = e^{xz+\zeta(\mathbf{t}_0,z)} \frac{\tau(x, \mathbf{t}_0 - 2[z^{-1}]_0)}{\tau(x, \mathbf{t}_0)}. \tag{2.60}$$

Comparing (2.59) and (2.58), we get the equation

$$\frac{1}{2z} \partial_x \left(e^{2xz} \frac{\tau^2(x, \mathbf{t}_0 - 2[z^{-1}]_0)}{\tau^2(x, \mathbf{t}_0)} \right) = e^{2xz} \left(\frac{\tau^{\text{KP}}(x, \dot{\mathbf{t}} - [z^{-1}])}{\tau^{\text{KP}}(x, \dot{\mathbf{t}})} \right)^2, \tag{2.61}$$

where we again use the short-hand notation $\dot{\mathbf{t}} = \{t_1, 0, t_3, 0, \dots\}$. Then using Eq. (2.47) we get that (2.61) is equivalent to the differential equation

$$\partial_x \varphi = -2z\varphi, \tag{2.62}$$

where

$$\varphi = \frac{\tau^2(x, \mathbf{t}_0 - 2[z^{-1}]_0)}{\tau^2(x, \mathbf{t}_0)} - \frac{\tau^{\text{KP}}(x, \dot{\mathbf{t}} - 2[z^{-1}]_0)}{\tau^{\text{KP}}(x, \dot{\mathbf{t}})}. \tag{2.63}$$

In (2.63), $\dot{\mathbf{t}} - 2[z^{-1}]_0 = \{t_1 - 2z^{-1}, 0, t_3 - \frac{2}{3}z^{-3}, 0, \dots\}$. The general solution of the differential equation (2.62) is

$$\varphi = c(z, t_3, t_5, \dots) e^{-2(x+t_1)z}$$

but from (2.63) it follows that φ is expanded in a power series as $\varphi = \varphi_1 z^{-1} + \varphi_2 z^{-2} + \dots$ as $z \rightarrow \infty$, and this means that c must be equal to 0. Therefore, $\varphi = 0$, i.e.

$$\frac{\tau^2(x, \mathbf{t}_0 - 2[z^{-1}]_0)}{\tau^2(x, \mathbf{t}_0)} = \frac{\tau^{\text{KP}}(x, \dot{\mathbf{t}} - 2[z^{-1}]_0)}{\tau^{\text{KP}}(x, \dot{\mathbf{t}})} \tag{2.64}$$

for all z . This is an identity on solutions to the KP/CKP hierarchies. It follows from (2.64) that $\tau^{\text{KP}} = \text{const} \cdot \tau^2$, i.e. $\tau(x, \mathbf{t}_0) = \sqrt{\tau^{\text{KP}}(x, \dot{\mathbf{t}})}$ is a tau-function of the CKP hierarchy. \square

Remark. Equation (2.61) is the CKP analog of the BKP statement that the corresponding τ^{KP} is a full square, i.e. $\tau = \sqrt{\tau^{\text{KP}}}$ is an entire function of its variables. In the CKP case $\partial_x \psi^2$ is a full square.

3. Algebraic–Geometrical Solutions to the KP and CKP Hierarchies

The algebraic–geometrical construction of quasi-periodic solutions to the CKP hierarchy briefly outlined in [6] is a reduction of the algebraic–geometrical construction of solutions to the KP hierarchy proposed in [11, 12]. The main goal of this section is to give a pure algebraic–geometrical proof of an identity for the Riemann theta-function of a curve with involution having at least one fixed point. This identity is an algebraic–geometrical incarnation of the relations between KP and CKP tau-functions discussed in Sect. 2.

3.1. Preliminaries. Let Γ be a smooth compact algebraic curve of genus g . We fix a canonical basis of cycles a_α, b_α ($\alpha = 1, \dots, g$) with the intersections $a_\alpha \circ a_\beta = b_\alpha \circ b_\beta = 0, a_\alpha \circ b_\beta = \delta_{\alpha\beta}$ and a basis of holomorphic differentials $d\omega_\alpha$ normalized by the condition $\oint_{a_\alpha} d\omega_\beta = \delta_{\alpha\beta}$. The period matrix is defined as

$$T_{\alpha\beta} = \oint_{b_\alpha} d\omega_\beta, \quad \alpha, \beta = 1, \dots, g. \tag{3.1}$$

It is a symmetric matrix with positively defined imaginary part. The Riemann theta-function is defined by the series

$$\theta(\vec{z}) = \theta(\vec{z}|T) = \sum_{\vec{n} \in \mathbb{Z}^g} e^{\pi i(\vec{n}, T\vec{n}) + 2\pi i(\vec{n}, \vec{z})}, \tag{3.2}$$

where $\vec{z} = (z_1, \dots, z_g)$ and $(\vec{n}, \vec{z}) = \sum_{\alpha=1}^g n_\alpha z_\alpha$.

The Jacobian of the curve Γ is the g -dimensional complex torus

$$J(\Gamma) = \mathbb{C}^g / \{2\pi i \vec{N} + 2\pi i T \vec{M}\}, \tag{3.3}$$

where \vec{N}, \vec{M} are g -dimensional vectors with integer components. Fix a point $Q_0 \in \Gamma$ and define the Abel map $\vec{A}(P), P \in \Gamma$ from Γ to $J(\Gamma)$, as

$$\vec{A}(P) = \vec{\omega}(P) = \int_{Q_0}^P d\vec{\omega}, \quad d\vec{\omega} = (d\omega_1, \dots, d\omega_g). \tag{3.4}$$

The Abel map can be extended to the group of divisors $\mathcal{D} = n_1 Q_1 + \dots + n_K Q_K$ as

$$\vec{A}(\mathcal{D}) = \sum_{i=1}^K n_i \int_{Q_0}^{Q_i} d\vec{\omega} = \sum_{i=1}^K n_i \vec{A}(Q_i). \tag{3.5}$$

Let $P_\infty \in \Gamma$ be a marked point and k^{-1} a local parameter in a neighborhood of the marked point ($k = \infty$ at P_∞). Let $d\Omega_j$ be abelian differentials of the second kind with the only pole at P_∞ of the form

$$d\Omega_j = dk^j + O(k^{-2})dk, \quad k \rightarrow \infty \tag{3.6}$$

normalized by the condition $\oint_{a_\alpha} d\Omega_j = 0$, and Ω_j be the (multi-valued) functions

$$\Omega_j(P) = \int_{Q_0}^P d\Omega_j + q_j,$$

where the constants q_j are chosen in such a way that $\Omega_i(P) = k^i + O(k^{-1})$, namely,

$$\Omega_i(P) = k^i + \sum_{j \geq 1} \frac{1}{j} \Omega_{ij} k^{-j}. \tag{3.7}$$

The Riemann identity implies that the matrix Ω_{ij} is symmetric: $\Omega_{ij} = \Omega_{ji}$.

Set

$$U_j^\alpha = \frac{1}{2\pi i} \oint_{b_\alpha} d\Omega_j, \quad \vec{U}_j = (U_j^1, \dots, U_j^g). \tag{3.8}$$

One can prove the following relation:

$$\vec{d}\omega = \sum_{j \geq 1} \vec{U}_j k^{-j-1} dk \tag{3.9}$$

or

$$\vec{A}(P) - \vec{A}(P_\infty) = \int_{P_\infty}^P d\vec{\omega} = - \sum_{j \geq 1} \frac{1}{j} \vec{U}_j k^{-j}. \tag{3.10}$$

We will also use the following fact [24,25]: for any non-special effective divisor $\mathcal{D} = Q_1 + \dots + Q_g$ of degree g the function

$$f(P) = \theta\left(\vec{A}(P) - \vec{A}(\mathcal{D}) - \vec{K}\right)$$

has exactly g zeros at the points Q_1, \dots, Q_g . Here $\vec{K} = (K_1, \dots, K_g)$ is the vector of Riemann's constants

$$K_\alpha = \pi i + \pi i T_{\alpha\alpha} - 2\pi i \sum_{\beta \neq \alpha} \oint_{a_\beta} \omega_\alpha(P) d\omega_\beta(P). \tag{3.11}$$

Let \mathcal{K} be the canonical class of divisors (the equivalence class of divisors of poles and zeros of abelian differentials on Γ), then one can show that

$$2\vec{K} = -\vec{A}(\mathcal{K}). \tag{3.12}$$

It is known that $\text{deg } \mathcal{K} = 2g - 2$. In particular, this means that holomorphic differentials have $2g - 2$ zeros on Γ .

We also need the bi-differential $d_P d_Q \Omega(P, Q)$ such that it is symmetric in P, Q , its only singularity is a second order pole at $P = Q$ and the integrals over a -cycles vanish. It is related to the differentials $d\Omega_j$ as follows:

$$\text{res}_{Q=P_\infty} \left(k^i(Q) d_P d_Q \Omega(P, Q) \right) = -d\Omega_i(P). \tag{3.13}$$

The expansion in the local parameters is

$$d_P d_Q \Omega(P, Q) = \left(\frac{1}{(k^{-1}(P) - k^{-1}(Q))^2} - \sum_{i,j \geq 1} \Omega_{ij} k^{1-i}(P) k^{1-j}(Q) \right) dk^{-1}(P) dk^{-1}(Q). \tag{3.14}$$

In fact this bi-differential can be expressed in terms of the odd theta-function

$$\theta_*(\vec{z}) = \theta \left[\begin{matrix} \vec{\delta}' \\ \vec{\delta}'' \end{matrix} \right] (\vec{z}) = \sum_{\vec{n} \in \mathbb{Z}^g} e^{\pi i (\vec{n} + \vec{\delta}', T(\vec{n} + \vec{\delta}'')) + 2\pi i (\vec{n} + \vec{\delta}', \vec{z} + \vec{\delta}'')},$$

where $(\vec{\delta}', \vec{\delta}'')$ is a non-singular odd theta-characteristics. One has:

$$d_P d_Q \Omega(P, Q) = d_P d_Q \log \theta_* \left(\vec{A}(P) - \vec{A}(Q) \right). \tag{3.15}$$

Calculating the double integral

$$\int_{P_\infty}^{P_1} \int_{Q_0}^{P_2} d_P d_Q \Omega(P, Q)$$

in two ways (using first (3.14) and then (3.15)), we obtain the equality

$$\begin{aligned} & \log \frac{(k^{-1}(P_1) - k^{-1}(P_2))k^{-1}(Q_0)}{(k^{-1}(P_1) - k^{-1}(Q_0))k^{-1}(P_2)} - \sum_{i,j \geq 1} \Omega_{ij} \frac{k^{-i}(P_1)k^{-j}(P_2)}{ij} \\ & + \sum_{i,j \geq 1} \Omega_{ij} \frac{k^{-i}(P_1)k^{-j}(Q_0)}{ij} \\ & = \log \frac{\theta_* \left(\vec{A}(P_2) - \vec{A}(P_1) \right) \theta_* \left(\vec{A}(P_\infty) \right)}{\theta_* \left(\vec{A}(P_2) - \vec{A}(P_\infty) \right) \theta_* \left(\vec{A}(P_1) \right)} \end{aligned}$$

Tending here $Q_0 \rightarrow P_\infty$, we arrive at the important relation

$$\exp \left(- \sum_{i,j \geq 1} \Omega_{ij} \frac{k_1^{-i} k_2^{-j}}{ij} \right) = \frac{C \theta_* \left(\vec{A}(P_1) - \vec{A}(P_2) \right)}{(k_1 - k_2) \theta_* \left(\vec{A}(P_1) - \vec{A}(P_\infty) \right) \theta_* \left(\vec{A}(P_2) - \vec{A}(P_\infty) \right)}, \tag{3.16}$$

where

$$C = \sum_{\alpha=1}^g U_1^\alpha \theta_{*,\alpha}(\vec{0}), \quad \theta_{*,\alpha}(\vec{0}) = \left. \frac{\partial \theta_*(\vec{z})}{\partial z_\alpha} \right|_{\vec{z}=0}$$

is a constant and $k_1 = k(P_1)$, $k_2 = k(P_2)$. In particular, tending $k_1 \rightarrow k_2$, we get

$$\exp \left(- \sum_{i,j \geq 1} \Omega_{ij} \frac{k^{-i-j}}{ij} \right) dk = \frac{C d\zeta}{\theta_*^2 \left(\vec{A}(P) - \vec{A}(P_\infty) \right)}, \tag{3.17}$$

where $d\zeta$ is the holomorphic differential

$$d\zeta = \sum_{\alpha=1}^g \theta_{*,\alpha}(\vec{0}) d\omega_\alpha. \tag{3.18}$$

As is explained in [25], the differential $d\zeta$ has double zeros at $g - 1$ points R_1, \dots, R_{g-1} while the function

$$f_*(P) = \theta_*\left(\vec{A}(P) - \vec{A}(P_\infty)\right)$$

has simple zeros at the same points R_i and P_∞ . Therefore, the differential in the right hand side of (3.17) has the only (second order) pole at P_∞ and no zeros. However, this differential is well-defined only on a covering of the curve Γ because it is not single-valued.

3.2. The Baker–Akhiezer function and the tau-function.

3.2.1. The Baker–Akhiezer function Let x, t_1, t_2, t_3, \dots be a set of complex parameters (here we assume that only a finite number of them are different from zero) and let Γ be a smooth genus g algebraic curve with fixed local coordinate $k^{-1}(P)$ in the neighbourhood of a fixed point P_∞ .

Lemma 3.1 [11, 12]. *Let $\mathcal{D} = Q_1 + \dots + Q_g$ be an effective non-special divisor of degree g . Then there is a unique function $\Psi_{BA}(x, \mathbf{t}, P)$ such that:*

- 1⁰. *As a function of $P \in \Gamma$ it is meromorphic away from the marked point P_∞ with poles at the points Q_s of multiplicity not greater than the multiplicity of Q_s in \mathcal{D} .*
- 2⁰. *In the neighbourhood of P_∞ it has the form*

$$Psi_{BA} = \exp\left(xk + \sum_{j \geq 1} t_j k^j\right) \left(1 + \xi_1 k^{-1} + \xi_2 k^{-2} + \dots\right), \quad k = k(P). \tag{3.19}$$

The function Ψ_{BA} is called (one-point) Baker–Akhiezer function. An easy corollary of the uniqueness of the Baker–Akhiezer function is

Theorem 3.1 [11, 12]. *Let Ψ_{BA} be the Baker–Akhiezer function defined by Lemma 3.1. Then for each $j = 1, 2, 3, \dots$ there is a unique differential operator B_j in x such that the equation*

$$(\partial_{t_j} - B_j)\Psi_{BA} = 0 \tag{3.20}$$

holds.

The operators B_j above can be easily expressed in terms of the dressing operator W for the Baker–Akhiezer function. Namely, the infinite series (3.19) can be represented as

$$\Psi_{BA} = W \exp\left(xk + \sum_{j \geq 1} t_j k^j\right), \tag{3.21}$$

where W is of the form (2.3). The corresponding Lax operator of the KP hierarchy is $\mathcal{L} = W\partial_x W^{-1}$. By the definition we have

$$\mathcal{L}\Psi_{BA} = k\Psi_{BA}. \tag{3.22}$$

The operator B_j in Theorem 3.1 was defined as the unique monic order j operator such that the congruence

$$(k^j - B_j)\Psi_{BA} = O(1/k) \exp\left(xk + \sum_{j \geq 1} t_j k^j\right)$$

holds. Using (3.22), it is easy to identify $B_j = \mathcal{L}_+^j$. Indeed

$$(k^j - \mathcal{L}_+^j)\Psi_{BA} = (\mathcal{L}^j - \mathcal{L}_+^j)\Psi_{BA} = \mathcal{L}_-^j \Psi_{BA} = O(1/k) \exp\left(xk + \sum_{j \geq 1} t_j k^j\right)$$

The compatibility conditions of Eq. (3.20) imply

Corollary 3.1 *The operators B_j defined by the BA function satisfies the equations*

$$[\partial_{t_j} - B_j, \partial_{t_l} - B_l] = 0. \tag{3.23}$$

It is the Zakharov–Shabat form (1.2) of the KP hierarchy. Note that Eq. (3.20) implies the evolution equation for the dressing operator:

$$\partial_{t_j} W = -(W \partial_x^j W^{-1})_- W, \tag{3.24}$$

where $(\dots)_-$ is the projection to negative powers of the operator ∂_x .

3.2.2. The dual Baker–Akhiezer function For further comparison with the tau-functional formulation of the KP hierarchy let us present the notion of the *dual* (adjoint) Baker–Akhiezer function introduced in [26] (see the details in [27, 34]).

First we define duality for divisors of degree g . For a generic effective degree g divisor $\mathcal{D} = Q_1 + \dots + Q_g$ define $d\Omega$ to be the abelian differential (unique up to a constant factor) with only a second order pole at P_∞ vanishing (with the corresponding multiplicities) at the points Q_s . The zero divisor of $d\Omega$ is of degree $2g$. Hence, it has other g zeros at some points $Q_1^\dagger, \dots, Q_g^\dagger$. The divisor

$$\mathcal{D}^\dagger = Q_1^\dagger + \dots + Q_g^\dagger$$

is called *dual to \mathcal{D}* . By the definition we have the equality

$$\mathcal{D} + \mathcal{D}^\dagger = \mathcal{K} + 2P_\infty \tag{3.25}$$

(where \mathcal{K} is the canonical class), which under the Abel transform takes the form

$$\vec{A}(\mathcal{D}) + \vec{A}(\mathcal{D}^\dagger) + 2\vec{K} - 2\vec{A}(P_\infty) = 0 \tag{3.26}$$

The dual (adjoint) Baker–Akhiezer function Ψ_{BA}^\dagger has the divisor of poles \mathcal{D}^\dagger and in the vicinity of P_∞ it has the form

$$\Psi_{BA}^\dagger = \exp\left(-xk - \sum_{j \geq 1} t_j k^j\right) \left(1 + \xi_1^\dagger k^{-1} + \xi_2^\dagger k^{-2} + \dots\right). \tag{3.27}$$

The differential $\Psi_{BA}(x, \mathbf{t}, P)\Psi_{BA}^\dagger(x, \mathbf{t}', P)d\Omega(P)$, where we have denoted the set of times as $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$ for brevity, is holomorphic everywhere on Γ except the

point P_∞ (because poles of the Baker–Akhiezer functions are cancelled by zeros of $d\Omega$). Therefore, its “residue” at this point is equal to zero, i.e.,

$$\oint_{C_\infty} \Psi_{BA}(x, \mathbf{t}, P) \Psi_{BA}^\dagger(x, \mathbf{t}', P) d\Omega(P) = 0 \tag{3.28}$$

for all \mathbf{t}, \mathbf{t}' , where C_∞ is a small contour around the point P_∞ . Equation (3.28) is equivalent to the equation

$$\operatorname{res}_{P_\infty} \left(\partial_x^i \Psi_{BA}(x, \mathbf{t}, P) \Psi_{BA}^\dagger(x, \mathbf{t}, P) \right) d\Omega(P) = 0, \quad i = 1, 2, 3, \dots \tag{3.29}$$

from which one can derive the following theorem (see [34] and [27] for more details).

Theorem 3.2. *The dual Baker–Akhiezer function is equal to*

$$\Psi_{BA}^\dagger = (W^\dagger)^{-1} \exp\left(-xk - \sum_{j \geq 1} t_j k^j\right) \tag{3.30}$$

and satisfies the adjoint equations

$$\mathcal{L}^\dagger \Psi_{BA}^\dagger = k \Psi_{BA}^\dagger, \quad -\partial_{t_j} \Psi_{BA}^\dagger = B_j^\dagger \Psi_{BA}^\dagger. \tag{3.31}$$

For completeness we outline here a direct proof of the theorem. Equation (3.31) immediately follow from (3.24) and (3.30):

$$\begin{aligned} -\partial_{t_j} \Psi_{BA}^\dagger &= \left(k^j (W^\dagger)^{-1} - (W^\dagger)^{-1} \partial_{t_j} W^\dagger (W^\dagger)^{-1} \right) \exp\left(-xk - \sum_{j \geq 1} t_j k^j\right) \\ &= \left(k^j (W^\dagger)^{-1} - ((W^\dagger)^{-1} (-\partial_x)^j W^\dagger)_- (W^\dagger)^{-1} \right) \exp\left(-xk - \sum_{j \geq 1} t_j k^j\right) \\ &= \left((\mathcal{L}^\dagger)^j - (\mathcal{L}^\dagger)_-^j \right) \Psi_{BA}^\dagger = B_j^\dagger \Psi_{BA}^\dagger. \end{aligned}$$

In order to prove (3.30) we note that Eq. (3.28) written in the local parameter k implies

$$b_m = \frac{1}{2\pi i} \partial_{x'}^m \oint_{C_\infty} \Psi_{BA}(x, \mathbf{t}, k) \Psi_{BA}^\dagger(x', \mathbf{t}, k) \varphi(k) \frac{dk}{2\pi i} \Big|_{x'=x} = 0 \quad \text{for all } m \geq 0.$$

Here

$$\varphi(k) = \frac{d\Omega}{dk} = \sum_{j \geq 0} \varphi_j k^{-j}.$$

We set

$$\Psi_{BA}^\dagger = V \exp\left(-xk - \sum_{j \geq 1} t_j k^j\right), \quad V = 1 + \xi_1^\dagger \partial_x^{-1} + \xi_2^\dagger \partial_x^{-2} + \dots$$

A calculation similar to the one given above in the proof of Proposition 2.1 yields:

$$b_m = \sum_{l=0}^m (-1)^{m-l} \frac{m!}{(m-l)!} \varphi_l \left(W V^\dagger \right)_{-m+l-1} = 0 \quad \text{for all } m \geq 0.$$

This is a homogeneous triangular system of linear equations for the coefficients $(WV^\dagger)_{-l}$. The unique solution is $(WV^\dagger)_{-l} = 0$ for all $l \geq 1$, hence $WV^\dagger = 1$, i.e. $V = (W^\dagger)^{-1}$. \square

The Baker–Akhiezer function and the dual Baker–Akhiezer function can be explicitly written in terms of the Riemann–theta function [11]. The corresponding formulas are well known and we do not present them here.

3.2.3. The tau-function Without loss of generality we can put $x = 0$ for simplicity. The dependence on x can be restored by the substitution $t_1 \rightarrow t_1 + x$.

The theta-functional formula for the Baker–Akhiezer function and the expansion (3.10) of the Abel map near P_∞ allows to reformulate the above presented construction of algebraic–geometrical construction in terms of the tau-functional formulation of the KP hierarchy. Namely, we have the following theorem.

Theorem 3.3 [11, 28] *The right hand side of the equation*

$$\tau^{\text{KP}}(\mathbf{t}) = \exp\left(-\frac{1}{2} \sum_{i,j \geq 1} \Omega_{ij} t_i t_j\right) \theta\left(\sum_{j \geq 1} \vec{U}_j t_j + \vec{Z}\right), \tag{3.32}$$

where the constant vector \vec{Z} is parameterized through the divisor \mathcal{D} as

$$\vec{Z} = -\vec{A}(\mathcal{D}) - \vec{K} + \vec{A}(P_\infty) \tag{3.33}$$

is the KP tau-function.

We omit the proof since it is well known.

Using (3.16), it is straightforward to check that the tau-function (3.32) does satisfy the Hirota–Miwa equation (2.45) which is the generating equation for the KP hierarchy. It appears to be equivalent to the Fay identity.

It is interesting to compare Eq. (3.28) and the bilinear relation (1.6) for the tau-function. They coincide if

$$\begin{aligned} d\Omega &= \frac{\tau^{\text{KP}}(-[k^{-1}])\tau^{\text{KP}}([k^{-1}])}{(\tau^{\text{KP}}(0))^2} dk \\ &= \frac{\theta(\vec{A}(P) - \vec{A}(\mathcal{D}) - \vec{K})\theta(\vec{A}(P) - \vec{A}(\mathcal{D}^\dagger) - \vec{K})}{\theta^2(\vec{A}(\mathcal{D}) + \vec{K} - \vec{A}(P_\infty))} \exp\left(-\sum_{i,j \geq 1} \Omega_{ij} \frac{k^{-i-j}}{ij}\right) dk. \end{aligned} \tag{3.34}$$

Using (3.17), we can rewrite this as

$$d\Omega = C \frac{\theta(\vec{A}(P) - \vec{A}(\mathcal{D}) - \vec{K})\theta(\vec{A}(P) - \vec{A}(\mathcal{D}^\dagger) - \vec{K})}{\theta_*^2(\vec{A}(P) - \vec{A}(P_\infty))} d\zeta, \tag{3.35}$$

where the holomorphic differential $d\zeta$ is given by (3.18). Its properties (see [25]) imply that the differential in the right hand side is a well-defined meromorphic differential on Γ with the only second order pole at P_∞ and $2g$ zeros at the points of the divisors $\mathcal{D}, \mathcal{D}^\dagger$. Therefore, it has all the properties of the differential $d\Omega$ and hence must be proportional to it. The equality (3.35) just reflects this fact.

Remark. The function Ψ_{BA} and the wave function Ψ^{KP} introduced in Sect. 2.3 (see (2.42)) differ by a normalization factor depending on $k(P)$. From (3.34) it follows that

$$\Psi_{BA}(x, \mathbf{t}, P)\Psi_{BA}^\dagger(x, \mathbf{t}', P)d\Omega = \Psi^{KP}(x, \mathbf{t}, k)\Psi^{\dagger KP}(x, \mathbf{t}', k)dk. \tag{3.36}$$

3.3. *Curves with involution: solutions to the CKP hierarchy.* Let Γ be a smooth genus g algebraic curve with involution ι having $2(n + 1) > 0$ fixed points. By the Riemann-Hurwitz formula $g = 2g_0 + n$ where g_0 is the genus of the factor-curve $\Gamma_0 = \Gamma/\iota$. It is known that on Γ there is a basis of a - and b -cycles with canonical intersection matrix: $a_i \cdot a_j = b_i \cdot b_j = 0, a_i \cdot b_j = \delta_{ij}$; and such that in this basis the action of the involution ι has the form

$$\iota(a_i) = a_{i+g_0}, \quad \iota(b_i) = b_{i+g_0}, \quad i = 1, \dots, g_0, \tag{3.37}$$

and

$$\iota(a_i) = -a_i, \quad \iota(b_i) = -b_i, \quad i = 2g_0 + 1, \dots, 2g_0 + n. \tag{3.38}$$

Let the marked point P_∞ on Γ be one of the fixed points of the involution, $\iota(P_\infty) = P_\infty$ and let $z = k^{-1}$ be a local coordinate in the neighborhood of P_∞ that is odd with respect to the involution, $\iota^*(k) = -k$. From the definition of the abelian differentials $d\Omega_j$ in Sect. 3.1 it follows that

$$d\Omega_j(\iota P) = (-1)^j d\Omega_j(P) \tag{3.39}$$

and, therefore,

$$\Omega_j(\iota P) = (-1)^j \Omega_j(P). \tag{3.40}$$

Suppose that the divisor \mathcal{D} satisfies the constraint

$$\mathcal{D} + \iota\mathcal{D} = \mathcal{K} + 2P_\infty. \tag{3.41}$$

Then for the Baker–Akhiezer function defined by Γ, P_∞ , the local coordinate k^{-1} and the divisor \mathcal{D} the equation

$$\Psi_{BA}^\dagger(t_1, 0, t_3, 0, \dots, P) = \Psi_{BA}(t_1, 0, t_3, 0, \dots, \iota P). \tag{3.42}$$

holds. The bilinear relation (3.28) takes the form

$$\oint_{C_\infty} \Psi_{BA}(x, \mathbf{t}_0, P)\Psi_{BA}(x, \mathbf{t}'_0, \iota P)d\Omega(P) = 0 \tag{3.43}$$

for all $\mathbf{t}_0, \mathbf{t}'_0$.

Using explicit formulas for the Baker–Akhiezer function and its dual we can write the relation (3.42) in the explicit form:

$$\begin{aligned} \Psi_{BA}(t_1, 0, t_3, 0, \dots, \iota P) &= \exp\left(\sum_{j \geq 1, j \text{ odd}} t_j \Omega_j(\iota P)\right) \\ &\quad \theta\left(\vec{A}(\iota P) + \sum_{j \geq 1, j \text{ odd}} \vec{U}_j t_j - \vec{A}(\mathcal{D}) - \vec{K}\right) \theta\left(\vec{A}(\mathcal{D}) + \vec{K} - \vec{A}(P_\infty)\right) \\ &\times \frac{\theta\left(\vec{A}(\iota P) - \vec{A}(\mathcal{D}) - \vec{K}\right) \theta\left(\sum_{j \geq 1, j \text{ odd}} \vec{U}_j t_j - \vec{A}(\mathcal{D}) - \vec{K} + \vec{A}(P_\infty)\right)}{\theta\left(\vec{A}(\iota P) - \vec{A}(\mathcal{D}) - \vec{K}\right) \theta\left(\sum_{j \geq 1, j \text{ odd}} \vec{U}_j t_j - \vec{A}(\mathcal{D}) - \vec{K} + \vec{A}(P_\infty)\right)} \end{aligned}$$

$$\begin{aligned}
 &= \Psi_{BA}^\dagger(t_1, 0, t_3, 0, \dots, P) = \exp\left(-\sum_{j \geq 1, j \text{ odd}} t_j \Omega_j(P)\right) \\
 &\quad \theta\left(\vec{A}(P) - \sum_{j \geq 1, j \text{ odd}} \vec{U}_j t_j + \vec{A}(\mathcal{D}) + \vec{K} - 2\vec{A}(P_\infty)\right) \theta\left(\vec{A}(\mathcal{D}) + \vec{K} - \vec{A}(P_\infty)\right) \\
 &\quad \times \frac{\theta\left(\vec{A}(P) + \vec{A}(\mathcal{D}) + \vec{K} - 2\vec{A}(P_\infty)\right) \theta\left(\sum_{j \geq 1, j \text{ odd}} \vec{U}_j t_j - \vec{A}(\mathcal{D}) - \vec{K} + \vec{A}(P_\infty)\right)}{\theta\left(\vec{A}(P) + \vec{A}(\mathcal{D}) + \vec{K} - 2\vec{A}(P_\infty)\right) \theta\left(\sum_{j \geq 1, j \text{ odd}} \vec{U}_j t_j - \vec{A}(\mathcal{D}) - \vec{K} + \vec{A}(P_\infty)\right)}.
 \end{aligned} \tag{3.44}$$

The tau-function of the CKP hierarchy is the square root of

$$\tau^{\text{KP}}(\mathbf{t}_0) = \exp\left(-\frac{1}{2} \sum_{i, j \geq 1, i, j \text{ odd}} \Omega_{ij} t_i t_j\right) \theta\left(\sum_{j \geq 1, j \text{ odd}} \vec{U}_j t_j - \vec{A}(\mathcal{D}) - \vec{K} + \vec{A}(P_\infty)\right), \tag{3.45}$$

where the divisor \mathcal{D} satisfies the condition (3.41). This condition guarantees that the functions Ψ_{BA} and Ψ (2.12) are related as

$$\Psi_{BA}(\mathbf{t}_0, P) \Psi_{BA}(\mathbf{t}'_0, \iota P) d\Omega(P) = \Psi(\mathbf{t}_0, k) \Psi(\mathbf{t}'_0, -k) dk \tag{3.46}$$

(cf. (3.36)) and, therefore, the bilinear relation for the CKP wave function is satisfied. This proves, in an indirect way, that the tau-function (3.45) with the even times included is the tau-function corresponding to the KP extension of a CKP solution. A direct verification of this fact starting from the explicit formula (3.45) is problematic.

The statement of the following theorem is in fact a corollary of Theorem 2.1 and the above identification of the square root of (3.45) with the tau-function of the CKP hierarchy but below we give its closed algebraic–geometrical proof.

Theorem 3.4. *Let Γ be a genus g smooth curve with holomorphic involution ι having at least one fixed point P_∞ and let Y be the locus in the Jacobian $\text{Jac}(\Gamma)$ spanned by the vectors such that $\vec{Z} = -\vec{A}(\mathcal{D}) - \vec{K}$, where the divisor \mathcal{D} satisfies the condition (3.41). Then for any point $Q \in \Gamma$ and $\vec{Z} \in Y$ the equation*

$$\begin{aligned}
 &\theta(\vec{Z}) \partial_1 \theta(\vec{A}(Q) - \vec{A}(\iota Q) + \vec{Z}) - \theta(\vec{A}(Q) - \vec{A}(\iota Q) + \vec{Z}) \partial_1 \theta(\vec{Z}) \\
 &\quad + 2\Omega_1(Q) \theta(\vec{Z}) \theta(\vec{A}(Q) - \vec{A}(\iota Q) + \vec{Z}) = C(Q) \theta^2(\vec{A}(Q) + \vec{Z})
 \end{aligned} \tag{3.47}$$

with

$$\partial_1 \theta(\vec{Z}) := \partial_t \theta(\vec{Z} + \vec{U}_1 t) \Big|_{t=0}$$

holds.

Proof. Let us fix a point $Q \in \Gamma$, an effective divisor \mathcal{D} of degree g and define the auxiliary Baker–Akhiezer function $\Psi_Q(\mathbf{t}_0, P)$ by the following properties:

- (1⁰) Outside P_∞ the singularities of Ψ_Q are poles at the divisor $\mathcal{D} + \iota Q$;
- (2⁰) It has simple zero at the point Q , i.e., $\Psi_Q(\mathbf{t}_0, Q) = 0$;
- (3⁰) In a small neighborhood of P_∞ the function Ψ_Q has the form

$$\Psi_Q(\mathbf{t}_0, P) = e^{\zeta(\mathbf{t}_0, k)} \left(1 + \sum_{j \geq 1} \xi_{j, Q}(\mathbf{t}_0) k^{-j}\right), \quad k = k(P). \tag{3.48}$$

The standard argument shows that this function is unique up to a common factor. The explicit formula for Ψ_Q in theta-functions is

$$\Psi_Q(\mathbf{t}_0, P) = \frac{\theta\left(\vec{A}(P) - \vec{A}(\iota Q) + \vec{A}(Q) + \vec{Z}_{\mathbf{t}_0}\right)\theta\left(\vec{Z}\right)}{\theta\left(\vec{A}(Q) - \vec{A}(\iota Q) + \vec{Z}_{\mathbf{t}_0}\right)\theta\left(\vec{A}(P) + \vec{Z}\right)} \exp\left(\Omega_0(P) + \sum_{j \geq 1, \text{ odd}} t_j \Omega_j(P)\right), \tag{3.49}$$

where $\vec{Z}_{\mathbf{t}_0} = \vec{Z} + \sum_{j \geq 1, \text{ odd}} U_j t_j$ and Ω_0 is the abelian integral of the normalized dipole differential $d\Omega_0$ with simple poles at the points $Q, \iota Q$ with residues ± 1 :

$$\Omega_0(P) = \int_{Q_0}^P d\Omega_0.$$

□

Remark. The standard Baker–Akhiezer function Ψ_{BA} corresponds to the case $Q = P_\infty$.

Consider the differential $\widetilde{d\Omega}(P) = \partial_{t_1} \Psi_Q(P) \Psi_Q(\iota P) d\Omega(P)$, where $d\Omega$ is the differential entering the bilinear relation (3.28). It is a meromorphic differential on Γ with the only pole at P_∞ . Hence it has no residue P_∞ . Computing the residue in terms of the coefficients of the expansion (3.48), we get

$$2\xi_{2,Q} - \xi_{1,Q}^2 + \partial_{t_1} \xi_{1,Q} + c_1 = 0, \tag{3.50}$$

where c_1 is a constant defined by the Laurent expansion of $d\Omega$ at P_∞ .

Consider now the differential $d\Omega_Q(P) = \Psi_Q(P) \Psi_{BA}(\iota P) d\Omega(P)$. It is a meromorphic differential with poles at P_∞ and ιQ . Therefore,

$$f_Q := \text{res}_{P_\infty} d\Omega_Q = \xi_{1,Q} - \xi_1 = -\text{res}_{\iota Q} d\Omega_Q = -\phi_Q \phi, \tag{3.51}$$

where

$$\phi_Q := \text{res}_{\iota Q} (\Psi_Q d\Omega), \quad \phi = \Psi_{BA}(\mathbf{t}_0, \iota Q). \tag{3.52}$$

The residue argument for the differential $\widetilde{d\Omega}_Q(P) = \partial_{t_1} \Psi_Q(P) \Psi_{BA}(\iota P) d\Omega(P)$ gives the relation

$$\xi_{2,Q} + \xi_2 - \xi_{1,Q} \xi_1 + \partial_{t_1} \xi_{1,Q} + c_1 = -(\partial_{t_1} \phi_Q) \phi. \tag{3.53}$$

Then, using (3.50), we obtain

$$\frac{1}{2} (f_Q^2 + \partial_{t_1} f_Q) = -(\partial_{t_1} \phi_Q) \phi. \tag{3.54}$$

From comparison of (3.51) and (3.54) it follows that

$$\partial_{t_1} \log \phi_Q = \frac{1}{2} (f_Q + \partial_{t_1} \log f_Q). \tag{3.55}$$

Recalling the definition of ϕ_Q and using formula (3.49), we get

$$\partial_{t_1} \log \phi_Q = \partial_{t_1} \log \left(\frac{\theta(\vec{A}(Q) + \vec{Z}_{t_0})}{\theta(\vec{A}(Q) - \vec{A}(tQ) + \vec{Z}_{t_0})} \right) + \Omega_1(tQ). \tag{3.56}$$

The expansion of (3.49) around P_∞ yields

$$f_Q = \partial_{t_1} \log \left(\frac{\theta(\vec{Z}_{t_0})}{\theta(\vec{A}(Q) - \vec{A}(tQ) + \vec{Z}_{t_0})} \right) + \Omega_{01}, \tag{3.57}$$

where Ω_{01} equals the coefficient at k^{-1} in the expansion of Ω_0 at P_∞ . The Riemann’s bilinear relation for the differentials $d\Omega_1$ and $d\Omega_0$ has the form

$$\Omega_{01} = \Omega_1(tQ) - \Omega_1(Q) = 2\Omega_1(tQ). \tag{3.58}$$

Therefore, Eqs. (3.54) and (3.57) imply

$$\partial_{t_1} \log \left(\frac{\theta^2(\vec{A}(Q) + \vec{Z}_{t_0})}{\theta(\vec{A}(Q) - \vec{A}(tQ) + \vec{Z}_{t_0})\theta(\vec{Z}_{t_0})} \right) = \partial_{t_1} \log f_Q. \tag{3.59}$$

Equations (3.57) and (3.59) with $t_0 = 0$ after integration in t_1 give (3.47) with constant $C(Q, \vec{Z})$ which is ∂_{t_1} -invariant, i.e. $C(Q, \vec{Z}) = C(Q, \vec{Z} + t_1\vec{U}_1)$ for any value of t_1 . For a generic curve the complex line $\vec{Z} + t_1\vec{U}_1$ is dense in the Jacobian. Hence, the integration constant C does not depend on \vec{Z} and depends on Q only. Since the matrix of b -periods depends analytically on the curve and C is independent of \vec{Z} for generic curve it is independent of \vec{Z} for any curve. □

3.4. Degeneration of algebraic–geometrical solutions: soliton solutions. The algebraic–geometrical integration scheme naturally extends to the case of singular curves. In particular, the case when Γ is the Riemann sphere $\mathbb{C}P^1$ with nodes (double points) corresponds to soliton solutions. N -soliton solutions of the CKP hierarchy are obtained by imposing certain constraints on the parameters of $2N$ -soliton solutions to the KP hierarchy. We recall that $\tau = \sqrt{\tau^{KP}}$, with “even” times t_{2k} put equal to zero and it is implied that the parameters of the KP tau-function τ^{KP} are chosen in a special way.

M -solutions of the KP hierarchy are constructed starting from a singular curve which is $\mathbb{C}P^1$ with M double points. Let z be the global coordinate. The Baker–Akhiezer function has simple poles at M points q_i . It has the form

$$\Psi^{KP}(\mathbf{t}, z) = \exp\left(\sum_{j \geq 1} t_j z^j\right) \left(1 + \sum_{l=1}^M \frac{y_l(\mathbf{t})}{z - q_l}\right). \tag{3.60}$$

Let us impose M linear conditions of the form

$$\operatorname{res}_{z=q_i} \left[\Psi^{KP}(\mathbf{t}, z) dz \right] = -\alpha_i (p_i - q_i) \Psi^{KP}(\mathbf{t}, p_i), \quad i = 1, \dots, M, \tag{3.61}$$

which mean that the points p_i, q_i are glued together forming a double point. Here α_i are complex parameters. These conditions make the Baker–Akhiezer function unique (up to a common multiplier). The conditions (3.61) are equivalent to the following linear system for y_l :

$$y_i + \sum_{l=1}^M \frac{\tilde{\alpha}_l y_l}{p_i - q_l} = -\tilde{\alpha}_i, \tag{3.62}$$

where

$$\tilde{\alpha}_i = \alpha_i (p_i - q_i) \exp\left(\sum_{j \geq 1} (p_i^j - q_i^j) t_j\right).$$

Solving this system, we obtain the Baker–Akhiezer function in the explicit form:

$$\Psi^{\text{KP}} = \frac{\begin{vmatrix} 1 & \frac{1}{z-q_1} & \frac{1}{z-q_2} & \cdots & \frac{1}{z-q_M} \\ \tilde{\alpha}_1 & 1 + \frac{\tilde{\alpha}_1}{p_1-q_1} & \frac{\tilde{\alpha}_1}{p_1-q_2} & \cdots & \frac{\tilde{\alpha}_1}{p_1-q_M} \\ \tilde{\alpha}_2 & \frac{\tilde{\alpha}_2}{p_2-q_1} & 1 + \frac{\tilde{\alpha}_2}{p_2-q_2} & \cdots & \frac{\tilde{\alpha}_2}{p_2-q_M} \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{\alpha}_M & \frac{\tilde{\alpha}_M}{p_M-q_1} & \frac{\tilde{\alpha}_M}{p_M-q_2} & \dots & 1 + \frac{\tilde{\alpha}_M}{p_M-q_M} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{\tilde{\alpha}_1}{p_1-q_1} & \frac{\tilde{\alpha}_1}{p_1-q_2} & \cdots & \frac{\tilde{\alpha}_1}{p_1-q_M} \\ \frac{\tilde{\alpha}_2}{p_2-q_1} & 1 + \frac{\tilde{\alpha}_2}{p_2-q_2} & \cdots & \frac{\tilde{\alpha}_2}{p_2-q_M} \\ \dots & \dots & \dots & \dots \\ \frac{\tilde{\alpha}_M}{p_M-q_1} & \frac{\tilde{\alpha}_M}{p_M-q_2} & \cdots & 1 + \frac{\tilde{\alpha}_M}{p_M-q_M} \end{vmatrix}} \exp\left(\sum_{j \geq 1} t_j z^j\right). \tag{3.63}$$

The denominator of this expression is the tau-function.

The general KP tau-function for M -soliton solution has $3M$ arbitrary parameters α_i, p_i, q_i ($i = 1, \dots, M$) and is given by

$$\tau^{\text{KP}}(x, \mathbf{t}) = \det_{1 \leq i, j \leq M} \left(\delta_{ij} + \alpha_i \frac{p_i - q_i}{p_i - q_j} \exp\left((p_i - q_i)x + \sum_{k \geq 1} (p_i^k - q_i^k) t_k\right) \right) \tag{3.64}$$

Let us denote this tau-function as

$$\tau^{\text{KP}} \left[\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{M-1} & \alpha_M \\ p_1, q_1 & p_2, q_2 & p_3, q_3 & p_4, q_4 & \cdots & p_{M-1}, q_{M-1} & p_M, q_M \end{matrix} \right].$$

The parameters p_i, q_i are sometimes called momenta of solitons.

In the CKP case we have the involution $z \rightarrow -z$ which means that the double points should be symmetric under the involution. The multi-soliton tau-function of the CKP hierarchy is the square root of the τ^{KP} specialized as

$$\tau^{\text{KP}} \left[\begin{matrix} \alpha_0 & \alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \cdots & \alpha_N & \alpha_N \\ p_0, -p_0 & p_1, -q_1 & q_1, -p_1 & p_2, -q_2 & q_2, -p_2 & \cdots & p_N, -q_N & q_N, -p_N \end{matrix} \right],$$

$$(3.65)$$

where it is assumed that even times evolution is suppressed ($t_{2k} = 0$ for all $k \geq 1$). Clearly, the total number of independent parameters is $3N + 2$. If $\alpha_0 = 0$, the tau-function (3.65) reduces to

$$\tau^{\text{KP}} \left[\begin{array}{cccccc} \alpha_1 & & \alpha_1 & & \alpha_2 & & \alpha_2 & & \cdots & & \alpha_N & & \alpha_N \end{array}; \begin{array}{cccccc} p_1, -q_1 & ; & q_1, -p_1 & ; & p_2, -q_2 & ; & q_2, -p_2 & ; & \cdots & ; & p_N, -q_N & ; & q_N, -p_N \end{array} \right], \tag{3.66}$$

and it is this tau-function which is usually called the N -soliton CKP tau-function in the literature (see, e.g. [6]). It is a specialization of $2N$ -soliton KP tau-function and has $3N$ free parameters.

The simplest example is one-soliton solution. The tau-function for one CKP soliton is the square root of a specialization of 2-soliton tau-function of the KP hierarchy:

$$\tau^{\text{KP}} = 1 + 2\alpha w - \frac{\alpha^2(p - q)^2}{4pq} w^2, \tag{3.67}$$

where

$$w = e^{(p+q)x + \zeta(t_0, p) + \zeta(t_0, q)}, \quad \zeta(t_0, z) \text{ is given by (2.13)}. \tag{3.68}$$

A direct calculation shows that $\partial_x \psi^2$ (where ψ is given by (2.17)) for the solution (3.67) is a full square for all z .

Remark. It is instructive to prove directly that the tau-functions (3.65) and (3.66) satisfy Eq. (2.48). Consider (3.66) first. We represent the tau-function as

$$\tau^{\text{KP}} = \det_{2N \times 2N} (I + HK),$$

where H is the diagonal matrix $W_{jk} = \delta_{jk} W_j$ with matrix elements

$$H_{2i-1} = \alpha_i (p_i + q_i) \exp \left((p_i + q_i)x + \sum_{k \geq 1} t_k (p_i^k - (-q_i)^k) \right), \quad i = 1, \dots, N,$$

$$H_{2i} = \alpha_i (p_i + q_i) \exp \left((p_i + q_i)x + \sum_{k \geq 1} t_k (q_i^k - (-p_i)^k) \right), \quad i = 1, \dots, N,$$

and K is the Cauchy matrix $K_{jk} = 1/(x_j - y_k)$ with $x_{2i-1} = -y_{2i} = p_i, x_{2i} = -y_{2i-1} = q_i, i = 1, \dots, N$. We have:

$$\begin{aligned} \partial_{t_{2m}} \log \tau^{\text{KP}} \Big|_{t_{2k}=0} &= \partial_{t_{2m}} \log \det(I + HK) \Big|_{t_{2k}=0} = \partial_{t_{2m}} \text{tr} \log (I + HK) \Big|_{t_{2k}=0} \\ &= \text{tr} \left[VHK(I + HK)^{-1} \right] = \text{tr} V - \text{tr} \left[V(I + HK)^{-1} \right], \end{aligned}$$

where V is the diagonal matrix $V_{jk} = \delta_{jk} V_j$ with the matrix elements

$$V_{2i-1} = -V_{2i} = p_i^{2m} - q_i^{2m}.$$

Note also that when all even times are put equal to zero, we have also $H_{2i-1} = -H_{2i}$. Obviously, $\text{tr} V = 0$. A careful inspection shows that $(I + HK)_{2i-1, 2i-1}^{-1} = (I + HK)_{2i, 2i}^{-1}$,

and, therefore, $\text{tr} \left[V(I + HK)^{-1} \right] = 0$, too, and the conditions (2.48) are satisfied. Indeed, permuting rows and columns, one can see that the diagonal $(2i - 1, 2i - 1)$ and $(2i, 2i)$ minors of the matrix $I + HK$ are equal. As for the tau-function (3.65) with $\alpha_0 \neq 0$, it is obvious that the additional pair of soliton momenta of the form $p_0, -p_0$ does not lead to any extra dependence on the even times, and so the conditions (2.48) are still satisfied.

4. Elliptic Solutions

By elliptic solutions of the CKP equation (2.10) we mean solutions u that are double-periodic in the complex plane of the variable x with periods $2\omega_1, 2\omega_2$, $\text{Im}(\omega_2/\omega_1) > 0$. Equations of motion for their poles and their algebraic integrability is an easy corollary of the established above relation between the CKP and KP hierarchies and the well-developed theory of elliptic solutions to the KP hierarchy, equivalent to the theory of the elliptic Calogero–Moser (eCM) system.

Namely, elliptic solutions of the CKP equation can be extended to elliptic solutions of the KP equation and further to the whole KP hierarchy. From that perspective the pole dynamics of the elliptic solutions of the CKP equation in t_3 is just the restriction of t_3 -dynamics generated by the Hamiltonian H_3 of the eCM system,

$$\begin{cases} \dot{x}_i = -3p_i^2 + 3 \sum_{j \neq i} \wp(x_i - x_j) - 6c \\ \dot{p}_i = -3 \sum_{j \neq i} (p_i + p_j) \wp'(x_i - x_j), \end{cases} \tag{4.1}$$

onto the locus of turning points $p_i = 0$ that is invariant under \mathbf{t}_0 flows of the eCM system, i.e.

$$\dot{x}_i = 3 \sum_{k \neq i} \wp(x_i - x_k) - 6c, \tag{4.2}$$

where c is a constant and dot means the t_3 -derivative. Here \wp is the Weierstrass \wp -function which is an even double-periodic function with periods $2\omega_1, 2\omega_2$ having second order poles at the lattice points $2\omega_1 m_1 + 2\omega_2 m_2$ with integer m_1, m_2 and

$$\wp(x) = \frac{1}{x^2} + O(x^2), \quad x \rightarrow 0.$$

For further use recall the definitions of the Weierstrass functions. The Weierstrass σ -function is given by the infinite product

$$\sigma(x) = \sigma(x | \omega_1, \omega_2) = x \prod_{s \neq 0} \left(1 - \frac{x}{s} \right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}, \quad s = 2\omega_1 m_1 + 2\omega_2 m_2, \quad m_1, m_2 \in \mathbb{Z}.$$

The Weierstrass ζ - and \wp -functions are connected with the σ -function as follows: $\zeta(x) = \sigma'(x)/\sigma(x)$, $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$.

The algebraic integrability of the eCM system established in [19] restricted to the locus of turning points can be stated as follows.

Theorem 4.1. For each set of constants $x_i^0 \neq x_j^0$ define the algebraic curve Γ by the characteristic equation $\det(zI - L) = 0$ for the matrix

$$L_{ii} = 0, \quad L_{ij} = -\Phi(x_i^0 - x_j^0, \lambda), \quad i \neq j, \tag{4.3}$$

where

$$\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda)\sigma(x)} e^{-\zeta(\lambda)x}. \tag{4.4}$$

Let P_∞ be the point on Γ that is the pre-image of $\lambda = 0$ in the neighborhood of which z has the expansion $z = -(n - 1)\lambda^{-1} + O(\lambda)$. Then the solution of (4.2) with the initial conditions $x_i(0) = x_i^0$ are roots $x_i(t_3)$ of the equation

$$\theta\left(\vec{U}_1 x_i + \vec{U}_3 t_3 + \vec{Z} \mid T\right) = 0. \tag{4.5}$$

Here $\theta(z \mid T)$ is Riemann theta-function defined by the matrix of b -periods of normalized holomorphic differentials on Γ ; the vectors \vec{U}_j are given by (3.8) with $d\Omega_j$ defined in (3.6); the vector \vec{Z} is in the locus Y defined in Theorem 3.4.

The elliptic solutions are particular cases of the general algebraic–geometrical solutions considered in the previous section. The corresponding spectral data are singled out by the following constraint: the vectors $2\omega_1 \vec{U}_1, 2\omega_2 \vec{U}_1$ are in the lattice of periods of the Jacobian of the spectral curve, where \vec{U}_1 is the vector of b -periods of the normalized differential with the only pole (of order 2) at the marked point P_∞ .

4.1. The generating problem. For completeness, in this section we present the scheme proposed in [19] which allows one to derive the equations of motion for poles of elliptic solutions to a variety of soliton equations together with their Lax-type representation (see more in [20]). With the help of this scheme we will get the Eq. (4.2) and the Lax matrix (4.3) directly without use of relations to the theory of the eCM system.

The elliptic solution of the CKP equation is an elliptic function with double poles at the points x_i :

$$u = -\frac{1}{2} \sum_{i=1}^n \wp(x - x_i) + c, \tag{4.6}$$

where c is a constant. The poles depend on the times t_3, t_5 (as well as on the higher times) and are assumed to be all distinct. The corresponding CKP tau-function has the form

$$\tau(x, \mathbf{t}_0) = C_0 e^{cx^2/2} \left(\prod_{i=1}^n \sigma(x - x_i(\mathbf{t}_0)) \right)^{1/2}. \tag{4.7}$$

In the rest of this section we denote $t_3 = t$. According to the scheme proposed in [19], the basic tool is the auxiliary linear problem $\partial_t \Psi = B_3 \Psi$ for the wave function Ψ , i.e.,

$$\partial_t \Psi = \partial_x^3 \Psi + 6u \partial_x \Psi + 3u' \Psi, \tag{4.8}$$

for which one can state the following problem: characterize an elliptic in x function u of the form (4.6) for which Eq. (4.8) has *double-Bloch solutions* $\Psi(x)$, i.e., solutions such that $\Psi(x + 2\omega_\alpha) = B_\alpha \Psi(x)$ with some Bloch multipliers B_α . Equations (2.17), (2.16) imply that the wave function has simple poles at the points x_i . Therefore, if a double-Bloch solution exists, then it is of the following pole ansatz form:

$$\Psi = e^{xz+tz^3} \sum_{i=1}^n c_i \Phi(x - x_i, \lambda), \tag{4.9}$$

where the coefficients c_i do not depend on x (but do depend on t, z and λ). Indeed, the function $\Phi(x, \lambda)$ given by formula (4.4) has the following monodromy properties:

$$\Phi(x + 2\omega_\alpha, \lambda) = e^{2(\zeta(\omega_\alpha)\lambda - \zeta(\lambda)\omega_\alpha)} \Phi(x, \lambda), \quad \alpha = 1, 2. \tag{4.10}$$

Therefore, the wave function Ψ given by (4.9) is a double-Bloch function with Bloch multipliers $B_\alpha = e^{2(\omega_\alpha z + \zeta(\omega_\alpha)\lambda - \zeta(\lambda)\omega_\alpha)}$ parameterized by z and λ .

In what follows we will often suppress the second argument of Φ writing simply $\Phi(x) = \Phi(x, \lambda)$. For further use note also that Φ has a simple pole at $x = 0$ with residue 1. The coefficients β_1, β_2 of its expansion

$$\Phi(x, \lambda) = \frac{1}{x} + \beta_1 x + \beta_2 x^2 + O(x^3) \quad \text{as } x \rightarrow 0,$$

are equal to

$$\beta_1 = -\frac{1}{2} \wp(\lambda), \quad \beta_2 = -\frac{1}{6} \wp'(\lambda). \tag{4.11}$$

The function Φ We will also need the x -derivatives $\Phi'(x, \lambda) = \partial_x \Phi(x, \lambda)$, $\Phi''(x, \lambda) = \partial_x^2 \Phi(x, \lambda)$ and so on.

Theorem 4.2 *The equations of motion (4.2) for poles x_i of elliptic solutions as functions of $t = t_3$ have the following commutation representation of the Manakov's triple kind:*

$$\dot{L} + [L, M] = 3D'(zI - L), \tag{4.12}$$

where

$$L_{ii} = 0, \quad L_{ij} = -\Phi(x_i - x_j, \lambda), \quad i \neq j; \tag{4.13}$$

the matrix M is defined by (4.18), and D' is the diagonal matrix $D'_{ik} = \delta_{ik} \sum_{j \neq i} \wp'(x_i - x_j)$.

Proof. Substituting (4.9) into (4.8) with $u = -\frac{1}{2} \sum_i \wp(x - x_i) + c$, we get:

$$\begin{aligned} & \sum_i \dot{c}_i \Phi(x - x_i) - \sum_i c_i \dot{x}_i \Phi'(x - x_i) = 3z^2 \sum_i c_i \Phi'(x - x_i) + 3z \sum_i c_i \Phi''(x - x_i) \\ & + \sum_i c_i \Phi'''(x - x_i) \\ & - 3z \left(\sum_k \wp(x - x_k) \right) \left(\sum_i c_i \Phi(x - x_i) \right) - 3 \left(\sum_k \wp'(x - x_k) \right) \left(\sum_i c_i \Phi'(x - x_i) \right) \end{aligned}$$

$$-\frac{3}{2} \left(\sum_k \wp'(x - x_k) \right) \left(\sum_i c_i \Phi(x - x_i) \right) + 6cz \sum_i c_i \Phi(x - x_i) + 6c \sum_i c_i \Phi'(x - x_i).$$

It is enough to cancel all poles in the fundamental domain which are at the points x_i (up to fourth order). It is easy to see that poles of the fourth order cancel identically. A direct calculation shows that the conditions of cancellation of third, second and first order poles have the form

$$zc_i = - \sum_{k \neq i} c_k \Phi(x_i - x_k), \tag{4.14}$$

$$c_i \dot{x}_i = -3z^2 c_i + 3c_i \sum_{k \neq i} \wp(x_i - x_k) - 3z \sum_{k \neq i} c_k \Phi(x_i - x_k) - 6cc_i, \tag{4.15}$$

$$\begin{aligned} \dot{c}_i &= -3(\beta_1 z + \beta_2) c_i - 3z c_i \sum_{k \neq i} \wp(x_i - x_k) + \frac{3}{2} c_i \sum_{k \neq i} \wp'(x_i - x_k) \\ &\quad - 3z \sum_{k \neq i} c_k \Phi'(x_i - x_k) - \frac{3}{2} \sum_{k \neq i} c_k \Phi''(x_i - x_k) + 6cz c_i \end{aligned} \tag{4.16}$$

which have to be valid for all $i = 1, \dots, n$. Substitution of (4.14) into (4.15) gives (4.2) (if the coefficients c_i are not identically zero). The conditions (4.14), (4.16) can be rewritten in the matrix form as linear problems for a vector $\mathbf{c} = (c_1, \dots, c_n)^T$:

$$\begin{cases} L\mathbf{c} = z\mathbf{c} \\ \dot{\mathbf{c}} = M\mathbf{c}, \end{cases} \tag{4.17}$$

where L is the matrix (4.13),

$$M = -3(\beta_1 z + \beta_2 - 2cz)I - 3zB - 3zD - \frac{3}{2}C + \frac{3}{2}D' \tag{4.18}$$

and the $n \times n$ matrices I, B, C, D , are given by $I_{ik} = \delta_{ik}$,

$$\begin{aligned} B_{ik} &= (1 - \delta_{ik})\Phi'(x_i - x_k), \\ C_{ik} &= (1 - \delta_{ik})\Phi''(x_i - x_k), \\ D_{ik} &= \delta_{ik} \sum_{j \neq i} \wp(x_i - x_j), \end{aligned} \tag{4.19}$$

The matrices L, B, C are off-diagonal while the matrices D, D' are diagonal.

The linear system (4.17) is overdetermined. Differentiating the first equation in (4.17) with respect to t , we see that the compatibility condition of the linear problems (4.17) is

$$(\dot{L} + [L, M])\mathbf{c} = 0. \tag{4.20}$$

One can prove the following matrix identity (see the ‘‘Appendix’’):

$$\dot{L} + [L, M] = 3D'(zI - L) - [\dot{X} - 3D, B], \tag{4.21}$$

where X is the diagonal matrix $X_{ik} = \delta_{ik}x_i$. Since $(zI - L)\mathbf{c} = 0$ according to (4.17) and $\dot{X} = 3D - 6cI$ according to (4.2), we see from (4.21) that the compatibility condition (4.20) is satisfied. From (4.21) it follows that the equations of motion have the commutation representation of the Manakov’s triple kind (4.12) [29]. \square

4.2. *The integrals of motion and the spectral curve.* It follows from Eq. (4.12) that the characteristic polynomial of the matrix L is an integral of motion. Indeed,

$$\begin{aligned} \frac{d}{dt} \log \det(L - zI) &= \frac{d}{dt} \operatorname{tr} \log(L - zI) \\ &= \operatorname{tr} \left[\dot{L}(L - zI)^{-1} \right] = -3 \operatorname{tr} D' = 0, \end{aligned} \tag{4.22}$$

where we have used Eq. (4.12) and the fact that $\operatorname{tr} D' = \sum_{i \neq j} \wp'(x_i - x_j) = 0$ (\wp' is an odd function). The expression $R(z, \lambda) = \det(zI - L(\lambda))$ is a polynomial in z of degree n . Its coefficients are integrals of motion (some of them may be trivial). For example:

$$\begin{aligned} n = 2 : \quad \det_{2 \times 2}(zI - L) &= z^2 + \wp(x_{12}) - \wp(\lambda), \\ n = 3 : \quad \det_{3 \times 3}(zI - L) &= z^3 + z \left(\wp(x_{12}) + \wp(x_{13}) + \wp(x_{23}) - 3\wp(\lambda) \right) - \wp'(\lambda), \end{aligned}$$

where $x_{ik} \equiv x_i - x_k$.

Remark. Although the Lax equation for matrices L, M does not hold, it follows from (4.22) that traces of the Lax matrix L (and therefore its eigenvalues) are integrals of motion: $\partial_t \operatorname{tr} L^m = 0, m \geq 1$. (This is equivalent to the equalities $\operatorname{tr}(D' L^m) = 0$ for $m \geq 1$ which are based on certain non-trivial identities for the \wp -function.) This means that the time evolution is an isospectral transformation of the Lax matrix L . Therefore, there should exist a matrix M_0 such that the Lax equation $\dot{L} + [L, M_0] = 0$ holds. In order to find it explicitly, we first note that by virtue of the matrix identity (B2) (see the ‘‘Appendix’’) we can write Eq. (4.12) in the form $\dot{L} + [L, \hat{M}] = -3D'L$, where

$$\hat{M} = M + 3z \left((\beta_1 - 2c)I + B + D \right) = -3\beta_2 I - \frac{3}{2} (C - D')$$

does not depend on z . Using again the identity (B2), one can see that

$$\begin{aligned} M_0 = \hat{M} - 3(B + D)L &= -3\beta_2 I - \frac{3}{2} (C - D') - 3(B + D)L \\ &= M + 3z(\beta_1 - 2c)I + 3(B + D)(zI - L) \end{aligned} \tag{4.23}$$

(β_1, β_2 are given in (4.11)).

The embedding into the Calogero–Moser dynamics discussed above implies that the integrals of motion I_k for the dynamical system (4.2) are restrictions of the Calogero–Moser integrals of motion to the subspace of the phase space with $p_i = 0$. For example:

$$I_2 = \sum_{i < j} \wp(x_{ij}),$$

$$I_4 = \sum_{i < j < k < l} \left[\wp(x_{ij})\wp(x_{kl}) + \wp(x_{ik})\wp(x_{jl}) + \wp(x_{il})\wp(x_{jk}) \right]. \tag{4.24}$$

The spectral curve Γ is defined by the equation $R(z, \lambda) = \det(zI - L(\lambda)) = 0$. It is an n -sheet covering of the elliptic curve \mathcal{E} uniformized by the variable λ and realized as a factor of the complex plane with respect to the lattice generated by $2\omega_1, 2\omega_2$. Since $L(-\lambda) = -L^T(\lambda)$, it is easy to see that the curve Γ is equipped with the holomorphic involution $\iota : (z, \lambda) \rightarrow (-z, -\lambda)$. As it was already mentioned, the equation of the spectral curve (the characteristic equation of the Lax matrix) is an integral of motion.

Proposition 4.1 [19]. *For generic values of x_i the spectral curve is smooth of genus $g = n$.*

4.3. *The wave function as the Baker–Akhiezer function on the spectral curve.* Let P be a point of the spectral curve Γ , i.e. $P = (z, \lambda)$, where z and λ are connected by the equation $R(z, \lambda) = 0$. To each point P of the curve there corresponds a single eigenvector $\mathbf{c}(0, P) = (c_1(0, P), \dots, c_n(0, P))^T$ of the matrix $L(t = 0, \lambda)$ normalized by the condition $c_1(0, P) = 1$. The non-normalized components c_i are equal to $\Delta_i(0, P)$, where $\Delta_i(0, P)$ are suitable minors of the matrix $zI - L(0, \lambda)$. They are holomorphic functions on Γ outside the points above $\lambda = 0$. After normalizing the first component, all other components $c_i(0, P)$ become meromorphic functions on Γ outside the points P_j located above $\lambda = 0$. Let \mathcal{D}' be the poles divisor of the vector \mathbf{c} with coordinates c_i . Unlike the spectra curve which is time-independent the divisor \mathcal{D}' depends on the initial data.

Lemma 4.1 *The sum of the divisors \mathcal{D}' and $\iota(\mathcal{D}')$ is the zero divisor of a holomorphic differential on the spectral curve, i.e. the equation*

$$\mathcal{D}' + \iota(\mathcal{D}') = \mathcal{K} \tag{4.25}$$

holds.

Proof. The idea of the proof goes back to the proof of Theorem 4 in [30]. Taking the differential of the eigenvalue equation $(zI - L(\lambda))\mathbf{c}(P) = 0$ and using the equation $\mathbf{c}^T(\iota P)(zI - L(\lambda)) = 0$, which follows from the definition of the involution, we get the equation

$$\mathbf{c}^T(\iota P)(dzI - dL(\lambda))\mathbf{c}(P) = 0,$$

or

$$\langle \mathbf{c}(\iota P), \mathbf{c}(P) \rangle dz = \langle \mathbf{c}(\iota P), L_\lambda \mathbf{c}(P) \rangle d\lambda, \tag{4.26}$$

where $L_\lambda = \partial L / \partial \lambda$ and $\langle \mathbf{c}(\iota P), \mathbf{c}(P) \rangle = \sum_i c_i(\iota P)c_i(P)$. For a generic initial data the spectral curve is smooth, i.e. the differentials dz and $d\lambda$ have no common zeros. Then from (4.26) it follows that the zeros of the differential $d\lambda$ (which are ramification points of the covering $\Gamma \rightarrow \mathcal{E}$) coincide with the zeros of the function $\langle \mathbf{c}(\iota P), \mathbf{c}(P) \rangle$. Therefore, the differential

$$d\Lambda = \frac{d\lambda}{\langle \mathbf{c}(\iota P), \mathbf{c}(P) \rangle} \tag{4.27}$$

is a holomorphic differential on the curve Γ . Its $2g - 2$ zeros at the points, where the vectors $\mathbf{c}(P)$ and $\mathbf{c}(\iota P)$ have poles. □

For completeness let us outline the arguments that ultimately lead to the proof of the algebraic integrability of equations (4.2).

A particular case of Theorem 2 in [19] is the following statement.

Theorem 4.3 *The function*

$$\hat{\Psi}(x, t, P) = e^{-\zeta(\lambda)x_1(0)} \sum_{i=1}^n c_i(t, P) \Phi(x - x_i, \lambda) e^{zx + z^3t} \tag{4.28}$$

is the one-point Baker–Akhiezer function on the spectral curve Γ with the marked point P_∞ (one of pre-images of $\lambda = 0$) corresponding to the divisor $\mathcal{D} = \mathcal{D}' + P_\infty$.

By definition the function $\hat{\Psi}$ has poles at $x_i(t)$. From the theta-functional formula for the Baker–Akhiezer function it follows that x_i are zeros of the second factor in the denominator, i.e. they are roots in x of the equation

$$\theta\left(\vec{U}_1x + \vec{U}_3t - \vec{A}(\mathcal{D}) - \vec{K} + \vec{A}(P_\infty)\right) = 0.$$

From Lemma 4.1 it follows that the pole divisor \mathcal{D} of the Baker–Akhiezer function satisfies the equation

$$\mathcal{D} + t\mathcal{D} - 2P_\infty = \mathcal{K}, \tag{4.29}$$

where \mathcal{K} is the canonical class. This is precisely the condition (3.41) on the divisor of poles of the Baker–Akhiezer function for algebraic–geometric solutions to the CKP equation. This completes the proof of (4.5) since (4.29) is equivalent to the condition for the vector \vec{Z} in Theorem 3.4.

4.4. Degenerations of elliptic solutions.

4.4.1. *Trigonometric solutions* In the degenerate case, when one of the periods tends to infinity, the elliptic solutions become trigonometric (hyperbolic). We consider trigonometric solutions which vanish at infinity:

$$u(x, \mathbf{t}) = -\frac{1}{2} \sum_{i=1}^n \frac{\gamma^2}{\sinh^2(\gamma(x - x_i(\mathbf{t})))},$$

where γ is a complex parameter. When γ is purely imaginary (respectively, real), one deals with trigonometric (respectively, hyperbolic) solutions. The equations of motion for the poles are

$$\dot{x}_i = 3 \sum_{k \neq i} \frac{\gamma^2}{\sinh^2(\gamma(x_i - x_k))} - \gamma^2. \tag{4.30}$$

Tending the spectral parameter λ to infinity, we find the Lax matrix in the form

$$L_{ij} = -\frac{\gamma(1 - \delta_{ij})}{\sinh(\gamma(x_i - x_j))}. \tag{4.31}$$

Note that it is antisymmetric.

As is shown in [31], the KP tau-function for trigonometric solutions has the following determinant representation:

$$\tau^{\text{KP}}(x, \mathbf{t}) = \det_{n \times n} \left(e^{2\gamma x} I - \exp \left(- \sum_{k \geq 1} t_k \mathcal{L}_k \right) e^{2\gamma X_0} \right) = \prod_{j=1}^n (e^{2\gamma x} - e^{2\gamma x_j(\mathbf{t})}), \tag{4.32}$$

where $X_0 = \text{diag} (x_1(0), \dots, x_n(0))$ and

$$\mathcal{L}_k = (L_0 + \gamma I)^k - (L_0 - \gamma I)^k, \quad L_0 = L(\mathbf{t} = 0). \tag{4.33}$$

We see that \mathcal{L}_k is a polynomial in L_0 of degree $k - 1$. If k is even (respectively, odd), \mathcal{L}_k contains only odd (respectively, even) powers of L_0 .

It is easy to see that this tau-function satisfies the conditions (2.48), and, therefore, gives rise to the CKP tau-function

$$\tau(x, \mathbf{t}_0) = \left(\det_{n \times n} \left(e^{2\gamma x} I - \exp \left(- \sum_{k \geq 1, k \text{ odd}} t_k \mathcal{L}_k \right) e^{2\gamma X_0} \right) \right)^{1/2}. \tag{4.34}$$

Indeed, we have

$$\begin{aligned} \partial_{t_{2m}} \log \tau^{\text{KP}} \Big|_{t_{2k}=0} &= \partial_{t_{2m}} \text{tr} \log \left(e^{2\gamma x} I - \exp \left(- \sum_{k \geq 1} t_k \mathcal{L}_k \right) e^{2\gamma X_0} \right) \Big|_{t_e=0} \\ &= \text{tr} \left[\mathcal{L}_{2m} \exp \left(- \sum_{k \geq 1, k \text{ odd}} t_k \mathcal{L}_k \right) \left(e^{2\gamma x} I - \exp \left(- \sum_{k \geq 1, k \text{ odd}} t_k \mathcal{L}_k \right) e^{2\gamma X_0} \right)^{-1} \right]. \end{aligned}$$

But this is zero for all $m \geq 1$ because $\text{tr} L_0^{2l-1} = 0$ for all $l \geq 1$ and, as it was said above, \mathcal{L}_{2m} contains only odd powers of L_0 while all other \mathcal{L}_k in this expression contain only even powers of L_0 .

4.4.2. Rational solutions In the most degenerate case, when both periods tend to infinity, $\wp(x) \rightarrow 1/x^2$ and the elliptic solutions become rational:

$$u(x, \mathbf{t}) = - \frac{1}{2} \sum_{i=1}^n \frac{1}{(x - x_i(\mathbf{t}))^2}.$$

This corresponds to the limit $\gamma \rightarrow 0$ in the trigonometric solutions. The equations of motion for the poles are

$$\dot{x}_i = 3 \sum_{k \neq i} \frac{1}{(x_i - x_k)^2}. \tag{4.35}$$

Tending the spectral parameter λ to infinity, $\lambda = \infty$, we find the (antisymmetric) Lax matrix in the form

$$L_{ij} = - \frac{1 - \delta_{ij}}{x_i - x_j}. \tag{4.36}$$

It is known that the KP tau-function for rational solutions has the following determinant representation (see, e.g. [32]):

$$\tau^{\text{KP}}(x, \mathbf{t}) = \det_{n \times n} \left(xI - X_0 + \sum_{k \geq 1} kt_k L_0^{k-1} \right) = \prod_{j=1}^n (x - x_j(\mathbf{t})), \tag{4.37}$$

where $X_0 = \text{diag}(x_1(0), \dots, x_n(0))$ and $L_0 = L(\mathbf{t} = 0)$. It is easy to see that this tau-function satisfies the conditions (2.48), and, therefore, gives rise to the CKP tau-function

$$\tau(x, \mathbf{t}_0) = \left(\det_{n \times n} \left(xI - X_0 + \sum_{k \geq 1, k \text{ odd}} kt_k L_0^{k-1} \right) \right)^{1/2}, \tag{4.38}$$

Indeed, we have

$$\begin{aligned} \partial_{t_{2m}} \log \tau^{\text{KP}} \Big|_{t_k=0} &= \partial_{t_{2m}} \text{tr} \log \left(xI - X_0 + \sum_{k \geq 1} kt_k L_0^{k-1} \right) \Big|_{t_e=0} \\ &= 2m \text{tr} \left[L_0^{2m-1} \left((x + t_1)I - X_0 + 3t_3 L_0^2 + 5t_5 L_0^4 + \dots \right)^{-1} \right] = 0 \end{aligned}$$

for all $m \geq 1$ because $\text{tr} L_0^{2l-1} = 0$ for all $l \geq 1$.

5. Concluding Remarks

The main result of this paper is the identification of the CKP hierarchy as the hierarchy of *odd times* flows of the KP hierarchy restricted onto the locus of its turning points. It suggests that a similar result might be valid for the BKP hierarchy. Namely, we conjecture that the BKP hierarchy can be identified with the restriction of odd times flows of the KP hierarchy onto the locus which in the Sato formulation is defined by the equation

$$(\mathcal{L}^3)_+ = \partial_x^3 + 6u \partial_x, \tag{5.1}$$

i.e. the coefficient at the zero power of ∂_x in \mathcal{L}^3 vanishes. In terms of the tau-function, this condition means that

$$(\partial_{t_2} + \partial_{t_1}^2) \log \tau^{\text{KP}} \Big|_{t_e=0} = 0 \tag{5.2}$$

for all t_1, t_3, t_5, \dots (see [33]). The latter is an analog of Eq. (2.48) defining turning points of the KP hierarchy.

Another interesting problem we plan to consider in the future is the Hamiltonian theory of equations of motion for poles of elliptic solutions to the CKP hierarchy. In Sect. 4 we have derived these equation in two ways. First, these equations can be obtained by restricting the higher equations of motion of the elliptic Calogero–Moser system onto the locus of its turning points. As a corollary of this, we have presented solutions of these equations in the implicit function form using theta-function of the spectral curve. The second approach to the equations of motion is via the “generating linear problem” scheme which allows us to define the corresponding spectral curve and to prove that it is time-independent in a direct way (i.e. without any reference to the elliptic Calogero–Moser system).

As it was shown earlier in [23], the phase space of the elliptic CKP system can be identified with the total space of the Prym varieties bundle over the space of the spectral curves. Under this identification the equations of motion become linear on the fibers. Such picture is characteristic for algebraically integrable Hamiltonian systems. However, the authors’s attempts to find the corresponding Hamiltonian formulation of Eq. (4.2) by a direct guess or by more advanced machinery proposed in [20, 34, 35] have failed so far.

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Appendix A: Proof of Lemma 2.1

In this appendix we give a sketch of proof of Lemma 2.1, i.e. we are going to prove that the conditions (2.53) and

$$\partial_x \partial_{t_4} \log \tau^{\text{KP}} \Big|_{t_e=0} = \partial_x \partial_{t_6} \log \tau^{\text{KP}} \Big|_{t_e=0} = \dots = 0. \tag{A1}$$

follow from the constraint

$$\partial_{t_2} \log \tau^{\text{KP}} \Big|_{t_e=0} = 0 \quad \text{for all } t_1, t_3, t_5, \dots \tag{A2}$$

(see (2.48)) provided τ^{KP} is a KP tau-function, i.e. satisfies all the equations of the KP hierarchy.

We use the representation of the KP hierarchy in the unfolded form suggested in [36, 37], see also section 3.2 of [38]. Set $F = \log \tau^{\text{KP}}$ and $F_{k_1, \dots, k_m} = \partial_{t_{k_1}} \dots \partial_{t_{k_m}} F$. Then the KP hierarchy can be written in the form

$$F_{k_1, \dots, k_m} = \sum_{n \geq 1} \sum R_{k_1, \dots, k_m}^{(n)} \begin{pmatrix} s_1 & \dots & s_n \\ r_1 & \dots & r_n \end{pmatrix} \partial_x^{r_1} F_{s_1} \dots \partial_x^{r_n} F_{s_n}, \tag{A3}$$

where $m \geq 2$ and $R_{k_1, \dots, k_m}^{(n)} \begin{pmatrix} s_1 & \dots & s_n \\ r_1 & \dots & r_n \end{pmatrix}$ are universal rational coefficients. The second sum is taken over all matrices $\begin{pmatrix} s_1 & \dots & s_n \\ r_1 & \dots & r_n \end{pmatrix}$ such that $s_i, r_i \geq 1$ with the conditions

$$\sum_{i=1}^n (s_i + r_i) = \sum_{i=1}^m k_i, \quad \sum_{i=1}^n r_i \geq n + m - 2. \tag{A4}$$

For example [36],

$$F_{2,3} = \frac{3}{2} \partial_x F_4 - \frac{3}{2} \partial_x^3 F_2 - 3 \partial_x F_2 \partial_x^2 F. \tag{A5}$$

From the fact that if $\tau^{\text{KP}}(x, \mathbf{t})$ is a tau-function, then $\tau^{\text{KP}}(-x, -\mathbf{t})$ is a tau-function, too (this is a corollary of the Hirota equations), it follows that

$$\text{if } \sum_{i=1}^n (r_i - 1) - m \equiv 1 \pmod{2}, \text{ then } R_{k_1, \dots, k_m}^{(n)} \begin{pmatrix} s_1 \cdots s_n \\ r_1 \cdots r_n \end{pmatrix} = 0. \tag{A6}$$

First we prove (A1). The proof is by induction. We assume that (A1) is true for $\partial_x F_2, \dots, \partial_x F_{2k}$ (this is certainly true if $k = 1$) and will deduce from (A3) that it is true for $k \rightarrow k+1$. From (A2) and (A3) at $m = 2$ we have (at $\mathbf{t}_e = 0$):

$$\begin{aligned} 0 = F_{2, 2k+1} &= \sum_{s_1+r_1=2k+3} R_{2, 2k+1}^{(1)} \begin{pmatrix} s_1 \\ r_1 \end{pmatrix} \partial_x^{r_1} F_{s_1} \\ &+ \sum_{s_1+s_2+r_1+r_2=2k+3} R_{2, 2k+1}^{(2)} \begin{pmatrix} s_1 & s_2 \\ r_1 & r_2 \end{pmatrix} \partial_x^{r_1} F_{s_1} \partial_x^{r_2} F_{s_2} + \dots \end{aligned} \tag{A7}$$

Separating the term with $r_1 = 1$ in the first sum in the right hand side of (A7), we write it as

$$0 = F_{2, 2k+1} = R_{2, 2k+1}^{(1)} \begin{pmatrix} 2k+2 \\ 1 \end{pmatrix} \partial_x F_{2k+2} + \text{all the rest.} \tag{A8}$$

Now, recalling the condition (A6), we see that the non-zero coefficients at the different terms in the right hand side are when $\sum_{i=1}^n s_i = n - 1 \pmod{2}$. From this it follows that for both odd and even n at least one of the s_i 's must be even (and less than $2k + 2$). Therefore, "all the rest" terms vanish by the induction assumption. Since the coefficient $R_{2, 2k+1}^{(1)} \begin{pmatrix} 2k+2 \\ 1 \end{pmatrix}$ is not equal to zero (see [36]), we conclude from (A8) that $\partial_x F_{2k+2} = 0$.

Next we are going to prove that if $\partial_x F_{2k} = 0$ for all $k \geq 1$ and all t_1, t_3, \dots , then $F_{k_1, \dots, k_m} = 0$ for all even k_1, \dots, k_m and odd $m \geq 3$. As soon as $m + 1$ and all k_i 's are even, we can, using (A4), rewrite the condition (A6) in the form

$$\sum_{i=1}^n s_i \equiv n \pmod{2}. \tag{A9}$$

But if at least one of s_i in (A3) is even, then the corresponding term vanishes because $F_{2k} = 0$ for all $k \geq 1$. Therefore, all the s_i 's must be odd, i.e., $s_i = 2l_i + 1$ and so the condition (A9) is satisfied which means that the coefficient $R_{k_1, \dots, k_m}^{(n)} \begin{pmatrix} s_1 \cdots s_n \\ r_1 \cdots r_n \end{pmatrix}$ vanishes. This proves that $F_{k_1, \dots, k_m} = 0$.

Appendix B: Proof of Eq. (4.21)

Here we prove the matrix identity (4.21).

First of all we note that $\dot{L}_{ik} = -(\dot{x}_i - \dot{x}_k)\Phi'(x_i - x_k)$, and, therefore, we have $\dot{L} = -[\dot{X}, B]$. To transform the commutators $[L, B] + [L, D]$, we use the identity

$$\Phi(x)\Phi'(y) - \Phi(y)\Phi'(x) = \Phi(x+y)(\wp(x) - \wp(y)). \quad (\text{B1})$$

With the help of it we get for $i \neq k$

$$\begin{aligned} & -\left([L, B] + [L, D]\right)_{ik} \\ &= \sum_{j \neq i, k} \Phi(x_i - x_j)\Phi'(x_j - x_k) - \sum_{j \neq k} \Phi'(x_i - x_j)\Phi(x_j - x_k) \\ & \quad + \Phi(x_i - x_k)\left(\sum_{j \neq k} \wp(x_j - x_k) - \sum_{j \neq i} \wp(x_i - x_j)\right) = 0, \end{aligned}$$

so we see that $[L, B] + [L, D]$ is a diagonal matrix. To find its matrix elements, we use the limit of (B1) at $y = -x$:

$$\Phi(x)\Phi'(-x) - \Phi(-x)\Phi'(x) = \wp'(x)$$

which leads to

$$\begin{aligned} & -\left([L, B] + [L, D]\right)_{ii} \\ &= \sum_{j \neq i} \left(\Phi(x_i - x_j)\Phi'(x_j - x_i) - \Phi'(x_i - x_j)\Phi(x_j - x_i)\right) = \sum_{j \neq i} \wp'(x_i - x_j) = D'_{ii}, \end{aligned}$$

so we finally obtain the matrix identity

$$[L, B] + [L, D] = -D'. \quad (\text{B2})$$

Combining the derivatives of (B1) w.r.t. x and y , we obtain the identity

$$\Phi(x)\Phi''(y) - \Phi(y)\Phi''(x) = 2\Phi'(x+y)(\wp(x) - \wp(y)) + \Phi(x+y)(\wp'(x) - \wp'(y)) \quad (\text{B3})$$

which allows us to prove the matrix identity

$$[L, C] = -2[D, B] + D'L + LD', \quad (\text{B4})$$

which is used, together with (B2), to transform $\dot{L} + [L, M]$ to the form (4.21).

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