

PAPER

Real-normalized differentials: limits on stable curves

To cite this article: S. Grushevsky *et al* 2019 *Russ. Math. Surv.* **74** 265

View the [article online](#) for updates and enhancements.

You may also like

- [ON PERIODS OF QUADRATIC DIFFERENTIALS](#)
A N Tyurin
- [Theta functions and non-linear equations](#)
B A Dubrovin
- [Classical and quantum Teichmüller spaces](#)
A.Yu. Vasiliev and A. G. Sergeev

To Sergei Novikov with admiration

Real-normalized differentials: limits on stable curves

S. Grushevsky, I. M. Krichever, and C. Norton

Abstract. We study the behaviour of real-normalized (RN) meromorphic differentials on Riemann surfaces under degeneration. We describe all possible limits of RN differentials on any stable curve. In particular we prove that the residues at the nodes are solutions of a suitable Kirchhoff problem on the dual graph of the curve. We further show that the limits of zeros of RN differentials are the divisor of zeros of a twisted differential—an explicitly constructed collection of RN differentials on the irreducible components of the stable curve, with higher order poles at some nodes. Our main tool is a new method for constructing differentials (in this paper, RN differentials, but the method is more general) on smooth Riemann surfaces, in a plumbing neighbourhood of a given stable curve. To accomplish this, we think of a smooth Riemann surface as the complement of a neighbourhood of the nodes in a stable curve, with boundary circles identified pairwise. Constructing a differential on a smooth surface with prescribed singularities is then reduced to a construction of a suitable normalized holomorphic differential with prescribed ‘jumps’ (mismatches) along the identified circles (seams). We solve this additive analogue of the multiplicative Riemann–Hilbert problem in a new way, by using iteratively the Cauchy integration kernels on the irreducible components of the stable curve, instead of using the Cauchy kernel on the plumbed smooth surface. As the stable curve is fixed, this provides explicit estimates for the differential constructed, and allows a precise degeneration analysis.

Bibliography: 22 titles.

Keywords: Riemann surfaces, Abelian differentials, boundary value problem, degenerations.

Contents

Introduction	266
1. Limits of solutions of the Kirchhoff problem	271
2. Notation for RN differentials and moduli of jet curves	278
3. Statement of results: limits of RN differentials	279

The research of the first author was supported in part by the National Science Foundation under grant DMS-15-01265, and by a Simons Fellowship in Mathematics (Simons Foundation grant #341858 to Samuel Grushevsky).

AMS 2010 *Mathematics Subject Classification*. Primary 14H10, 14H15, 30F30; Secondary 32G15.

4. Plumbing setup for Riemann surfaces	283
5. The jump problem	285
6. The RN differential in plumbing coordinates	298
7. The limit RN differential: proof of Theorem 3.7	305
8. Limits of zeros of RN differentials	306
9. Appendix: m -balanced approximation	318
Bibliography	322

Introduction

A smooth *jet curve* X is a Riemann surface C with distinct marked points $p_1, \dots, p_n \in C$, and with prescribed singular parts $\sigma_1, \dots, \sigma_n$ of a meromorphic differential at these points. If each prescribed residue r_ℓ at each p_ℓ is purely imaginary, and $\sum r_\ell = 0$, then there exists a unique meromorphic differential Ψ on C with singular part σ_ℓ at each p_ℓ , holomorphic on $C \setminus \{p_1, \dots, p_n\}$, and such that all periods of Ψ are real. This differential is called the *real-normalized* (RN) meromorphic differential, and this paper is one in a series investigating its properties and using it to study the geometry of the moduli space of curves. Here we focus on the behaviour of the RN differential as the Riemann surface degenerates to a stable singular curve.

The jump problem. Our main technical tool is a new analytic method for studying the behaviour of differentials on Riemann surfaces under degeneration. This is done by working explicitly in plumbing coordinates, and we need to introduce some notation to describe it; this setup will be defined in full detail in §4. Fix a nodal curve C ; its dual graph Γ has vertices, denoted by v , corresponding to irreducible components C^v of the normalization of C , and (unoriented) edges, which we denote by $|e|$, corresponding to nodes $q_{|e|} \in C$. We will write e for edges of Γ together with a choice of orientation. If an oriented edge e starts from a vertex v , we say that it corresponds to a pre-image $q_e \in C^v$ of a node $q_{|e|} \in C$. We write $-e$ for the edge e with the opposite orientation, $|e|$ for the corresponding unoriented edge, and write E and $|E|$ for the sets of oriented and unoriented edges, respectively.

Plumbing gives a way to understand versal deformations of C in the Deligne–Mumford compactification — that is, coordinates on $\overline{\mathcal{M}}_{g,n}$ transverse to the boundary stratum containing C . To define plumbing coordinates $\underline{s} = (s_1, \dots, s_{\#|E|})$, one fixes once and for all a local coordinate z_e on the normalization of C near each q_e . Let $\widehat{C}_{\underline{s}}$ be the complement in C of the union of the disks $\{|z_e| < \sqrt{|s_{|e|}|\}$ around each q_e . Then $\widehat{C}_{\underline{s}}$ is a Riemann surface with boundary components $\gamma_e := \{|z_e| = \sqrt{|s_{|e|}|\}$. The compact Riemann surface $C_{\underline{s}}$ is obtained from $\widehat{C}_{\underline{s}}$ by identifying each pair of boundaries γ_e and γ_{-e} via the map $I_e: z_e \mapsto s_{|e|}/z_e$. The complex structure on $C_{\underline{s}}$ is obtained by declaring a function on $C_{\underline{s}}$ to be holomorphic if it is holomorphic outside all ‘seams’, and *continuous* on each seam $\gamma_{|e|} \subset C_{\underline{s}}$; $C_{\underline{s}}$ is smooth if and only if each $s_{|e|}$ is non-zero.

Then a differential on $C_{\underline{s}}$ is the same as a differential on $\widehat{C}_{\underline{s}}$ such that its boundary values on γ_e and γ_{-e} match under the pullback by I_e ; this is to say, there is no ‘jump’ on the seam $\gamma_{|e|}$. Our approach to constructing such a differential on

$C_{\underline{s}}$ with prescribed singular parts is novel. We start with a collection of meromorphic differentials on the irreducible components C^v with prescribed singular parts. Then, of course, the boundary values of this collection on the seams do not agree, so there are non-zero ‘jumps’. We then construct explicitly a suitably normalized collection of holomorphic differentials on C^v such that their jumps are precisely equal to those of the original collection of meromorphic differentials. Subtracting this collection of holomorphic differentials from the collection of meromorphic differentials then gives a differential on $\widehat{C}_{\underline{s}}$ with no jumps, that is, a meromorphic differential on the smooth surface $C_{\underline{s}}$ with prescribed singularities.

The problem of constructing a differential with prescribed jumps is an additive analog of a well-known general problem, known variously as the Riemann–Hilbert problem, or the Riemann boundary value problem. We use the name *jump problem* for the version of the problem that is relevant for us. It is the question of constructing a suitably normalized differential on a Riemann surface with boundary, with prescribed differences of boundary values on pairwise identified boundary circles. Equivalently, this is the problem of constructing a differential on a compact Riemann surface, defined on the complement of a set of disjoint closed loops, with prescribed jumps from one side of each loop to the other side. As a matter of language, we will talk about the jump problem either on $\widehat{C}_{\underline{s}}$ or on $C_{\underline{s}}$, as is more convenient in each case.

Classically (see [19], [22]), the jump problem is solved by integrating the jumps with respect to the suitably normalized Cauchy integration kernel on the surface, that is, on $C_{\underline{s}}$ in our case. In this classical approach it appears very difficult to determine the behaviour of the solution under degeneration, as the Cauchy kernel varies with \underline{s} , and degenerates as $\underline{s} \rightarrow 0$. The technical core of our paper is a new method for solving the jump problem, which allows explicit estimates for the solution under degeneration.

Instead, we view $\widehat{C}_{\underline{s}}$ as a subset of the normalization of C , and try to obtain the solution of the jump problem by integrating with respect to the suitably normalized Cauchy kernels of C^v , which are thus *independent* of \underline{s} . We then compute the jumps on the seams of $C_{\underline{s}}$ of the convolution of arbitrary initial data on the seams with the Cauchy kernels on C^v . While the jumps on $\gamma_{|e|}$ of this convolution are of course not equal to the initial data, the condition for the jumps to be equal to the prescribed data amounts to an integral equation on the functions that are convolved with the Cauchy kernels on C^v . We show that this integral equation can be solved, by showing that the norm of the corresponding integral operator is sufficiently close to 1, so that it can be formally inverted, as the sum of an iteratively defined series.

Our main technical result is this construction in plumbing coordinates, Proposition 5.2, and the bound for it, Proposition 5.3. This allows us to construct and estimate the RN differentials in an entire neighbourhood of C in the moduli space.

While we apply this machinery to study the limits of RN differentials and their zeros, it can also be used, for example, to study the behaviour of a normalized basis of holomorphic differentials. Hu and the third author [11] used our approach to the jump problem to extend and re-prove the results of Yamada on degenerations of period matrices.

Limits of RN differentials. We use this explicit construction of RN differentials to understand their degenerations, in plumbing coordinates. Our first result is on limits of RN differentials with arbitrary residues. As our setup is real-analytic, we state it for degenerating sequences (not families). Let $\{X_k\}$ be a sequence of smooth jet curves converging to a stable jet curve X , with underlying smooth curves C_k , with plumbing coordinates \underline{s}_k , converging to a nodal curve C , whose dual graph is Γ . The rough version of our first result is as follows.

Theorem 0.1 (Theorem 3.7 and Proposition 3.14). *Let $\{X_k\}$ be a sequence of smooth jet curves converging to a stable jet curve X . The limit RN differential $\Psi := \lim_{k \rightarrow \infty} \Psi_k$ exists if and only if the solutions of the flow Kirchhoff problem on Γ , with inflows $ir_{\ell,k}$ and resistances $\log |\underline{s}_k|$, converge. If the limit Ψ exists, then $\Psi|_{C^v}$ is the RN differential with prescribed singularities at those marked points p_ℓ that lie on C^v , and with simple poles at the pre-images of the nodes, with residues given by the limit of solutions of the flow Kirchhoff problem on Γ , with inflows ir_ℓ and resistances $\log |\underline{s}_k|$.*

Thus the existence of the limit RN differential is controlled by the existence of the limit of solutions of a flow Kirchhoff problem on the dual graph of the stable curve (see Definition 1.2 for the precise general statement of the Kirchhoff problem). Surprisingly, it seems that the classical problem of determining and parameterizing all possible limits of solutions of the flow Kirchhoff problem as some resistances approach zero has not been addressed previously. In Lemma 1.13 we show that if the resistances (that is, in our case, $\log |\underline{s}_k|$) converge in a suitable iterated real oriented blowup $\mathbb{S}_+^{\#|E|-1}$ of the non-negative sector of the real sphere $S^{\#|E|-1} := (\mathbb{R}_{\geq 0}^{\#|E|} \setminus \{0\})/\mathbb{R}_{>0}$ (see Definition 1.10), then the solutions of the flow Kirchhoff problem converge. We will call such degenerating sequences *admissible*, and will show that in an admissible sequence the limit of solutions of the flow Kirchhoff problem is given by the solution of what we call the multi-scale Kirchhoff problem (Definition 1.11), with resistance given as a point in $\mathbb{S}_+^{\#|E|-1}$. The full notation is rather involved, and we thus postpone the precise statement of the theorem to §3, where it appears as Theorem 3.7, stated using the notation on the Kirchhoff problem, developed in §1.

This theorem on limits of RN differentials is proven by applying the jump problem. We start with a collection of RN differentials on the irreducible components C^v of the nodal curve C , whose residues at the nodes are given by solutions to the limiting multi-scale flow Kirchhoff problem. We then construct the solution of the jump problem with a normalization condition which ensures that all periods over cycles which do not intersect a neighbourhood of the nodes are real. We finally find an explicit perturbation of the residues, as a series expansion in plumbing parameters (see (6.11)), such that the resulting differential is actually RN. The final part of the argument relies on the estimates for the solution of the jump problem ensuring these terms disappear in the limit.

Limits of zeros of RN differentials. The zeros of differentials play a crucial role in various questions on moduli. In Teichmüller dynamics one studies the orbits on the stratum: the moduli of Riemann surfaces together with a holomorphic differential with a prescribed configuration of zeros. For possible applications

to Teichmüller dynamics, and for applications of common zeros of RN differentials to cusps of plane curves in our upcoming work [10], it is natural to study the limits of zeros of differentials under degeneration. The difficulty is that the limit differential Ψ may be identically zero on some irreducible component C^v . Algebro-geometrically, one approaches this by considering aspects of limit linear series — which, however, are not yet fully developed for an arbitrary stable curve, though see [18] for recent progress. In [2] the problem is dealt with by deforming, in plumbing coordinates or using flat surface constructions, differentials on irreducible components of the stable curves that have zeros as prescribed.

Our approach to locating the zeros of RN differentials is again via the jump problem, with a further improvement resulting from starting from a better approximation of the solution. Indeed, to determine limits of RN differentials, we started with a collection of RN differentials on C^v that we postulate the limit to be, and then construct the differentials in a neighbourhood by using the jump problem. In doing this, we could be starting with an identically zero differential on some C^v , if that is what the limit RN differential on C^v is. Instead, we now start with a collection of non-identically zero RN differentials on C^v which provide a better approximation to the full RN differential on $C_{\underline{s}}$. The key estimate (8.19) implies that the solution of the jump problem posed with these improved approximations vanishes to higher order than scaling required to determine the first non-zero term, and thus the location of the zeros in the limit.

As before, we then need to obtain a bound showing that the solution of the jump problem is smaller than the original RN differentials, and thus disappears in the limit. For this to be the case, we need to ensure that the suitably rescaled RN differentials locally near q_e and q_{-e} are such that the singular part of the differential on one side cancels the lowest order terms of the (holomorphic) differential on the other. This is the concept of so-called *balanced* differentials, developed in § 8.

We will call a sequence of degenerating smooth jet curves *jet-convergent* if the singular parts at the nodes of the differentials constructed using this balanced condition converge, after suitable rescaling (see Definition 8.8 for the precise statement). As the limits of differentials are unchanged under rescaling, it will follow that the limits of zeros of RN differentials exist in jet-convergent sequences, and are the zeros of the originally taken collection of not identically zero RN differentials on C^v .

A rough statement of our main result on limits of zeros is thus the following.

Theorem 0.2 (Theorem 8.12 and Corollary 8.20). *Any admissible sequence $\{X_k\}$ of smooth jet curves converging to a stable jet curve X has a jet-convergent subsequence. For any jet-convergent sequence the limits of zeros of RN differentials exist. These limits of zeros are the divisor of zeros on X of a twisted RN differential constructed from the jet-convergent subsequence. In particular, the residues of the twisted RN differential arise from a suitable force Kirchhoff problem.*

By a twisted RN differential here we mean a collection of non-trivial RN differentials Φ^v on the irreducible components $\{C^v\}$, with prescribed singularities at p_ℓ , and with higher order poles at some pre-images of the nodes. The divisor of zeros of such a twisted differential is the set of all its zeros, with multiplicity, away from the nodes, together with the set of nodes counted with suitable multiplicities. The precise statement of these results requires developing the notion of balanced

differentials, and related machinery, and obtaining the necessary bounds for the solution of the jump problem. This is done in § 8, where the precise version of our main result is given as Theorem 8.12.

Since our construction approximates the RN differential on any jet curve in the neighbourhood of a given stable jet curve, it describes *all* possible limits of zeros, and thus in fact constructs a compactification of the moduli space of jet curves on which the limits of zeros of RN differentials exist. This compactification can be described as a suitable real blowup, and merits an independent study.

Related work. The question of describing the closures of strata of Riemann surfaces together with a meromorphic differential is currently under intense investigation, for instance, in [6], [4], [5], and [2], and the answer there is also in terms of twisted meromorphic differentials on the stable curve.

However, our analytic approach via the jump problem is completely different from the methods employed there and, in particular, allows us to describe the RN differential with arbitrary precision on *any* smooth jet curve in a plumbing neighbourhood. As a byproduct we get for example an explicit description of the residues in the limit via the Kirchhoff problem.

History of the project. The integral $F := \text{Im} \int \Psi$ of the RN differential is a single-valued harmonic function on $C \setminus \{p_1, \dots, p_n\}$. In this guise, as harmonic functions which are potentials of the electromagnetic field created by point charges at the marked points, the RN differentials with simple poles have been studied since at least the time of Maxwell. A variant of the general notion of RN differentials already appears in [20], while their study in full generality was initiated by the second author in [12] and [13], where the relationship with the Whitham perturbation theory of soliton equations was also established. In [8] and [9] the first and second author applied RN differentials to obtain a new proof of the theorem of Diaz on complete complex subvarieties of the moduli space of curves \mathcal{M}_g , and established a relationship with the loci of spectral curves of the elliptic Calogero–Moser system, while in [14] the second author used RN differentials to prove a conjecture of Arbarello on subvarieties of \mathcal{M}_g .

The rough version of the results of this paper, with an approach not using the jump problem, and not yielding the full statement of Theorem 8.12, appeared in the third author’s Stony Brook PhD dissertation [17] defended in August 2014. We then developed the current approach to the problem using our solution of the jump problem. The current paper, and our proof, are completely independent of the concurrent and independent progress on the compactifications of strata of differentials with prescribed zeros, in [6], [4], [5], and [2].

Structure of the paper. First, in § 1 we give the statement of the Kirchhoff problem on a general graph, and investigate the properties of its solutions, proving that they are a priori bounded, and constructing the blowup $\mathbb{S}_+^{\#E-1}$ such that convergence of resistances there implies convergence of solutions of the flow Kirchhoff problem. This section is elementary and does not deal with Riemann surfaces and differentials. The setup and the lemmas from it are essential to stating the main results of the paper.

In §2 we recall the notation for the spaces of jet curves and RN differentials. In §3 we develop the notation for degenerating sequences and give the precise statement of the main Theorem 3.7 on limits of RN differentials. In §4 we recall the plumbing coordinates and the notation. Section 5 contains the technical core of our construction: we pose the jump problem, and use Cauchy kernels independent of plumbing parameters to construct an almost real-normalized (ARN) solution, with a bound on its norm. In §6 we use the ARN solution of the jump problem to construct the RN differential explicitly in plumbing coordinates, as the sum of a recursively defined series, and effectively bound the terms of these series. In §7 we determine the behaviour of this construction of the RN differential in a degenerating sequence, proving the main theorem, Theorem 3.7, on limit RN differentials.

In §8 we introduce the notion of two differentials balancing (canceling up to order m under the map $z \mapsto sz^{-1}$) at a node to construct a better approximation to the RN differential recursively. Starting from a collection of balanced differentials on C^v , we show that the ARN solution of the corresponding jump problem is smaller than the differentials themselves, and thus in the limit the balanced differentials dominate — this yields the main Theorem 8.12 on limits of zeros of RN differentials.

In the appendix (§9) we formalize this notion of a collection of differentials on C^v that are close to a differential on the plumbed surface, by introducing the notion of an m th order approximation. While this setup is not necessary for our main proofs, the method can be used to study the behaviour of degenerating differentials with arbitrary precision, as will be investigated elsewhere.

The second author thanks Columbia University for hospitality in January–March 2016, when much work on this paper was done. We are grateful to Scott Wolpert for carefully reading the third author’s PhD dissertation, and for many useful discussions and comments on the topics surrounding plumbing.

1. Limits of solutions of the Kirchhoff problem

In this section we pose the Kirchhoff problem on an arbitrary graph, in the generality that we require, and investigate the limits of its solutions. This setup will be used to state our main results.

Notation 1.1. We denote by Γ a graph, which is a collection of vertices $v \in V(\Gamma)$, and a collection of edges $|e| \in |E|(\Gamma)$, allowing loops and parallel edges. We further denote by $E(\Gamma)$ the set of oriented edges e , writing $-e$ for the same edge as e , but with the opposite orientation, and writing $|e| = |-e|$ for the corresponding unoriented edge. Our graphs also have legs, that is, half-edges attached to some vertices. For an oriented edge e we denote by $v(e)$ the vertex that is its target, and for a vertex v we denote by E_v the set of all edges pointing to it, that is,

$$E_v = \{e \in E(\Gamma) : v(e) = v\}.$$

We denote by $\#E$, $\#|E|$, and $\#V$ the cardinalities of the corresponding sets. We will use the underline for the elements of $\#E$ -dimensional or $\#|E|$ -dimensional vector spaces, for example \underline{c} will mean the collection of numbers c_e for all $e \in E$ and $\underline{\rho}$ will mean the collection of numbers $\rho_{|e|}$ (we will specify in each case whether the oriented or unoriented edges are taken).

Definition 1.2 (Kirchhoff problem). The *general Kirchhoff problem* for a graph Γ is the following. As initial data, to every leg ℓ one assigns a real number $f_\ell \in \mathbb{R}$, thought of as the in/outflow of current, and to every unoriented edge $|e|$ of Γ , one assigns a positive real number $\rho_{|e|} \in \mathbb{R}_+$, thought of as resistance. In addition, one chooses a class in the first cohomology group of the graph, $\mathcal{E} \in H^1(\Gamma, \mathbb{R})$, thought of as the electromotive force.

The Kirchhoff problem is then to find for each oriented edge $e \in E(\Gamma)$ a real number c_e (the electrical current) such that the set of all c_e satisfies the following three conditions:

- (0) $c_e = -c_{-e}$ for any $e \in E(\Gamma)$;
- (1) the total current flow at any vertex is zero: for any $v \in V(\Gamma)$,

$$\sum_{e \in E_v} c_e = -\sum' f_\ell, \tag{1.1}$$

where the sum on the right is taken over the half-edges ℓ incident to v ;

- (2) for any oriented cycle of edges $\gamma \subset \Gamma$ the total voltage drop is equal to the electromotive force along the cycle:

$$\sum_{e \in \gamma} c_e \rho_{|e|} = \mathcal{E}_\gamma := \langle \mathcal{E}, \gamma \rangle. \tag{1.2}$$

In modern terminology, the Kirchhoff problem is to find a 1-form on the graph with prescribed periods over cycles. In physics, this is the problem of determining the flow of the electrical current. Physically, it is classically known that the current flows, and in a unique way.

Fact 1.3. *For any connected graph Γ , if the sum of all f_ℓ is equal to zero, the general Kirchhoff problem has a unique solution.*

We think of the Kirchhoff problem as a system of non-homogeneous linear equations on $\underline{c} = \{c_e\}$, with the right-hand side given by the flows f_ℓ and electromotive force \mathcal{E} . As such, its solution is linear in the initial data, and is given as the sum of the solutions of the problem with only f 's or only \mathcal{E} present; we study these two special cases separately.

Definition 1.4. The *flow Kirchhoff problem* is the special case of the general Kirchhoff problem when the electromotive force \mathcal{E} is zero.

The (*electromotive*) *force Kirchhoff problem* is the special case of the general Kirchhoff problem when all the in/outflows are zero, that is, when all $f_\ell = 0$.

Remark 1.5. The solution of the flow Kirchhoff problem is unchanged if $\underline{\rho} = \{\rho_{|e|}\}$ is rescaled by some $\mu \in \mathbb{R}_+$, while $\{f_\ell\}$ are unchanged. Thus it is natural to think of the initial data of the flow Kirchhoff problem as a point $\mathbb{P}\underline{\rho} \in S_{>0}^{\#|E|-1}$, where we denote by $S_{>0}^{\#|E|-1} := \mathbb{R}_+^{\#|E|} / \mathbb{R}_+$ the positive octant of the real sphere.

The solution of the force Kirchhoff problem is homogeneous under rescaling resistances: if all resistances $\rho_{|e|}$ are rescaled by $\mu \in \mathbb{R}_+$ while \mathcal{E} is unchanged, then \underline{c} is rescaled by μ^{-1} . Therefore, if thinking of the initial data as $\mathbb{P}\underline{\rho} \in S_{>0}^{\#|E|-1}$, then the solution is also only defined as a projective point.

One crucial feature of our setup is that since all resistances are positive reals, the currents solving the Kirchhoff problem can be a priori bounded. As hinted at by the homogeneity, it is natural to expect a bound for the flow Kirchhoff problem independent of ρ , and a bound for the force problem that is linear in $1/\rho$. We prove these two a priori bounds – which we could not find in the literature – by elementary arguments.

Lemma 1.6. *For a given graph Γ and given inflows $\{f_\ell\}$, for any edge e of Γ the solution c_e of the corresponding flow Kirchhoff problem satisfies*

$$|c_e| \leq \frac{1}{2} \sum_{\ell} |f_\ell| \tag{1.3}$$

for any resistances $\rho \in \mathbb{R}_+^{\#|E|}$.

Proof. We prove the statement by induction on the number k of vertices of Γ . If $k = 1$, then every edge e is a loop, and thus by condition (2) of the Kirchhoff problem (that is, equation (1.2)) $c_e = 0$, so that the inequality is trivially satisfied. Suppose now that the statement holds for any graph with k vertices. For a graph with $k+1$ vertices we claim that there must exist a vertex v such that $c_e \geq 0$ for any $e \in E_v$. Indeed, suppose for contradiction that such a vertex did not exist. Then starting from an arbitrary vertex we follow some edge originating from it such that the current is negative, get to another vertex, and repeat. Then eventually we must return to a vertex that we have already visited, and thus we will have constructed an oriented cycle of edges in Γ such that $c_e < 0$ for any edge in the cycle. However, since all the $\rho_{|e|}$ are positive real numbers, the sum $\sum c_e \rho_{|e|}$ over this cycle would be negative, contradicting condition (2) of the flow Kirchhoff problem, as there is no force on the right-hand side there.

Thus there exists a vertex $v \in V(\Gamma)$ such that $c_e \geq 0$ for any $e \in E_v$. Condition (1) of the Kirchhoff problem (that is, equation (1.1)) at v then gives

$$\sum_{e \in E_v} |c_e| = \sum_{e \in E_v} c_e = - \sum_{\ell: p_\ell \in C^v} f_\ell. \tag{1.4}$$

Since the sum of all inflows equals zero we have

$$- \sum_{\ell: p_\ell \in C^v} f_\ell = -\frac{1}{2} \left(\sum_{\ell: p_\ell \in C^v} f_\ell - \sum_{\ell: p_\ell \notin C^v} f_\ell \right) \leq \frac{1}{2} \sum_{\ell} |f_\ell|.$$

Hence the inequality (1.3) holds for any $e \in E_v$.

The currents $\{c_e: e \notin E_v\}$ are a solution of the flow Kirchhoff problem on the graph Γ' whose vertices are $V(\Gamma) \setminus v$, and with additional legs obtained by replacing each oriented edge $e \in E_v$ by a leg attached to $v(-e)$, with inflow c_e in that new leg. By the inductive assumption for the graph Γ' we have for any $e \in \Gamma'$ the inequality

$$|c_e| \leq \frac{1}{2} \left(\sum_{\ell: p_\ell \notin C^v} |f_\ell| + \sum_{e \in E_v} c_e \right).$$

Combining this with (1.4) implies (1.3) for all edges of the original graph Γ . \square

The bound for solutions of the force Kirchhoff problem is as follows.

Lemma 1.7. *For a given graph Γ and given electromotive force \mathcal{E} , for any edge e of Γ the solution c_e of the force Kirchhoff problem for any resistances $\rho \in \mathbb{R}_+^{\#|E|}$ satisfies*

$$|c_e| \leq \frac{N|\mathcal{E}|}{\min_{|e| \in |E|} \rho_{|e|}}, \tag{1.5}$$

where $|\mathcal{E}|$ denotes the maximum value of \mathcal{E} on simple loops in Γ , and N is the rank of $H_1(\Gamma)$.

Proof. We will prove the lemma by induction on N . If $N = 0$ then there are no cycles and it is easy to see that in this case all currents c_e are zero. In order to prove the induction step, first note that by condition (1) of the force Kirchhoff problem, in the absence of in and outflows, for any vertex $v \in V(\Gamma)$ there must exist some edge e_+ such that $c_{e_+} \geq 0$. As in the proof of the previous lemma, in going along such edges we must eventually return back to a vertex already passed, and the first time we do so, we have constructed a simple oriented loop $\gamma \subset \Gamma$ such that $c_{e'} \geq 0$ for any $e' \in \gamma$. Hence for any $e_0 \in \gamma$ we have the estimate

$$c_{e_0} \rho_{|e_0|} \leq \sum_{e' \in \gamma} c_{e'} \rho_{|e'|} = \mathcal{E}_\gamma \leq |\mathcal{E}|, \tag{1.6}$$

which is stronger than the required bound (1.3).

Consider the graph Γ' obtained from Γ by cutting the edge e_0 and attaching to the vertices $v(e_0)$ and $v(-e_0)$ new legs with inflows c_{e_0} and c_{-e_0} , respectively. The solution of the force Kirchhoff problem on Γ restricted to all edges of Γ' coincides with the solution \tilde{c}_e of the general Kirchhoff problem on Γ' with the same force as before on all cycles that did not pass through e_0 , and with these inflows in the two new legs. Since the general Kirchhoff problem is the linear combination of the flow and force Kirchhoff problems, we can write $\tilde{c}_e = c'_e + c''_e$, where c'_e and c''_e are the solutions of the corresponding flow and force problems. For c'_e we can use the previous lemma, while for c''_e we use the inductive assumption, obtaining respectively the bounds

$$|c'_e| \leq c_{e_0} \quad \text{and} \quad |c''_e| \leq \frac{(N-1)|\mathcal{E}|}{\min_{|e| \in |E|} \rho_{|e|}}. \tag{1.7}$$

Combining these estimates with (1.6) implies the needed bound (1.5). \square

Remark 1.8. We note that as resistances $\rho_{|e|}$ go to infinity, the bound for solutions of the force problem goes to zero, which implies that for the general Kirchhoff problem the limit of solutions is given by the solutions to the corresponding flow Kirchhoff problem. This explains why only the solution of the flow Kirchhoff problem appears in our statement of Theorem 3.7 on limits of RN differentials, while the force Kirchhoff problem is used to construct the RN differential explicitly in plumbing coordinates, in §6, essentially as corrections to the solution of the flow Kirchhoff problem.

Remark 1.9. If \underline{c} is the solution of the flow Kirchhoff problem, then every vertex v can be assigned a voltage potential $V_v \in \mathbb{R}$ such that Ohm's law

$$V_{v(e)} = V_{v(-e)} + c_e \rho_{|e|}$$

holds for any edge e . The voltage potential on a connected graph is unique up to a global additive constant, while its existence is equivalent to condition (2) of the Kirchhoff problem. The voltage potential then induces a full (non-strict) order on the vertices of the graph, which it is natural to call the chronological order (motivated by construction of operator quantization of bosonic string in [15]).

This order is very different from the one considered in [2] and the order of vanishing stratification that we introduce in Definition 8.8 below. The chronological order is a weak full order on the set $C^{(0)}$ of non-null irreducible components — that is, on what would be the set of top level components in the terminology of [2]. The chronological ordering is only present in our RN setup, when all the currents are real.

We now investigate the limits of solutions of the Kirchhoff problem as resistances vary. The flow Kirchhoff problem is a system of non-homogeneous linear equations on the currents \underline{c} with coefficients $\underline{\rho}$, invariant under scaling $\underline{\rho}$ by \mathbb{R}_+ . Therefore, the solution of the flow Kirchhoff problem depends continuously on $\mathbb{P}\underline{\rho}$. Given a sequence of resistances $\mathbb{P}\underline{\rho}_k$ that converges in $S_{>0}^{\#|E|-1}$, it thus follows that the solutions of the corresponding flow Kirchhoff problems converge. Since $S_{>0}^{\#|E|-1}$ is not compact, we will also need to investigate when the solutions of the flow Kirchhoff problems converge if $\mathbb{P}\underline{\rho}_k$ do not converge in $S_{>0}^{\#|E|-1}$. The simplest compactification is $S_{\geq 0}^{\#|E|-1}$ — the closed octant of the sphere where the coordinates are required to be non-negative. However, convergence of $\mathbb{P}\underline{\rho}_k$ in $S_{\geq 0}^{\#|E|-1}$ does not guarantee convergence of the corresponding solutions of the Kirchhoff problem: to see this we note that if for some oriented cycle of edges all resistances are zero, then an arbitrary constant can be added to all the flows in the cycle. This indicates that convergence of resistances in a certain blowup of $S_{\geq 0}^{\#|E|-1}$ is required to guarantee convergence of solutions of the flow Kirchhoff problem. The necessary blowup is in fact the real oriented blowup of the union of real coordinate planes intersected with the non-negative sector of the real sphere. We refer to [7] for a detailed definition and a survey of properties of the real oriented blowup of complex manifolds. For our purposes we give a direct iterative definition, which will also allow us to write down explicitly the analytic conditions for a sequence to converge in the blowup.

Definition 1.10. We denote by $S_{\geq 0}^{N-1} := (\mathbb{R}_{\geq 0}^N \setminus \{0\})/\mathbb{R}_+$ the non-negative sector of the real sphere. The *positive blow-up of the sphere*, denoted by \mathbb{S}_+^{N-1} , is the blowup

$$\pi : \mathbb{S}_+^{N-1} \rightarrow S_{\geq 0}^{N-1}$$

defined recursively in N as follows. We let \mathbb{S}_+^0 be a point. Given the definition of \mathbb{S}_+^j for all $0 < j < N$, we define \mathbb{S}_+^N to be the result of blowing up every coordinate subspace $\{0\}^j \times S_{>0}^{N-j-1}$ to $\mathbb{S}_+^j \times S_{>0}^{N-j-1}$ (for all possible renumberings of coordinates).

Recursively, this means that \mathbb{S}_+^{N-1} is the disjoint union over all subsets $P \subset \{1, \dots, N\}$ of the products $S_{>0}^{\#P-1} \times \mathbb{S}_+^{N-\#P-1}$, where the sphere records those coordinates that are non-zero, and the second factor records the corresponding recursive blowup.

Explicitly, denote by Z_N the set of all partitions of the set $\{1, \dots, N\}$ into numbered subsets: $\{P\} \in Z_N$ is a decomposition

$$\{1, \dots, N\} = P_1 \sqcup \dots \sqcup P_l.$$

Then, as a set, \mathbb{S}_+^{N-1} is the disjoint union over all $\{P\} \in Z_n$ of the products of positive sectors of the sphere:

$$\mathbb{S}_+^{N-1} = \bigsqcup_{\{P\} \in Z_N} \prod_{j=1}^l S_{>0}^{N_j-1}, \tag{1.8}$$

where $N_j := \#P_j$. The topology on \mathbb{S}_+^{N-1} is such that a point

$$\mathbb{P}\underline{\rho} = (x_1^{(1)} : \dots : x_{N_1}^{(1)}) \times \dots \times (x_1^{(l)} : \dots : x_{N_l}^{(l)}) \in \prod_{j=1}^l S_{>0}^{N_j-1} \subset \mathbb{S}_+^{N-1}$$

is the limit as $k \rightarrow \infty$ of a sequence of points

$$(y_1[k] : \dots : y_N[k]) \in S_{>0}^{N-1}$$

if and only if the following conditions hold:

$$\begin{aligned} \lim_{k \rightarrow \infty} (y_a[k]x_b^{(j)} - y_b[k]x_a^{(j)}) &= 0 \quad \text{for any } a, b \in P_j, 1 \leq j \leq l, \\ \lim_{k \rightarrow \infty} \frac{y_a[k]}{y_b[k]} &= 0 \quad \text{for any } a \in P_j, b \in P_{j'}, 1 \leq j' < j \leq l. \end{aligned} \tag{1.9}$$

The case $l = 1$ and $P_1 = \{1, \dots, N\}$ corresponds to the open dense subset $S_{>0}^{N-1} \subset \mathbb{S}_+^{N-1}$. The contraction

$$\pi : \mathbb{S}_+^{N-1} \rightarrow S_{\geq 0}^{N-1}$$

is defined by sending $\mathbb{P}\underline{\rho}$ to a point where all $x_i^{(j)}$ for $j > 1$ are replaced by zeros, while all $x_i^{(1)}$ are unchanged. The map π is thus an isomorphism on $S_{>0}^{N-1}$, and we think of \mathbb{S}_+^{N-1} as a recursive real oriented blowup of $S_{>0}^{N-1}$ (see, for instance, [1], §X.9, for a discussion of real oriented blowups). It can be seen that in fact \mathbb{S}_+^{N-1} is a real manifold with corners, but all that matters for us is that \mathbb{S}_+^{N-1} is a compact topological space containing $S_{>0}^{N-1}$ as a dense open subset.

We will show that convergence of resistances in $\mathbb{S}_+^{\#|E|-1}$ implies convergence of solutions of the flow Kirchhoff problem, and that the limits of solutions are solutions of the multi-scale Kirchhoff problem, which we now define. For a given point $\mathbb{P}\underline{\rho} \in \mathbb{S}_+^{\#|E|-1}$, let

$$x := (x_{i_1} : \dots : x_{i_{\#P_1}}) \in S_{>0}^{\#P_1}$$

be the ‘largest factor’ in (1.8) corresponding to P_1 , and let $\mathbb{P}\underline{\rho}' \in \mathbb{S}_+^{\#|E|-\#P_1}$ correspond to the product of all other factors, so that we think of $\mathbb{P}\underline{\rho} \in \mathbb{S}_+^{\#|E|-1}$ as

$$x \times \mathbb{P}\underline{\rho}' \in \mathbb{S}_{>0}^{\#P_1} \times \mathbb{S}_+^{\#|E|-\#P_1}.$$

Definition 1.11. The *multi-scale flow Kirchhoff problem* on a graph Γ with inflows $\{f_\ell\}$ and generalized resistance $\mathbb{P}\underline{\rho} \in \mathbb{S}_+^{\#|E|-1}$, is posed recursively as follows.

Let Γ_1 be the graph obtained from Γ by contracting all edges $|e|$ for $|e| \notin P_1$. Let $\{c_e\}_{|e| \in P_1}$ be the solution of the flow Kirchhoff problem on Γ_1 with inflows f_ℓ and resistances x .

Let Γ_2 be the (possibly disconnected) graph whose edges are all edges $|e| \in |E|(\Gamma) \setminus P_1$ whose vertices are all the endpoints of such edges and whose legs are the original legs that connect at these vertices, together with a new leg for each edge e such that $|e| \in P_1$ and $v(e) \in V(\Gamma_2)$. Then on every connected component of Γ_2 we pose recursively the multi-scale flow Kirchhoff problem with the inflows being $\{f_\ell\}$ for the original legs attached to Γ_2 and c_e for each new leg, and with resistances $\mathbb{P}\underline{\rho}'$ (note that condition (1) for the flow Kirchhoff problem on Γ_1 ensures that the sum of the inflows for every connected component of Γ_2 is then equal to zero).

The *solution* to the multi-scale flow Kirchhoff problem on Γ is then defined to be the union of $\{c_e : |e| \in P_1\}$ and of the recursively defined solution of the multi-scale flow Kirchhoff problem on each connected component of Γ_2 .

Remark 1.12. In terms of stable curves, if Γ is the dual graph of a stable curve C , then Γ_1 is the graph of the smoothing of C at all the nodes except those indexed by P_1 , while Γ_2 is the graph of the partial normalization of all the nodes except those indexed by P_1 .

We now prove that if resistances converge in $\mathbb{S}_+^{\#|E|-1}$, then solutions of the flow Kirchhoff problem converge to the solution of the multi-scale Kirchhoff problem.

Lemma 1.13. *For a fixed graph Γ , if a sequence of inflows $f_{\ell,k}$ converges to f_ℓ and for a sequence of non-zero resistances $\{\underline{\rho}_k\}$ the projectivized resistances $\mathbb{P}\underline{\rho}_k \in \mathbb{S}_{>0}^{\#|E|-1}$ converge to some $\mathbb{P}\underline{\rho} \in \mathbb{S}_+^{\#|E|-1}$, then the solutions \underline{c}_k of the flow Kirchhoff problems with resistances $\underline{\rho}_k$ and inflows $f_{\ell,k}$ converge. Moreover, the limit of \underline{c}_k is the solution of the multi-scale Kirchhoff problem with inflows f_ℓ and generalized resistance $\mathbb{P}\underline{\rho}$.*

Proof. We will prove the lemma by induction on the number of levels of the multi-scale problem (that is, on the number l of factors in (1.8)). If $\mathbb{P}\underline{\rho}_k$ converge to some $\mathbb{P}\underline{\rho} \in \mathbb{S}_{>0}^{\#|E|-1}$, then the statement is obvious, since the flow Kirchhoff problem is simply a system of non-degenerate linear equations, and solutions depend continuously on the parameters $\underline{\rho}_k$ and inflows $f_{\ell,k}$.

Now suppose that $\pi(\underline{\rho}) = (x, 0)$, where x corresponds to the P_1 factor, and denote by $|x|$ the minimal absolute value of coordinates of x . Then by rescaling each $\underline{\rho}_k$ by a suitable positive real number, we can assume that $\underline{\rho}_k = (x_k, \underline{\rho}'_k)$ converge to $(x, 0)$, while $\mathbb{P}\underline{\rho}_k$ converge to $\mathbb{P}\underline{\rho}$. Then for k sufficiently large we know that the absolute value of each coordinate of x_k is bounded below by $|x|/2$, while for any $t > 0$ there exists a sufficiently large K such that for any $k > K$ the absolute value of each

coordinate of ρ'_k is less than t . Given any simple oriented loop $\gamma \subset \Gamma$, let $\gamma_1 \subset \Gamma_1$ be the loop obtained by contracting those edges that are not in P_1 . Then equation (2) in the flow Kirchhoff problem on Γ is

$$\sum_{e \in \gamma_1} c_{e,k} x_{e,k} + \sum_{e' \in \gamma \setminus \gamma_1} c_{e',k} \rho_{e',k} = 0. \tag{1.10}$$

Let $\{\tilde{c}_{e,k} : e \in E(\Gamma_1)\}$ be the solution of the flow Kirchhoff problem on Γ_1 with inflows $f_{\ell,k}$ and resistances x_k . Then condition (2) of the Kirchhoff problem gives

$$\sum_{e \in \gamma_1} \tilde{c}_{e,k} x_{e,k} = 0. \tag{1.11}$$

From (1.10) and (1.11) it follows that

$$\sum_{e \in \gamma_1} (c_{e,k} - \tilde{c}_{e,k}) x_{e,k} = - \sum_{e' \in \gamma \setminus \gamma_1} c_{e',k} \rho_{e',k}. \tag{1.12}$$

The set $(c_{e,k} - \tilde{c}_{e,k})$ for $e \in E(\Gamma_1)$ is the solution of the general Kirchhoff problem on Γ_1 with the electromotive force defined by the right-hand side of (1.12) and inflows at every vertex $v \in \Gamma_1$ such that $v = v(e)$, $|e| \notin P_1$, equal to $\sum_{e \notin P_1, v(e)=v} c_{e,k}$. Let $d'_{e,k}$ and $d''_{e,k}$ be the solutions of the corresponding flow and force problems. Since

$$|c_{e,k}| \leq \frac{1}{2} \sum_{\ell} |f_{\ell,k}|,$$

we can use (1.3) and (1.5) to conclude that there is a constant M such that for any $e \in E(\Gamma_1)$

$$|c_{e,k} - \tilde{c}_{e,k}| < Mt|x|^{-1}. \tag{1.13}$$

Since $t > 0$ could be chosen arbitrary, and the rest of the right-hand side is a constant, this implies that, as $k \rightarrow \infty$, the solutions $c_{e,k}$ and $\tilde{c}_{e,k}$ on the edges of Γ_1 have the same limit. By the inductive assumption, the solutions of the Kirchhoff problem on Γ_2 converge to the solutions of the multi-scale problem on Γ_2 with the additional inflows equal $\tilde{c}_{e \in \Gamma_1}$. \square

2. Notation for RN differentials and moduli of jet curves

We follow the (slightly adjusted, in anticipation of [10]) notation and the real-normalized differentials setup of [8] and [9], which we now review.

Definition 2.1. The *singular part* of a meromorphic differential at a point p on a Riemann surface C is the equivalence class of meromorphic differentials ω in a neighbourhood of p , with the equivalence $\omega \sim \omega'$ if and only if $\omega' - \omega$ is holomorphic at p .

Definition 2.2. For $m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}$ we let $\mathcal{M}_{g,n}^{m_1, \dots, m_n}$ be the moduli space of smooth genus g complex curves C with n distinct labelled marked points p_1, \dots, p_n together with a singular part σ_{ℓ} of a meromorphic differential with pole of order exactly $m_{\ell} + 1$ at each point p_{ℓ} , such that each residue r_{ℓ} is purely imaginary, and the sum of all residues is equal to zero.

We similarly denote by $\mathcal{M}_{g,n}^{\leq m_1, \dots, \leq m_n}$ the moduli space where each σ_ℓ is a singular part of order up to $m_\ell + 1$, and at least one of the singular parts is non-zero, with the same condition on the residues.

We call points of $\mathcal{M}_{g,n}^{m_1, \dots, m_n}$ or of $\mathcal{M}_{g,n}^{\leq m_1, \dots, \leq m_n}$ smooth jet curves.

We will always denote jet curves by X , with C denoting the underlying smooth curve. To keep the notation manageable, we will always suppress the marked points in our notation for curves and families of curves.

The reason for the name of a jet curve is that the datum of a singular part is equivalent to the datum of a jet of a local coordinate in which the meromorphic differential can be written in the standard form as $(z^{-m} + rz^{-1}) dz$.

We think of $\mathcal{M}_{g,n}^{m_1, \dots, m_n} \subset \mathcal{M}_{g,n}^{\leq m_1, \dots, \leq m_n}$ as fibrations over $\mathcal{M}_{g,n}$ with fibers

$$\prod_{\ell} (\mathbb{C}^{m_\ell} \setminus \mathbb{C}^{m_\ell - 1}) \times \mathbb{R}^{n-1} \quad \text{and} \quad (\mathbb{C}^{\sum m_\ell} \times \mathbb{R}^{n-1}) \setminus \{0, 0\}$$

respectively.

As easily follows from the positive-definiteness of the imaginary part of the period matrix, for any $X \in \mathcal{M}_{g,n}^{\leq m_1, \dots, \leq m_n}$ there exists a unique meromorphic differential

$$\Psi_X \in H^0\left(C, K_C + \sum (m_\ell + 1)p_\ell\right)$$

with prescribed singular parts σ_ℓ at p_ℓ , with residues $r_\ell \in i\mathbb{R}$ at p_ℓ and with all periods real.

Definition 2.3. For any $X \in \mathcal{M}_{g,n}^{\leq m_1, \dots, \leq m_n}$ we call $\Psi = \Psi_X$ the associated real-normalized (RN for short) differential.

Remark 2.4. The datum of a real-normalized differential is equivalent to the datum of the harmonic function

$$F(p) := \text{Im} \int^p \Psi$$

on the punctured Riemann surface $C \setminus \{p_1, \dots, p_n\}$, defined up to an additive constant. Indeed, given any such harmonic function F , the RN differential is given by $d(F^* + iF)$, where F^* denotes the harmonic conjugate function to F .

Notation 2.5. From now on, we will fix g, n and m_1, \dots, m_n , with all $m_\ell \geq 0$, and write simply \mathcal{M} for $\mathcal{M}_{g,n}^{\leq m_1, \dots, \leq m_n}$.

Since in the Deligne–Mumford compactification the marked points on stable nodal curves are not allowed to coincide with the nodes, the holomorphic fibration $\mathcal{M} \rightarrow \mathcal{M}_{g,n}$ extends to a holomorphic fibration over the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, which we denote by $\overline{\mathcal{M}}$. We will call $X \in \overline{\mathcal{M}}$ stable jet curves.

3. Statement of results: limits of RN differentials

Our first goal is to give the precise statement of the theorem on limits of RN differentials – this will be Theorem 3.7, which is the precise version of Theorem 0.1. As the RN differential does not depend holomorphically on the moduli, we work with sequences of smooth curves degenerating to a stable curve, rather than with algebraic families of smooth curves degenerating to a stable one.

Notation 3.1. For a stable curve

$$(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n},$$

its *dual graph* Γ has vertices v that correspond to *normalizations* C^v of irreducible components of C , edges $|e|$ that correspond to nodes $q_{|e|}$ of C , oriented edges e that correspond to pre-images q_e of the nodes (as points on the normalization \tilde{C} of C), and legs ℓ that correspond to the marked points p_ℓ , attached to the vertex v such that $p_\ell \in C^v$. So E_v is the set of all pre-images of the nodes that are contained in C^v , and q_e and q_{-e} are the two pre-images on \tilde{C} of a node $q_{|e|}$ of C .

Notation 3.2. From now on, we always work with a sequence $\{X_k\} \subset \mathcal{M}$ of smooth jet curves such that X_k converge as $k \rightarrow \infty$ to some stable jet curve $X \in \partial \overline{\mathcal{M}}$. We denote by $\{C_k\} \subset \mathcal{M}_{g,n}$ the underlying sequence of smooth curves with distinct marked points (which, recall, we suppress in notation systematically), which then must converge to the stable curve $C \in \partial \overline{\mathcal{M}}_{g,n}$ underlying the stable jet curve X .

The limit

$$\Psi := \lim_{k \rightarrow \infty} \Psi_{X_k},$$

if it exists, will be called the *limit RN differential* in such a sequence. By abuse of notation we will speak of the singularities of differentials at points $p_\ell \in C_k$ without labelling the dependence of p_ℓ on k . We will write Ψ_k for Ψ_{X_k} .

If the limit RN differential Ψ exists in a given sequence, then

$$\Psi \in H^0\left(C, \omega_C\left(\sum (m_\ell + 1)p_\ell\right)\right).$$

This is to say, Ψ has prescribed singularities at every point $p_\ell \in C$, and has at most simple poles at the nodes of C , with opposite residues. By abuse of notation, we denote by Ψ^v the pullback to the normalization C^v of the restriction of Ψ to the corresponding irreducible component of C . Thus each Ψ^v is determined uniquely by its residues at q_e for all $e \in E_v$, and its singular parts at those p_ℓ that are contained in C^v .

In [8], §5, we showed that limits of RN differentials whose only singularity is one double pole do not develop residues at the nodes of the stable curve. The proof applies verbatim to the case of RN differentials with a single pole of arbitrary order, and by \mathbb{R} -linearity (of the dependence of the RN differential on its singular parts) it further extends to the general case of any differential without residues, that is, ‘of the second kind’ in classical terminology, giving the following result.

Theorem 3.3 (see [8]). *If all the residues $r_\ell(X_k)$ are zero, then the limit RN differential Ψ exists in any degenerating sequence $\{X_k\} \rightarrow X$, and on any C^v the restriction Ψ^v of the limit RN differential is an RN differential on C^v with prescribed singular parts at those marked points p_ℓ that lie on C^v , and no other singularities, including at the nodes.*

Remark 3.4. This statement is a priori surprising, as for example it follows that the limit RN differential of the second kind is identically zero on any C^v that contains no marked points. This is clearly false for general r_ℓ , as one sees by considering the

case of $n = 2$, with two simple poles that are on different components of the stable curve: then by the residue theorem there must appear a simple pole at some node between these components.

For limits of RN differentials ‘of the third kind’ — that is, with arbitrary residues r_ℓ — one can easily see that the residues of the limit RN differential may depend on the degenerating sequence, and our main theorem on limit RN differentials is a necessary condition for the existence of a limit RN differential, and the determination of its residues. Such an explicit construction is not available in the literature for the closures of the strata studied in [6], [4], [5], and [2].

We work in plumbing coordinates near the boundary of the moduli space, which are recalled and discussed in detail in §4. To state the results, recall that the plumbing parameter $s_{|e|}$ corresponds to locally opening up the node $q_{|e|} \in C$ given in local coordinates by $xy = 0$ to $xy = s_{|e|}$. The plumbing parameters for every node, together with the coordinates on the moduli space where \tilde{C} lives give local coordinates near the boundary point C of $\overline{\mathcal{M}}_{g,n}$. Since $\overline{\mathcal{M}}$ is a bundle over $\overline{\mathcal{M}}_{g,n}$, local coordinates near $X \in \partial\overline{\mathcal{M}}$ are given by the local coordinates on $\overline{\mathcal{M}}_{g,n}$ near C , together with local coordinates for the fiber of $\overline{\mathcal{M}} \rightarrow \overline{\mathcal{M}}_{g,n}$.

Definition 3.5. The *log-plumbing coordinates* of a smooth point C' in a neighbourhood of $C \in \partial\overline{\mathcal{M}}_{g,n}$ are given by the point

$$\underline{\rho}(C') := \{-\log |s_{|e|}|\} \in \mathbb{R}_+^{\#|E|}.$$

The *projectivized log-plumbing coordinates* of C' are given by the point

$$\mathbb{P}\underline{\rho}(C') \in \mathbb{R}_+^{\#|E|} / \mathbb{R}_+ = S_{>0}^{\#|E|-1}.$$

Recall that in §1 we proved that convergence of projectivized resistances in the blowup $S_+^{\#|E|-1}$ of $S_{\geq 0}^{\#|E|-1}$ implies convergence of solutions of the flow Kirchhoff problem; we thus make the following definition.

Definition 3.6. A sequence $\{C_k\} \subset \mathcal{M}_{g,n}$ converging to $C \in \partial\overline{\mathcal{M}}_{g,n}$ is called *admissible* if there exists a limit

$$\underline{\rho} := \lim_{k \rightarrow \infty} \mathbb{P}\underline{\rho}(C_k) \in S_+^{\#|E|-1} \quad \text{as } k \rightarrow \infty$$

of the projectivized log-plumbing coordinates $\mathbb{P}\underline{\rho}(C_k)$ of C_k . The point $\mathbb{P}\underline{\rho}$ is then called the *rates of degeneration* of the sequence $\{C_k\}$.

Our main result on limit RN differentials is that their residues are given by limits of solutions of the flow Kirchhoff problem, which by Lemma 1.13 is the solution of the multi-scale Kirchhoff problem.

Theorem 3.7. *Let $\{X_k\} \subset \mathcal{M}$ be a sequence of smooth jet curves converging to a stable jet curve X . Then the limit RN differential $\Psi = \lim_{k \rightarrow \infty} \Psi_{X_k}$ exists if and only if the solutions $c_{e,k}$ of the flow Kirchhoff problems with inflows $ir_{\ell,k}$ and resistances $\underline{\rho}_k$ converge. If the limit RN differential exists, then on any C^v the limit Ψ^v is the RN differential with prescribed singularities at the marked points p_ℓ contained in C^v , and with simple poles of residue i times the limit of the solutions of the flow Kirchhoff problem.*

This theorem will be proven in § 7. Since we have studied the limits of solutions of the Kirchhoff problem in § 1, Lemma 1.13 implies the following.

Corollary 3.8. *If the sequence $\{X_k\}$ is admissible with rates of degeneration $\mathbb{P}\underline{\rho} \in \mathbb{S}_+^{\#|E|-1}$, then the limit RN differential Ψ exists, and its residues are given by the solution of the multi-scale Kirchhoff problem with generalized resistances $\mathbb{P}\underline{\rho}$, for each $e \in E_v$.*

Remark 3.9. For the case of limit RN differentials of the second kind, all f_ℓ are zero, and thus for any $\mathbb{P}\underline{\rho}_k$ the set of all currents $c_e = 0$ is the unique solution of the flow Kirchhoff problem, so that the solutions converge for any sequence, and in the limit all currents are still equal to zero. In particular, in this case the limit is the same in all admissible sequences; indeed, while convergence of resistances in $\mathbb{S}_+^{\#|E|-1}$ implies convergence of solutions of the Kirchhoff problem, many such limits may be the same. We do not claim that $\mathbb{S}_+^{\#|E|-1}$ is the minimal blowup of $\mathbb{S}_{>0}^{\#|E|-1}$ onto which the solutions of the flow Kirchhoff problem extend continuously.

Remark 3.10. The special case of this theorem when C has geometric genus zero (that is, each C^v is a rational curve) and the rates of degeneration lie in $\mathbb{S}_{>0}^{\#|E|-1}$ (that is, no blowup to $\mathbb{S}_+^{\#|E|-1}$ is necessary) was studied by Lang [16], who obtained for this case a version of this theorem from a completely different viewpoint and with very different methods.

Remark 3.11. For degenerating algebraic 1-parameter families such as used in [2], each plumbing coordinate s_e has the form t^{n_e} for some integer $n_e > 0$, and thus any subsequence of such a family is admissible, with rates of degeneration $\mathbb{P}\underline{\rho} \in \mathbb{S}_{>0}^{\#|E|-1}$. In particular, for such an algebraic family, there is no need to blow up the sphere.

Remark 3.12. The meaning of condition (1) of the Kirchhoff problem in terms of differentials is clear: it serves to ensure that the residue theorem is satisfied for each Ψ^v . The meaning of condition (2) is less transparent. In fact if a collection of RN differentials Ψ on the components C^v were to have arbitrary residues at the nodes, then the imaginary parts of its periods over cycles passing through the nodes will diverge logarithmically, as computed in Lemma 6.5. Condition (2) is precisely to guarantee that the logarithmic divergences cancel, so that the imaginary parts of periods of the limit RN differential on the singular stable curve C are finite.

Since the space $\mathbb{S}_+^{\#|E|-1}$ is compact, it follows that all possible limit RN differentials on C are obtained in this way.

Lemma 3.13. *Any sequence of smooth jet curves $\{X_k\}$ converging to a stable jet curve $X \in \partial\mathcal{M}$ contains an admissible subsequence.*

Proof. The space $\mathbb{S}_+^{\#|E|-1}$ is compact, and thus the sequence $\{\mathbb{P}\underline{\rho}(C_k)\} \subset \mathbb{S}_+^{\#|E|-1}$ must contain a convergent subsequence, which by definition corresponds to an admissible sequence of smooth curves. \square

Proposition 3.14. *Let $\{X_k\} \subset \mathcal{M}$ be a sequence of smooth jet curves converging to a stable jet curve X . If the limit RN differential exists, then it is given by a collection of RN differentials on C^v with prescribed singularities at p_ℓ and with*

the residue at q_e being i times the solution of the multi-scale flow Kirchhoff problem for some $\mathbb{P}_\rho \in \mathbb{S}_+^{\#|E|-1}$.

Proof. By Lemma 3.13, the sequence $\{X_k\}$ must contain an admissible subsequence, with resistances converging to some $\mathbb{P}_\rho \in \mathbb{S}_+^{\#|E|-1}$. By Lemma 1.13, in such a subsequence there exists a limit of solutions of the flow Kirchhoff problem, and it is given by the solution of the multi-scale Kirchhoff problem with resistances \mathbb{P}_ρ . Finally, by Theorem 3.7, the convergence of solutions of the flow Kirchhoff problem implies the existence of the limit RN differential in this subsequence, of the form claimed. Since the limit RN differential is assumed to exist for all of the original sequence, it must be of the form claimed. \square

This completes the statement of our results on limits of RN differentials. The full details and statements of our results on limits of zeros of RN differentials will be given in § 8, after the main technical tool of solving the jump problem is introduced.

4. Plumbing setup for Riemann surfaces

We now recall the full details of the plumbing construction discussed in the introduction, and we fix the notation that will be used throughout the rest of the paper and in all the proofs.

Definition 4.1 (standard plumbing). Let $q_1, q_2 \in C$ (with C a possibly disconnected Riemann surface) be two distinct points. Let z_1, z_2 be local coordinates on C near q_1, q_2 such that $z_j(q_j) = 0$ and furthermore sufficiently small so that the inverse maps z_j^{-1} embed the unit disk into \mathbb{C} as disjoint neighbourhoods $V_j := \{|z_j| < 1\} \subset C$ of q_j . Then for any $s \in \mathbb{C}$ with $|s| < 1$ we denote by

$$U_j = U_j^s := \{|z_j| < \sqrt{|s|}\} \subset V_j$$

the corresponding disks and denote by

$$\gamma_j := \partial U_j$$

their boundary circles, which we orient negatively with respect to U_j . The *standard plumbing C_s with parameter s* is the Riemann surface

$$C_s := [C \setminus (U_1 \sqcup U_2)] / (\gamma_1 \sim \gamma_2),$$

where γ_1 is identified with γ_2 via the diffeomorphism $I(z_1) := s/z_1$. The structure of a Riemann surface on C_s is defined by saying that a function on C_s is holomorphic, if it is holomorphic on the complement of the *seam* γ (the image of γ_1 and γ_2) and continuous along the seam.

Definition 4.2 (plumbing coordinates on moduli). Local plumbing coordinates on $\overline{\mathcal{M}}_{g,n}$ near a stable curve $C \in \partial \overline{\mathcal{M}}_{g,n}$ are defined as follows. Let \tilde{C} be the normalization of C , which is a smooth (possibly disconnected) Riemann surface with marked points p_ℓ , and also with all the pre-images of the nodes as marked points.

We think of \tilde{C} as a point in a suitable Cartesian product of moduli spaces of curves with marked points. Let $u = (u_1, \dots, u_x)$ be some local coordinates on this

product of moduli spaces; we write \tilde{C}_u for the (possibly disconnected) curve in this moduli space with coordinates u , so that all the coordinates of \tilde{C} are $u_i = 0$. Choose, for all sufficiently small u , a holomorphically varying family of local coordinates z_e in the neighbourhood on \tilde{C}_u of every pre-image q_e of every node of C_u , scaled (by dividing by a large real number) to be sufficiently small so that the unit disks in these coordinates are all disjoint on \tilde{C}_u .

Then u together with a set of plumbing parameters

$$s := \{s_{|e|}\} \in \Delta^{\#|E|},$$

for $\Delta \subset \mathbb{C}$ a sufficiently small disk, give local coordinates on $\overline{\mathcal{M}}_{g,n}$ near C (see [3]).

Remark 4.3. Different versions of plumbing are available in the literature. First of all, one usually considers the neighbourhoods $V_j = \{|z_j| < \varepsilon\}$ in the local coordinates, for some sufficiently small ε ; by rescaling z_j by a real number this is of course equivalent to our setup.

The plumbing that we use, by identifying the boundaries of two cut out disks directly, is perhaps the earliest one, going back to [3]. It is clearly seen to be equivalent to cutting out closed disks of radii $|s|$ around q_j , and then identifying two boundary annuli in this open Riemann surface: if one has identified along the annuli, then one can alternatively cut the glued surface along the middle circle of the resulting glued annulus, and switch to our viewpoint.

In [1], [21], and [2], plumbing using a plumbing fixture is performed — which is the analytic description of the algebraic versal deformation coordinates. This third kind of plumbing can also easily be seen to be equivalent to the original version that we use, by cutting the plumbing fixture $xy = t$ along the circle $|x| = |y|$. The advantage of the approach using a fixed plumbing fixture is the ability to see explicitly the algebraic structure of the degenerating family of Riemann surfaces as the node forms, and to interpret plumbing coordinates as versal deformations of nodal curves.

Remark 4.4. The version of plumbing that we use is most suited to understanding limits of 1-forms under degeneration. Indeed, in our setup if $\{C_k\}$ is a sequence of smooth Riemann surfaces converging to C , then each C_k is obtained by identifying the boundaries of a subset of C . Thus we can interpret a sequence Φ_k of meromorphic differentials on C_k as a sequence of differentials on a sequence of growing subsets of C , tending to all of C as $k \rightarrow \infty$. Therefore, the limit $\lim_{k \rightarrow \infty} \Phi_k$, if it exists, automatically makes sense as a collection of meromorphic differentials Φ^v on the irreducible components C^v of C .

To keep the notation manageable, we will write $\tilde{C} := \tilde{C}_{u,0}$ for the normalization of a nodal curve, and will consistently drop u when no confusion is possible. We write \tilde{C} as the union of its connected components

$$\tilde{C} = \bigcup C^v$$

indexed by vertices $v \in V(\Gamma)$ of the dual graph Γ of C . Recall that we write e for an *oriented* edge of Γ and $|e|$ for the unoriented edge. We write

$$q_e \in C^{v(e)}, \quad v(e) := \text{target}(e),$$

for the corresponding pre-image q_e of the node $q_{|e|} = q_e \sim q_{-e}$ of C . To simplify notation we will also write $s_e = s_{-e} = s_{|e|}$. We then write

$$\widehat{C}_{\underline{s}} := \widetilde{C} \setminus \bigcup_e U_e^{s_e}$$

for the closed Riemann surface with boundary obtained by removing these open disks from \widetilde{C} . Identifying for each $|e| \in |E|(\Gamma)$ the boundaries $\gamma_e^{s_e}$ and $\gamma_{-e}^{s_e}$ of $\widehat{C}_{\underline{s}}$ via the map I_e sending z_e to s_e/z_e gives precisely the plumbed Riemann surface $C_{u,\underline{s}}$. When speaking of a 1-form ω on \widetilde{C} or on \widehat{C} , we mean a collection of 1-forms ω^v on the set of connected components of \widetilde{C} or of \widehat{C} .

Since $\overline{\mathcal{M}}$ is the total space of a fibration over $\overline{\mathcal{M}}_{g,n}$, local coordinates on it near some $X \in \partial\overline{\mathcal{M}}$ are given by u, \underline{s} , together with some local coordinates w for the fiber of the fibration. We will thus write a stable jet curve with these coordinates as $X_{w,u,\underline{s}}$, with $C_{u,\underline{s}}$ as the underlying stable curve.

A meromorphic differential Φ on $\widehat{C}_{\underline{s}}$ (which is, recall, the shorthand for a collection of meromorphic differentials Φ^v on $\widehat{C}_{\underline{s}}^v$) glues to define a meromorphic differential on $C_{\underline{s}}$ if and only if

$$\Phi^{v(e)}|_{\gamma_e} = I_e^*(\Phi^{v(-e)}|_{\gamma_{-e}}) \tag{4.1}$$

for all e .

Remark 4.5. Of course not every differential Φ on $\widehat{C}_{\underline{s}}$ satisfies (4.1) and glues to a differential on $C_{\underline{s}}$. One standard setup is for differentials with simple poles at pre-images of the nodes, with opposite residues. Choosing the coordinate z_e near q_e such that locally $\Phi^{v(e)} = a_e dz_e/z_e$, with $a_e = -a_{-e}$, and performing plumbing in these coordinates, one constructs a glued differential on $C_{\underline{s}}$. More generally, one can choose standard coordinates associated to a differential to glue a zero of order k to a pole of order $k + 2$ with no residue, as discussed and applied in [6], [4], and [2]. As a result, and with much further work to deal with the residues appearing, one constructs a meromorphic differential on some smooth Riemann surface $C_{\underline{s}}$ near C , in plumbing coordinates. However, since the local coordinates z_e depend on the differential, it is hard to ensure from this viewpoint that all suitable differentials on all smooth Riemann surfaces in a neighbourhood can be obtained in this way.

Our approach is direct and analytic. We start with any collection of fixed local coordinates z_e near q_e (for any u), and thus with fixed plumbing coordinates on the moduli space. Given any Φ on $\widehat{C}_{\underline{s}}$, we will subtract from it another differential ω on $\widehat{C}_{\underline{s}}$ such that their difference satisfies (4.1), and thus defines a differential on $C_{\underline{s}}$. The condition for ω must then be that its ‘jumps’ on γ_e are the same as for Φ , and we construct it by explicitly solving the jump problem.

5. The jump problem

Given a compact Riemann surface with a collection of closed loops in it, the jump problem is the problem of constructing a holomorphic differential on the complement of these loops, such that it extends continuously to each loop from the two

sides, and its boundary values there differ by a prescribed jump. Equivalently, we think of the jump problem as posed on a Riemann surface with boundary, where the boundary components are identified pairwise, and the solution of the jump problem is a differential on the interior that extends continuously to the boundary, and such that the differences of its boundary values are the prescribed ‘jumps’. The classical approach to solving the jump problem is surveyed in [22] and explained in full detail in [19]. The jump problem is solved by integrating the jumps with respect to the Cauchy kernel on the Riemann surface.

We are interested in constructing RN differentials in plumbing coordinates, and thus in solving the jump problem on $C_{\underline{s}}$. Since the Cauchy kernel on $C_{\underline{s}}$ depends on \underline{s} , determining the behaviour of the solution of the jump problem under degeneration as $\underline{s} \rightarrow 0$ is hard and has not been accomplished in the literature. Instead, we use the Cauchy kernel on the normalization \tilde{C} of the nodal curve, to construct differentials on \tilde{C} with prescribed jumps along the seams, considered as closed loops on \tilde{C} . By an explicit control of the constructed solution of the jump problem in small neighbourhoods of the nodes, we can then correct this solution, in an iterative way, to eventually construct the desired solution of the jump problem on $C_{\underline{s}}$. As this only uses the Cauchy kernel on \tilde{C} , which is independent of \underline{s} , we can determine the behaviour of the solution under degeneration. Our interest in the current paper is in solving the jump problem to construct RN differentials; our method was then used by Hu and the third author in [11] to study a normalized basis of differentials, which turns out to be easier as holomorphic dependence on parameters can be used.

Throughout this section, we will only work on smooth jet curves, that is, all s_e are always assumed to be non-zero. For convenience, we write

$$|\underline{s}| := \max_e |s_e|.$$

The jump problem is an additive analogue of the (multiplicative) Riemann–Hilbert problem posed on a Riemann surface $\widehat{C}_{u,\underline{s}}$ with $\#E$ boundary components. The initial data for the jump problem is a set $\underline{\phi}$ of complex-valued smooth 1-forms ϕ_e on γ_e , which we call *jumps*. The jumps are required to satisfy

$$\phi_e = -I_e^*(\phi_{-e}) \quad \text{and} \quad \int_{\gamma_e} \phi_e = 0 \quad \text{for all } e \in E.$$

Definition 5.1. The *jump problem* is to find a holomorphic 1-form ω on the interior of $\widehat{C}_{u,\underline{s}}$ that extends continuously to every boundary component γ_e of $\widehat{C}_{u,\underline{s}}$, and such that the boundary extensions have jumps ϕ_e , that is, satisfy for any e the equation

$$\omega|_{\gamma_e} - I_e^*(\omega|_{\gamma_{-e}}) = \phi_e.$$

Equivalently, the jump problem is the problem of constructing differentials on $C_{u,\underline{s}}$ continuous away from the seams $\gamma_{|e|}$ and with prescribed differences of boundary values on the two sides of each seam.

The solution of the jump problem is never unique: the pullback to $\widehat{C}_{u,\underline{s}}$ of any holomorphic differential on $C_{u,\underline{s}}$ has zero jumps, and can be added to any solution to produce another solution. Our main technical tool is an explicit construction of a suitably normalized solution, which we will call ARN, with explicit bounds for it.

Proposition 5.2. *There exists a constant t independent of u , such that for any $|s| < t$ and any $\underline{\phi}$, the jump problem has a unique solution ω on $\widehat{C}_{u,s}$ satisfying*

- (a) $\int_{\gamma_e} \omega = 0$ for any node e ;
- (b) $\int_{\gamma} \omega \in \mathbb{R}$ for any cycle $\gamma \in H_1(\widetilde{C}_u, \mathbb{Z})$.

The solution ω is given explicitly as the sum $\xi + \chi$, with ξ and χ being the restrictions to $\widehat{C}_{u,s}$ of the integrals (5.7) and (5.18), where the smooth real 1-forms h_e and g_e on γ_e are defined as the sums of the series (5.26).

We will call this ω the *almost real-normalized (ARN) solution* of the jump problem. Note that the condition (b) on γ is equivalent to taking $\gamma \in H_1(C_{u,s}, \mathbb{Z})$ not intersecting any of the seams.

The importance of the proposition is the explicit construction, which will eventually allow us to give estimates for the ARN solution as the curve degenerates.

Proof. The proof of this proposition will occupy the bulk of this section.

1) *Uniqueness of the ARN solution.* Suppose that ω_1 and ω_2 were two different ARN solutions of the jump problem with the same initial data. Then $\omega := \omega_1 - \omega_2$ would have zero jumps, and would thus be a holomorphic 1-form on $C_{u,s}$ with zero integral over any seam γ_e , and with real integrals over any path contained in $\widehat{C}_{u,s}^v$.

To deduce that ω is identically zero we use the Stokes' theorem on each $\widehat{C}_{u,s}^v$, and then sum over the components (a similar method will be used again later on). Choose an arbitrary point $p_0 \in C^v$; then

$$F^v(p) := \text{Im} \int_{p_0}^p \omega|_{C^v}$$

is a single-valued real harmonic function on $\widehat{C}_{u,s}^v$, since all the periods of ω^v are real. The harmonic conjugate function

$$F^{v*}(p) = \text{Re} \int_{p_0}^p \omega$$

is multiple-valued, but locally defined up to an additive constant, and thus we can still write $\omega^v = dF^{v*} + i dF^v$. We use Stokes' theorem to compute the L^2 -norm of ω^v on C^v :

$$\begin{aligned} \frac{i}{2} \int_{\widehat{C}_{u,s}^v} \omega^v \wedge \bar{\omega}^v &= \frac{i}{2} \int_{\widehat{C}_{u,s}^v} (dF^{v*} + i dF^v) \wedge (dF^{v*} - i dF^v) \\ &= \int_{\widehat{C}_{u,s}^v} dF^{v*} \wedge dF^v = - \sum_{e \in E_v} \int_{\gamma_e} F^v dF^{v*}, \end{aligned} \tag{5.1}$$

where we have used the fact that F^v is a well-defined single-valued function on $\widehat{C}_{u,s}^v$ and that the boundary of $\widehat{C}_{u,s}^v$ is the collection of γ_e for all $e \in E_v$.

We now take the sum of these equalities over all v ; the summands on the right come in pairs $\{e, -e\}$. Since ω is a holomorphic differential on the plumbed surface $C_{u, \underline{s}}$, the restrictions $F^{v(e)}|_{\gamma_e}$ and $I_e^*(F^{v(-e)}|_{\gamma_{-e}})$ on any seam differ by some constant of integration C_e , while the restrictions of the differentials $dF^{v(e)*}|_{\gamma_e}$ and $dF^{v(-e)*}|_{\gamma_{-e}}$ are equal under pullback by I_e . We thus compute

$$\int_{\gamma_e} F^{v(e)} dF^{v(e)*} + \int_{\gamma_{-e}} F^{v(-e)} dF^{v(-e)*} = C_e \int_{\gamma_e} dF^{v(e)*},$$

where we recall that the map $I_e: \gamma_e \rightarrow \gamma_{-e}$ is orientation-reversing. Since $\int_{\gamma_e} \omega^{v(e)} = 0$ by the definition of the ARN solution, it follows that also

$$\int_{\gamma_e} dF^{v(e)*} = 0,$$

and thus, finally, the sum of (5.1) for all v vanishes. Altogether it then follows that

$$\int_{C_{u, \underline{s}}} \omega \wedge \bar{\omega} = 0,$$

which implies that ω is identically equal to zero.

2) *Construction of the ARN solution.* We first recall the notion of Cauchy kernels, then use appropriate versions of the kernel to essentially deal with the real and imaginary parts of the initial data, and then construct the ARN solution by explicitly writing the inverse of the relevant integral operator as the sum of a convergent series.

3) *The Cauchy kernels.* Recall that for a genus g compact Riemann surface \mathcal{C} with a symplectic basis $\{A_k, B_k\}$ of $H_1(\mathcal{C}, \mathbb{Z})$, the normalized basis of the space of holomorphic differentials on \mathcal{C} is prescribed by the condition

$$\int_{A_k} \omega_j = \delta_{j,k}.$$

Denote by $\tau_{j,k} = \int_{B_k} \omega_j$ the period matrix of \mathcal{C} , and by $\theta(z) = \theta(\tau, z)$ the corresponding theta function. We denote by A the Abel map of \mathcal{C} to \mathbb{C}^g sending q_0 to zero, and denote the corresponding Riemann constant by κ . Then for any sufficiently general collection of fixed g points $q_0, \dots, q_{g-1} \in \mathcal{C}$ the *normalized Cauchy kernel* is defined as

$$K_{\mathcal{C}}(p, q) := \frac{1}{2\pi i} d_p \log \frac{\theta(A(p) - A(q) - Z)}{\theta(A(p) - Z)}, \tag{5.2}$$

where

$$Z := \sum_{j=1}^{g-1} A(q_j) + \kappa$$

and by d_p we mean the exterior differential with respect to p , for q fixed, so that the result is a differential form in p . For q fixed, $K_{\mathcal{C}}$ is then a meromorphic differential

in the variable p with simple poles with residues $\pm(2\pi i)^{-1}$ at $p = q$ and at $p = q_0$, respectively. For p fixed, $K_{\mathcal{C}}$ is a multi-valued meromorphic function of q with the only pole at $q = p$.

The Cauchy kernel $K_{\mathcal{C}}$ is normalized in the sense that all its A -periods are zero:

$$\int_{p \in A_k} K_{\mathcal{C}}(p, q) = 0.$$

On the sphere, the Cauchy kernel becomes simply

$$K_{\mathbb{P}^1}(p, q) = \frac{1}{2\pi i} \frac{dp}{p - q}.$$

The Cauchy kernel is used to solve the jump problem, as we now recall (see [19]). Given a smooth closed simple curve $\gamma \subset \mathcal{C}$ and a smooth 1-form ϕ on γ , the Cauchy integral transform $\int_{q \in \gamma} K_{\mathcal{C}}(p, q)\phi(q)$ defines a holomorphic 1-form in the variable p , for $p \notin \gamma$ (the holomorphicity for $p \neq q_0$ follows from the holomorphicity of the kernel $K_{\mathcal{C}}$ in p , so that we can then differentiate under the integral sign, while the holomorphicity at $p = q_0$ follows from the fact that the residue of the Cauchy kernel is independent of q , while $\int_{\gamma_e} \phi = 0$). The Sokhotski–Plemelj formula is the statement that the boundary values of this expression for $z \in \gamma$ are

$$\lim_{p' \rightarrow p \in \gamma} \int_{q \in \gamma} K_{\mathcal{C}}(p', q)\phi(q) = \pm \frac{1}{2}\phi(p) + \int_{q \in \gamma} K_{\mathcal{C}}(p, q)\phi(q), \tag{5.3}$$

where we used the classical notation \int for the Cauchy principal value of the singular integral. Locally the neighbourhood of $\gamma \subset \mathcal{C}$ looks like an annulus with γ being the middle circle, and then the limit is taken for p' lying in a fixed component of the complement of γ in this annulus. The sign in the Sokhotski–Plemelj formula is the orientation of the contour γ as the boundary component of the corresponding half of the annulus. The Sokhotski–Plemelj formula implies that the integral transform with respect to the normalized Cauchy kernel solves the jump problem.

To continue the construction of the RN differential, instead of the A -normalized Cauchy kernel we will now introduce the suitably real-normalized Cauchy kernel, defined as follows:

$$K_{\mathcal{C}}^{\text{re}}(p, q) := K_{\mathcal{C}}(p, q) - \sum_{k=1}^g \alpha_k(q)\omega_k(p), \tag{5.4}$$

where α_k are the coordinates of the vector $\alpha(q) := (\text{Im } \tau)^{-1} \text{Im } A(q)$. For q fixed, $K_{\mathcal{C}}^{\text{re}}$ is also a meromorphic differential in p with simple poles at $p = q$ and $p = q_0$ with residues $\pm(2\pi i)^{-1}$. For p fixed, $K_{\mathcal{C}}^{\text{re}}$ is a *single-valued real-analytic* function of the variable q , away from $q = p$. It can easily be checked from the monodromy properties of the theta function that all periods of $K_{\mathcal{C}}^{\text{re}}$ are real:

$$\int_{p \in \gamma} K_{\mathcal{C}}^{\text{re}}(p, q) \in \mathbb{R} \quad \forall \gamma \in H_1(\mathcal{C}, \mathbb{Z}).$$

Similarly, we introduce

$$K_{\mathcal{C}}^{\text{im}}(p, q) := iK_{\mathcal{C}}(p, q) - i \sum_{k=1}^g \beta_k(q)\omega_k(p), \tag{5.5}$$

where β_k are the coordinates of the vector $\beta(q) := (\text{Im } \tau)^{-1} \text{Re } A(q)$. For q fixed, $K_{\mathcal{C}}^{\text{im}}$ is a meromorphic differential in p with simple poles at $p = q$ and $p = q_0$ with residues $\pm(2\pi)^{-1}$. For p fixed, $K_{\mathcal{C}}^{\text{im}}$ is a multi-valued real-analytic function of q away from $q = p$. We note that

$$\text{Im} \int_{p \in \gamma} K_{\mathcal{C}}^{\text{im}}(p, q) \in \mathbb{Z} \quad \forall \gamma \in H_1(\mathcal{C}, \mathbb{Z}).$$

Analogues of the Sokhotski–Plemelj formula hold for K^{re} and K^{im} . For these analogues, in the right-hand side of (5.3) the Cauchy principal value of the integral transform of ϕ with respect to $K_{\mathcal{C}}^{\text{re}}$ or $K_{\mathcal{C}}^{\text{im}}$, correspondingly, should be taken, while for $K_{\mathcal{C}}^{\text{im}}$, ϕ is also multiplied by i . To prove this, note that $K_{\mathcal{C}}^{\text{re}}$ and $K_{\mathcal{C}}^{\text{im}}$ differ from $K_{\mathcal{C}}$ by adding some differential $\omega(p, q)$ which is holomorphic in p (while K^{im} further multiplies by i), so that the Cauchy principal value of the integral transform of ϕ with respect to this added holomorphic differential is simply the value of the integral, which thus contributes $\int_{q \in \gamma} \omega(p, q)\phi(q)$ on both sides of the Sokhotski–Plemelj formula (5.3).

4) *The integral transform with respect to K^{re} .* We now apply the Cauchy kernels to obtain a solution of the jump problem, using K^{re} and K^{im} to ensure the ARN condition.

For any set $\underline{h} = \{h_e\}$ of real-valued smooth 1-forms on $\{\gamma_e\}$ such that $h_e = -I_e^*(h_{-e})$ for any e and all periods are zero, that is,

$$\int_{q \in \gamma_e} h_e(q) = 0, \tag{5.6}$$

we define for $p \in C^v$ the Cauchy integral transform

$$\xi^v(p) := \sum_{e \in E_v} \int_{q \in \gamma_e} K_{C^v}^{\text{re}}(p, q)h_e(q). \tag{5.7}$$

We emphasize that the formula above is an integral transform on the normalization C^v of the irreducible component of $C_{u,0}$. In particular the kernel K_v^{re} is independent of \underline{s} . A priori, ξ^v can have a pole at $p = q_0$. However, since the residue of K_v^{re} at $p = q_0$ is equal to $-1/(2\pi i)$ for any q , the residue of ξ^v at $p = q_0$ is equal to

$$-\frac{1}{2\pi i} \sum_{e \in E_v} \int_{q \in \gamma_e} h_e(q),$$

which vanishes by (5.6). We thus view ξ^v as a collection of holomorphic differentials ξ_e on each U_e for $e \in E_v$, and the holomorphic differential $\widehat{\xi}^v$ on $\widehat{C}_{\underline{s}}^v$. We observe

that for any closed path $\gamma \subset \widehat{C}^v$ the integral

$$\begin{aligned} \int_{p \in \gamma} \widehat{\xi}^v &= \sum_{e \in E_v} \int_{p \in \gamma} \int_{q \in \gamma_e} K_{C^v}^{\text{re}}(p, q) h_e(q) \\ &= \sum_{e \in E_v} \int_{q \in \gamma_e} \left(\int_{p \in \gamma} K_{C^v}^{\text{re}}(p, q) \right) h_e(q) \end{aligned}$$

is real, since $\int_{p \in \gamma} K_{C^v}^{\text{re}}(p, q) \in \mathbb{R}$, and h_e is real-valued.

We now study the singular part of the kernel in more detail. For $e, e' \in E_v$, $e \neq e'$, we write $z_e = z_e(p) \in V_e$ and $w_{e'} = w_{e'}(q) \in V_{e'}$ for the local coordinates in these disks, and we write

$$\mathbf{K}_v^{\text{re}}(z_e, w_{e'}) dz_e := K_v^{\text{re}}(p, q). \tag{5.8}$$

For $p, q \in V_e$, let \mathbf{K}_v^{re} be the *holomorphic* part of K_v^{re} :

$$\mathbf{K}_v^{\text{re}}(z_e, w_e) dz_e := K_v^{\text{re}}(p, q) - \frac{dz_e}{2\pi i(z_e - w_e)}. \tag{5.9}$$

We define the integral operator \mathcal{K}^{re} by

$$(\mathcal{K}^{\text{re}} \underline{h})_e(z_e) := dz_e \sum_{e' \in E_{v(e)}} \int_{w_{e'} \in \gamma_{e'}} \mathbf{K}_{v(e)}^{\text{re}}(z_e, w_{e'}) h_{e'}(w_{e'}). \tag{5.10}$$

Since each \mathbf{K}_v^{re} is holomorphic in z_e , the operator \mathcal{K}^{re} sends a collection \underline{h} of *real-valued* 1-forms on the set of seams γ_e to a collection of *holomorphic* 1-forms on the disks V_e —again, holomorphicity is simply due to the fact that each \mathbf{K}_v^{re} is holomorphic in z_e . By definition, for any $z_e \in V_e$ we have

$$\xi^v(z_e) = (\mathcal{K}^{\text{re}} \underline{h})_e(z_e) + dz_e \int_{w_e \in \gamma_e} \frac{h_e(w_e)}{2\pi i(z_e - w_e)}. \tag{5.11}$$

The integral in this expression is singular, and its boundary values for $z_e \in \gamma_e$ are given by the Sokhotski–Plemelj formula (5.3):

$$\widehat{\xi}^v(z_e) = (\mathcal{K}^{\text{re}} \underline{h})_e(z_e) + \frac{1}{2} h_e(z_e) + dz_e \int_{w_e \in \gamma_e} \frac{h_e(w_e)}{2\pi i(z_e - w_e)} \tag{5.12}$$

and

$$\xi_e(z_e) = (\mathcal{K}^{\text{re}} \underline{h})_e(z_e) - \frac{1}{2} h_e(z_e) + dz_e \int_{w_e \in \gamma_e} \frac{h_e(w_e)}{2\pi i(z_e - w_e)}. \tag{5.13}$$

Since ξ_e is holomorphic on U_e , its integral over $\gamma_e = \partial U_e$ vanishes by the residue theorem. Using assumption (5.6), we thus obtain

$$\int_{p \in \gamma_e} \widehat{\xi}^v(p) = \int_{p \in \gamma_e} \xi_e(p) + \int_{p \in \gamma_e} h_e(p) = 0. \tag{5.14}$$

To use this for solving the jump problem on \widehat{C}_s , we need to compare $\widehat{\xi}^{v(e)}|_{\gamma_e}$ to $I_e^*(\widehat{\xi}^{v(-e)})|_{\gamma_e}$. Using $I_e^*(h_{-e}) = -h_e$ and recalling that I_e^* is orientation-reversing, we compute

$$\begin{aligned} & \frac{dz_e}{2\pi i} \cdot \oint_{w_e \in \gamma_e} \frac{h_e(w_e)}{z_e - w_e} - \frac{d(sz_e^{-1})}{2\pi i} \cdot \oint_{w_e \in \gamma_e} \frac{h_e(w_e)}{sz_e^{-1} - sw_e^{-1}} \\ &= \frac{dz_e}{2\pi i z_e} \int_{w_e \in \gamma_e} h_e(w_e) = 0 \end{aligned} \tag{5.15}$$

for the jump of the singular integral in (5.12). Using again that $h_e = -I_e^*(h_{-e})$, we can write the jump of $\widehat{\xi}$ at a point $z_e \in \gamma_e$ as

$$(\widehat{\xi}^{v(e)} - I_e^*(\widehat{\xi}^{v(-e)}))(z_e) = (h_e + (\mathcal{K}^{re}\underline{h})_e - I_e^*((\mathcal{K}^{re}\underline{h})_{-e}))(z_e). \tag{5.16}$$

We define the matrix-valued (with indices $e \in E$) operator $\underline{\mathcal{K}}^{re}$ by

$$(\underline{\mathcal{K}}^{re}\underline{h})_e := (\mathcal{K}^{re}\underline{h})_e - I_e^*((\mathcal{K}^{re}\underline{h})_{-e}) \tag{5.17}$$

and then reinterpret equation (5.16) as saying that the jump of $\widehat{\xi}$ across γ_e is equal to $[(\underline{I} + \underline{\mathcal{K}}^{re})\underline{h}]_e$, where \underline{I} is the identity matrix.

5) *The integral transform with respect to K^{im} .* We now perform an analogous construction starting from K^{im} , to deal with the imaginary part of the jumps while preserving the reality of the periods. For any set \underline{g} of real-valued smooth 1-forms g_e on the seams γ_e such that

$$g_e = -I_e^*(g_{-e}) \quad \text{and} \quad \int_{\gamma_e} g_e = 0,$$

for any e , we define the Cauchy integral

$$\chi^v(p) := \sum_{e \in E_v} \int_{q \in \gamma_e} K_v^{im}(p, q) g_e(q). \tag{5.18}$$

Recall that, unlike K_v^{re} , the kernel K_v^{im} is a multi-valued function of q . However, we claim that χ^v is well-defined. Indeed, the difference of any two values of K_v^{im} is some holomorphic differential $\omega(p, q)$, and from the definition of K_v^{im} it follows that for any q

$$\text{Im} \int_{p \in \gamma} \omega(p, q) \in \mathbb{Z} \quad \text{for any cycle } \gamma \subset C^v.$$

Thus for γ fixed and q varying, $\text{Im} \int_{p \in \gamma} \omega(p, q) \in \mathbb{Z}$ is locally constant in q , and so

$$\text{Im} \int_{p \in \gamma} \partial_q \omega(p, q) = 0 \quad \text{for any } \gamma.$$

But then $\partial_q \omega(p, q)$, considered as a function of p , is a holomorphic real-normalized differential—which is thus identically zero, so that $\omega(p, q)$ is independent of q . It then follows that

$$\int_{q \in \gamma_e} \omega(p, q) g_e(q) = \omega(p) \int_{q \in \gamma_e} g_e(q) = 0.$$

Hence, finally, the multivaluedness of K_v^{im} cancels in the definition of $\chi^v(p)$.

We again think of each χ^v as a collection of holomorphic 1-forms χ_e on each disk U_e for $e \in E_v$, and a holomorphic 1-form $\widehat{\chi}^v$ on \widehat{C}_s^v . The Sokhotski–Plemelj formula in this case yields for $z_e \in \gamma_e$

$$\widehat{\chi}^v(z_e) = (\mathcal{K}^{\text{im}} \underline{g})_e(z_e) + \frac{i}{2} g_e(z_e) + dz_e \int_{w_e \in \gamma_e} \frac{ig_e(w_e)}{2\pi(z_e - w_e)} \tag{5.19}$$

and

$$\chi_e(z_e) = (\mathcal{K}^{\text{im}} \underline{g})_e(z_e) - \frac{i}{2} g_e(z_e) + dz_e \int_{w_e \in \gamma_e} \frac{ig_e(w_e)}{2\pi(z_e - w_e)}. \tag{5.20}$$

Similarly to the case of \mathcal{K}^{re} , this implies $\int_{\gamma_e} \chi^v = 0$ for any $e \in E_v$, analogously to (5.14). Moreover, although the periods of K_v^{im} are not real, the period of $\widehat{\chi}^v$ over any cycle $\gamma \subset \widehat{C}^v$ is real by the same argument as above: the imaginary part of any period of K_v^{im} is an integer. This integer is then independent of q , and thus when it is multiplied by $g_e(q)$, the integral over γ_e vanishes since $\int_{q \in \gamma_e} g_e(q) = 0$.

Similarly to (5.16), we compute the jump of $\widehat{\chi}$ at $z_e \in \gamma_e$ to be

$$(\widehat{\chi}^{v(e)} - I_e^*(\widehat{\chi}^{v(-e)}))(z_e) = (ig_e + (\mathcal{K}^{\text{im}} \underline{g})_e - I_e^*(\mathcal{K}^{\text{im}} \underline{g})_{-e})(z_e), \tag{5.21}$$

and we interpret the right-hand side as the operator $i\underline{I} + \mathcal{K}^{\text{im}}$ applied to \underline{g} .

6) *The ARN solution as a recursively defined series.* We now combine the pieces above to construct the ARN solution of the original jump problem on $C_{u,s}$, with initial data $\underline{\phi}$. The jumps of $\widehat{\xi}$ and $\widehat{\chi}$ are given by (5.16) and (5.21). Therefore, $\omega := \widehat{\xi} + \widehat{\chi}$ is the ARN solution of the jump problem on $C_{u,s}$ with initial data $\underline{\phi}$ if the jumps are correct, that is, if \underline{h} and \underline{g} satisfy the linear integral equation

$$\underline{\phi} = (\underline{I} + \mathcal{K}^{\text{re}}) \underline{h} + (i\underline{I} + \mathcal{K}^{\text{im}}) \underline{g}. \tag{5.22}$$

To keep track of the reality of \underline{h} and \underline{g} , we write out the real and imaginary parts of this equation separately, so it becomes

$$\begin{pmatrix} \text{Re } \underline{\phi} \\ \text{Im } \underline{\phi} \end{pmatrix} = (\underline{I} + \underline{\mathcal{K}}) \begin{pmatrix} \underline{h} \\ \underline{g} \end{pmatrix}, \tag{5.23}$$

where

$$\underline{\mathcal{K}} := \begin{pmatrix} \text{Re } \mathcal{K}^{\text{re}} & \text{Re } \mathcal{K}^{\text{im}} \\ \text{Im } \mathcal{K}^{\text{re}} & \text{Im } \mathcal{K}^{\text{im}} \end{pmatrix} \tag{5.24}$$

is now a real matrix-valued integral operator. Thus, finally, our goal is to show that for any given $\underline{\phi}$ the linear integral equation (5.22) has a solution $\underline{h}, \underline{g}$. If the norm of the operator $\underline{\mathcal{K}}$ is sufficiently small, then the inverse of the operator $\underline{I} + \underline{\mathcal{K}}$ is given by the sum of the convergent series $\underline{I} + \sum_{l=1}^{\infty} (-\underline{\mathcal{K}})^l$, so that (5.23) is solved by the convergent series

$$\underline{h} := \sum_{l=0}^{\infty} (-1)^l \underline{h}^{(l)} \quad \text{and} \quad \underline{g} := \sum_{l=0}^{\infty} (-1)^l \underline{g}^{(l)} \tag{5.25}$$

with the leading terms $\underline{h}^0 := \operatorname{Re} \underline{\phi}$ and $\underline{g}^0 := \operatorname{Im} \underline{\phi}$, and the higher order terms defined recurrently by

$$\underline{h}^{(l)} + i \underline{g}^{(l)} := \underline{\mathcal{X}}^{\operatorname{re}} \underline{h}^{(l-1)} + \underline{\mathcal{X}}^{\operatorname{im}} \underline{g}^{(l-1)}. \tag{5.26}$$

7) *Bounds for the terms of the series.* To complete the construction of the ARN solution, it thus remains to show that the series (5.25) with terms defined by (5.26) are indeed convergent. For this, we estimate the norm of the integral operator $\underline{\mathcal{X}}$. The explicit recursive bounds that we obtain for the terms of the series (5.26) will be crucial in our further analysis of the behaviour the ARN solution as $\underline{s} \rightarrow 0$, yielding eventually Proposition 5.3, going beyond proving the convergence of the series (5.26).

We define the L^∞ -norm of a collection of 1-forms $\phi_e = \tilde{\phi}_e dz_e$ on the seams by

$$|\underline{\phi}|_{\underline{s}} := \max_e \sup_{z_e \in \gamma_e^{s_e}} \left| \frac{2\pi \phi_e}{d \log z_e} \right| = \max_e \left(2\pi \sqrt{|s_e|} \sup_{z_e \in \gamma_e^{s_e}} |\tilde{\phi}_e| \right) \tag{5.27}$$

and our goal is to bound $|\underline{h}^{(l)} + i \underline{g}^{(l)}|_{\underline{s}}$ for the terms recursively defined by (5.26).

Since $\mathbf{K}^{\operatorname{re}}(p, q)$ is a real-analytic function of q , there exists a constant M_1 such that

$$|\mathbf{K}_v^{\operatorname{re}}(z_e, w_{e'}) - \mathbf{K}_v^{\operatorname{re}}(z_e, 0)| < M_1 |w_{e'}| \tag{5.28}$$

holds for all v , all $e, e' \in E_v$, and any $z_e \in V_e$ and $w_{e'} \in V_{e'}$. Since $\mathbf{K}^{\operatorname{re}}(p, q)$ depends real-analytically on the moduli coordinate u , the constant M_1 can be chosen to be the same for all u in some neighbourhood of $u = 0$. As we emphasized above and proved in equation (5.14), $\int_{p \in \gamma_e} (\underline{\mathcal{X}}^{\operatorname{re}} \underline{h})_e(p) = 0$ for any \underline{h} such that $\int_{p \in \gamma_e} h_e(p) = 0$ for any e . A similar analysis applies to $\underline{\mathcal{X}}^{\operatorname{im}} \underline{g}$, and it thus follows from the recurrent definition (5.26) that

$$\int_{p \in \gamma_e} h_e^{(l)}(p) = \int_{p \in \gamma_e} g_e^{(l)}(p) = 0 \quad \text{for all } e \text{ and } l.$$

On multiplying (5.28) by $h_{e'}^{(l)}$ and integrating over $\gamma_{e'}$ the contribution from $\mathbf{K}(z_e, 0)$ vanishes, so that for any $z_e \in V_e$ we obtain

$$\begin{aligned} \left| \int_{w_{e'} \in \gamma_{e'}} \mathbf{K}^{\operatorname{re}}(z_e, w_{e'}) h_{e'}^{(l)}(w_{e'}) \right| &< M_1 \sqrt{|s_e|} \sup_{w_{e'} \in \gamma_{e'}} \left| \frac{2\pi i h_{e'}^{(l)}}{d \log w_{e'}} \right| \\ &\leq M_1 \sqrt{|\underline{s}|} |\underline{h}^{(l)}|_{\underline{s}}. \end{aligned} \tag{5.29}$$

By recalling the definition of $\underline{\mathcal{X}}^{\operatorname{re}}$, the bound above implies the same bound for it; the bound for $\underline{\mathcal{X}}^{\operatorname{im}}$ is obtained analogously. Adding these inequalities and applying them recursively with respect to l , it follows that there exists a constant M_2 such that

$$|\underline{h}^{(l)} + i \underline{g}^{(l)}|_{\underline{s}} < (M_2 \sqrt{|\underline{s}|})^l |\underline{\phi}|_{\underline{s}} \tag{5.30}$$

for any l . Thus for $\sqrt{|\underline{s}|} < (2M_2)^{-1}$ the terms of the series \underline{h} and \underline{g} are bounded by the geometric sequence with ratio less than $1/2$. Since the sum of such a geometric

series is less than 2, for further use we record that we have proven the crucial inequality

$$|\underline{h} + i\underline{g}|_{\underline{s}} < 2|\underline{\phi}|_{\underline{s}} \tag{5.31}$$

for any $|\underline{s}| < (2M_2)^{-2}$. To finish the proof of convergence we note that in particular the sum of the left-hand sides of (5.30) for all l converges, and thus the convergent series (5.25) give a solution of the linear integral equation (5.23). Therefore, the sum of the corresponding Cauchy integrals $\widehat{\xi} + \widehat{\chi}$ gives the ARN solution of the original jump problem with initial data $\underline{\phi}$, finally proving Proposition 5.2. \square

Our main interest is the behaviour of the ARN solution as $\underline{s} \rightarrow 0$, and the setup is as follows. Let

$$\underline{f} = \{f_e\} = \{\tilde{f}_e(z_e) dz_e\}$$

be a collection of holomorphic 1-forms defined on the unit disks V_e . Then for any $|\underline{s}| < 1$ we consider the jump problem with initial data

$$\{\phi_e\} := \{(f_e - I_e^* f_{-e})|_{\gamma_e^{s_e}}\}.$$

Note that by the residue theorem, $\int_{\gamma_e} f_e = 0$ for any e , so this ϕ_e can be used as initial data for the jump problem. The fact that ϕ_e is the restriction to the seam of a holomorphic 1-form will allow us to explicitly compute the Cauchy principal value appearing in the Sokhotski–Plemelj formula, while we will crucially use the fact that our Cauchy kernels K_v^{re} and K_v^{im} are taken on C^v , and are independent of \underline{s} .

We define the L^∞ -norm of \underline{f} by

$$|\underline{f}| := 2\pi \max_e \sup_{|z_e|=1} \left| \frac{f_e}{d \log z_e} \right| = 2\pi \max_e \sup_{|z_e|=1} |\tilde{f}_e|. \tag{5.32}$$

The Schwarz lemma on the disk $U_e = \{|z_e| < \sqrt{|s_e}|\}$ implies

$$|\underline{f}|_{\underline{s}} := |\underline{f}|_{\gamma_e^{s_e}} \leq |\underline{f}| (\sqrt{|s|})^{\text{ord } \underline{f}}, \tag{5.33}$$

where we have written $\text{ord } \underline{f} := \min_e (\text{ord}_{q_e} f_e)$. We further write, for any irreducible component C^v ,

$$|\underline{s}_v| := \max_{e \in E_v} |s_e|.$$

We finally denote the usual L^2 -norm of a differential on $\widehat{C}_{\underline{s}}^v$ by

$$\|\omega\|_{\widehat{C}_{\underline{s}}^v}^2 := \frac{i}{2} \int_{\widehat{C}_{\underline{s}}^v} \omega^v \wedge \bar{\omega}^v.$$

Our main bound is then the following.

Proposition 5.3 (a bound for the ARN solution). *For a fixed \underline{f} and any sufficiently small $|\underline{s}|$, let $\omega_{\underline{s}}$ be the ARN solution of the jump problem with initial data*

$$\{\phi_e\} := \{(f_e - I_e^* f_{-e})|_{\gamma_e^{s_e}}\}.$$

Then there exists a constant M independent of sufficiently small u and \underline{s} such that for any irreducible component C^v the following inequality holds:

$$\|\omega_{\underline{s}}\|_{\hat{C}_{\underline{s}}^v} \leq M|\underline{f}|(\sqrt{|\underline{s}_v|})^{\text{ord } \underline{f}+1}. \tag{5.34}$$

Since $C_{\underline{1}}^v := C^v \setminus \bigcup_{e \in E_v} V_e$ is a compact set on which L^2 - and L^1 -norms can be bounded in terms of each other, we have the following.

Corollary 5.4. *For any fixed path $\gamma \subset C_{\underline{1}}^v$ there exists a constant M_γ such that the following inequality holds for all sufficiently small u :*

$$\left| \int_\gamma \omega_{\underline{s}} \right| \leq M_\gamma |\underline{f}| (\sqrt{|\underline{s}_v|})^{\text{ord } \underline{f}+1}.$$

To prove Proposition 5.3 we first obtain a pointwise bound.

Lemma 5.5. *In the setup as above, there exists a constant M_3 independent of u and \underline{s} such that for any $z_e \in \gamma_e^{\text{se}}$ and with the ARN solution written as $\omega_{\underline{s}}^v = \tilde{\omega}^v(z_e) dz_e$, the following inequality holds:*

$$|\tilde{\omega}^v(z_e)| < M_3 |\underline{f}| (\sqrt{|\underline{s}_v|})^{\text{ord } \underline{f}}. \tag{5.35}$$

Proof. To obtain this bound, we return to the details of the construction of the ARN solution; using the fact that the initial data is the restriction of a holomorphic 1-form will eventually allow us to evaluate the singular integral by the residue theorem. Writing the ARN solution for any $z_e \in V_e$ as the sum of (5.11) and the similar formula for χ gives

$$\omega^v(z_e) = (\mathcal{K}^{\text{re}} \underline{h})_e(z_e) + (\mathcal{K}^{\text{im}} \underline{g})_e(z_e) + \frac{dz_e}{2\pi i} \int_{w_e \in \gamma_e} \frac{h_e(w_e) + ig_e(w_e)}{z_e - w_e}.$$

The first two summands are holomorphic functions of $z_e \in V_e$ (see the discussion after formula (5.10)). For the singular integral, recall that \underline{h} and \underline{g} are given by the sums of the series (5.25), with the terms defined recursively by

$$\begin{aligned} (\underline{h}^{(l)} + i\underline{g}^{(l)})_e &= (\mathcal{K}^{\text{re}} \underline{h}^{(l-1)} + \mathcal{K}^{\text{im}} \underline{g}^{(l-1)})_e \\ &= (\mathcal{K}^{\text{re}} \underline{h}^{(l-1)} + \mathcal{K}^{\text{im}} \underline{g}^{(l-1)})_e - I_e^* \left((\mathcal{K}^{\text{re}} \underline{h}^{(l-1)} + \mathcal{K}^{\text{im}} \underline{g}^{(l-1)})_{-e} \right), \end{aligned}$$

where we have recalled the definition of the operator \mathcal{K}^{re} in formula (5.17) (and \mathcal{K}^{im} is similar). In this recursive definition we clearly see a holomorphic form on V_e and a pullback of a holomorphic form on V_{-e} under I_e^* . To make use of this, we define the series

$$\varphi_e := \sum_{l=0}^{\infty} (-1)^l \varphi_e^{(l)}$$

with the first term $\varphi_e^{(0)} = f_e$ and the further terms simply being the holomorphic forms appearing in the recursion above:

$$\begin{aligned} \varphi_e^{(l)} &:= \mathcal{K}^{\text{re}} \underline{h}^{(l-1)} + \mathcal{K}^{\text{im}} \underline{g}^{(l-1)} \\ &= dz_e \sum_{e' \in E_v} \int_{w_{e'} \in \gamma_{e'}} [\mathbf{K}_v^{\text{re}}(z_e, w_{e'}) h_{e'}^{(l-1)}(w_{e'}) + \mathbf{K}_v^{\text{im}}(z_e, w_{e'}) g_{e'}^{(l-1)}(w_{e'})]. \end{aligned} \tag{5.36}$$

Now $\varphi_e^{(0)}$ is a holomorphic form on V_e , and so is every $\varphi_e^{(l)}$; moreover, equation (5.30) gives a bound on $|\varphi_e^{(l)}|$ for $l \geq 1$, which shows that for $|\underline{s}|$ sufficiently small the series defining φ_e converge uniformly – and thus their sum φ_e is also a holomorphic 1-form on V_e . Moreover, the bound (5.30) combined with the Schwarz inequality (5.33) implies that

$$|\underline{\varphi}|_{\underline{s}} < 2|\underline{f}|_{\underline{s}} \leq 2|\underline{f}|(\sqrt{|\underline{s}_v|})^{\text{ord } \underline{f}}. \tag{5.37}$$

Finally, we rewrite the ARN solution as

$$\begin{aligned} \omega^v(z_e) &= (\mathcal{K}^{\text{re}} \underline{h})_e(z_e) + (\mathcal{K}^{\text{im}} \underline{g})_e(z_e) \\ &+ \frac{dz_e}{2\pi i} \int_{|w_e|=\sqrt{|s_e|}} \frac{\varphi_e(w_e) - I_e^*(\varphi_{-e})(w_e)}{z_e - w_e}, \end{aligned} \tag{5.38}$$

and now estimate each term in this formula. Since φ_e is holomorphic, by the residue theorem we compute the singular integral to be

$$\frac{dz_e}{2\pi i} \int_{|w_e|=\sqrt{|s_e|}} \frac{\varphi_e(w_e) - I_e^*(\varphi_{-e})(w_e)}{z_e - w_e} = -I_e^*(\varphi_{-e})(z_e) \tag{5.39}$$

for any $|z_e| > \sqrt{|s_e|}$. Indeed, since each φ_e is holomorphic for $w_e \in V_e$, the first term in (5.38) has no residues in the disk $|w_e| < \sqrt{|s_e|}$. The second term is equal to (recall that I_e is orientation-reversing)

$$\begin{aligned} \frac{dz_e}{2\pi i} \int_{|w_e|=\sqrt{|s_e|}} \frac{\varphi_{-e}(w_{-e})}{z_e - s_e w_{-e}^{-1}} &= -\frac{dz_e}{2\pi i z_e} \int_{|w_{-e}|=\sqrt{|s_e|}} \frac{w_{-e} \varphi_{-e}(w_{-e})}{s_e z_e^{-1} - w_{-e}} \\ &= \frac{s_e dz_e}{z_e^2} \tilde{\varphi}_{-e}(s_e z_e^{-1}) = -I_e^*(\varphi_{-e}). \end{aligned} \tag{5.40}$$

Equation (5.40) extends continuously to $|z_e| = \sqrt{|s_e|}$. Thus equation (5.37) implies that the singular integral in (5.38) is bounded by $2|\underline{f}|(\sqrt{|\underline{s}_v|})^{\text{ord } \underline{f}}$, which is the order of the bound that we want.

On the other hand, we have the bound (5.29) for the norm of \mathcal{K}^{re} , and an analogous bound for the norm of \mathcal{K}^{im} . Then combining the bound (5.31) with the Schwarz inequality gives

$$|\mathcal{K}^{\text{re}} \underline{h} + \mathcal{K}^{\text{im}} \underline{g}|_{\underline{s}} \leq M'(\sqrt{|\underline{s}_v|})^{\text{ord } \underline{f}+1},$$

which is of smaller order than the bound for the singular integral. We have thus obtained bounds for both summands, giving ω^v in formula (5.38), and the lemma is proven. \square

Now we are ready to prove the bound for the L^2 -norm of ω .

Proof of Proposition 5.3. As in the proof of uniqueness of the ARN differential, we use the Stokes’ theorem expression (5.1) for the L^2 -norm of ω^v . For a given e , we thus need to compute

$$\int_{z_e \in \gamma_e} F^{v(e)}(z_e) dF^{v(e)*}(z_e) = \int_{\gamma_e} F^{v(e)}(z_e) \cdot (\omega^v(z_e) + \bar{\omega}^v(z_e)). \tag{5.41}$$

Recall that since $\int_{z_e \in \gamma_e} \omega^{v(e)}(z_e) = 0$, it follows that this integral does not depend on the choice of the constant of integration for the definition of $F^{v(e)}$, so we can pick any point $z_e^0 \in \gamma_e$ and replace $F^{v(e)}(z_e)$ on the right-hand side of (5.41) by $F^{v(e)}(z_e) - F^{v(e)}(z_e^0)$. From Lemma 5.5 we have the pointwise bound

$$|\tilde{\omega}^v| \leq M_3 |f| (\sqrt{|\underline{s}|})^{\text{ord } \underline{f}}$$

for some constant M_3 independent of u and \underline{s} . As the length of the arc from z_e^0 to z_e is at most $2\pi\sqrt{|s_e|}$, it follows by integrating $\tilde{\omega}$ that

$$F^{v(e)}(z_e) - F^{v(e)}(z_e^0) \leq 2\pi M_3 |f| (\sqrt{|\underline{s}|})^{\text{ord } \underline{f} + 1}.$$

We thus obtain

$$\int_{z_e \in \gamma_e} F^{v(e)}(z_e) dF^{v(e)*}(z_e) \leq 2\pi M_3 |f| (\sqrt{|\underline{s}|})^{\text{ord } \underline{f} + 1} \int_{\gamma_e} |2\omega^v(z_e)|.$$

Lemma 5.5 again gives a pointwise bound for the integrand, while integrating over the seam introduces another factor of $2\pi\sqrt{|s_e|}$, so that by summing over all e we finally obtain

$$\|\omega^v\|^2 \leq 4\pi^2 M_3^2 |f| \left(\sqrt{\max_{e \in E_v} |s_e|} \right)^{2 \text{ord } \underline{f} + 2} \cdot \#E_v. \tag{5.42}$$

Thus, finally, there exists a constant M such that inequality (5.34) holds. \square

6. The RN differential in plumbing coordinates

In this section we construct explicitly the RN differential on any smooth jet curve $X_{w,u,\underline{s}}$ in plumbing coordinates. The construction starts with a collection of RN differentials on C^v with prescribed singular parts at p_ℓ and with residues at the nodes given by a solution of the *flow* Kirchhoff problem. Note that this differential is strictly speaking not real-normalized: already for the case of an irreducible nodal curve and a differential with no residues, when there are no residues in the limit either, the period over a loop passing through a node is equal to the integral of the RN integral on the normalization from one pre-image of the node to the other, which may not be real.

We then use the ARN solution of the jump problem with the initial data matching the jumps of this collection of differentials on C^v to construct a differential on $C_{u,\underline{s}}$. Since the ARN solutions have zero periods, the resulting differential will not be real-normalized. We thus proceed recurrently, using RN differentials with residues solving the *force* Kirchhoff problem with the electromotive force being equal to the imaginary part of the periods of the differential constructed at the previous step. The bound of the ARN solution of the jump problem from Proposition 5.3 is used crucially to show that this construction converges.

We continue to use the notation for plumbed surfaces and the jump problem as in the previous section.

Notation 6.1. For a fixed smooth jet curve $X_{w,u,s}$, given any collection of real numbers $\underline{c} = \{c_e\}_{e \in E(\Gamma)}$ satisfying conditions (0) and (1) of the Kirchhoff problem, we denote by

$$\Phi(\underline{c}) = \{\Phi^v(\underline{c})\}_{v \in V(\Gamma)}$$

the collection of RN differentials on C_u^v with prescribed singular parts σ_ℓ and prescribed residues r_ℓ (encoded by the coordinate w) at the marked points p_ℓ , with residue ic_e at the pre-image q_e of every node, and holomorphic elsewhere.

Lemma 6.2. For any \underline{c} satisfying conditions (0) and (1) in the Kirchhoff problem, let

$$f_e(\underline{c}) := \Phi^{v(e)}(\underline{c})|_{V_e} - \frac{ic_e}{z_e} dz_e$$

be the collection of holomorphic differentials on V_e . Let $\omega(\underline{c}) = \{\omega^v(\underline{c})\}$ be the ARN solution on $\widehat{C}_{u,s}$ to the jump problem with initial data $(f_e(\underline{c}) - I_e^* f_{-e}(\underline{c}))|_{\gamma_e^{s_e}}$. Then the collection of differentials

$$\Psi^v(\underline{c}) := \Phi^v(\underline{c}) - \omega^v(\underline{c}) \tag{6.1}$$

is the unique meromorphic differential on $C_{u,s}$ with singularities σ_ℓ and residues r_ℓ at p_ℓ prescribed by the coordinate w , and holomorphic elsewhere, such that

$$\int_{\gamma_e} \Psi(\underline{c}) = 2\pi c_e \text{ for any } e \text{ and } \int_{\gamma} \Psi(\underline{c}) \in \mathbb{R} \text{ for any cycle } \gamma \in H_1(\widehat{C}_{u,s}, \mathbb{Z}).$$

Again, recall that the last condition is equivalent to

$$\int_{\gamma} \Psi(\underline{c}) \in \mathbb{R}$$

for any $\gamma \in H_1(C_{u,s}, \mathbb{Z})$ not intersecting the seams. As in the previous section, we will drop w and u in the notation from now on.

Proof. Since

$$I_e^* \left(\frac{ic_{-e}}{z_{-e}} dz_{-e} \right) = -\frac{ic_{-e}}{z_e} dz_e = \frac{ic_e}{z_e} dz_e,$$

the jumps of $\omega(\underline{c})$ on the seams are equal to those of $\Phi(\underline{c})$, and thus by construction the differential $\Psi(\underline{c})$ on \widehat{C}_s has no jumps. Thus $\Psi(\underline{c})$ defines a meromorphic differential on C_s with prescribed singularities at p_ℓ , and holomorphic elsewhere (the simple pole of $\Phi^{v(e)}(\underline{c})$ at q_e is cut out by plumbing). Since $\int_{\gamma_e} \omega^{v(e)}(\underline{c}) = 0$ for any e , by definition of the ARN solution the residue theorem on U_e yields

$$\int_{\gamma_e} \Psi(\underline{c}) = -2\pi i \text{Res}_{q_e} \Phi^v(\underline{c}) = 2\pi c_e$$

(for the sign, recall that γ_e is oriented as the boundary of $\widehat{C}^{v(e)}$). Since $\Phi^v(\underline{c})$ is real-normalized on C^v , while the period of $\omega^v(\underline{c})$ over any cycle on \widehat{C}_s^v is real, it follows that the integral of $\Psi(\underline{c})$ over any cycle on \widehat{C}_s is also real.

The uniqueness of $\Psi(\underline{c})$ follows from the uniqueness of the ARN solution of the jump problem. Indeed, if Ψ' is another such differential, then $\Psi(\underline{c}) - \Psi'$ is

a holomorphic differential on $C_{\underline{s}}$ with zero periods over the seams, and with real periods over all cycles not intersecting the seams. Hence it is the ARN solution of the jump problem with zero initial data, which must then be identically zero. \square

Notation 6.3. The construction above can be applied to the case when there are no prescribed singularities. In this case suppose that $\underline{c}' = \{c'_e\}$ satisfy conditions (0) and (1) of the Kirchhoff problem with no inflow, that is, when all f_ℓ are equal to zero. We denote by $\Omega(\underline{c}')$ the differential constructed on $C_{\underline{s}}$ by the above procedure.

Then $\Omega(\underline{c}')$ is the unique *holomorphic* differential on $C_{\underline{s}}$ such that $\int_{\gamma_e} \Omega(\underline{c}') = 2\pi c'_e$ for any e and $\int_{\gamma} \Omega(\underline{c}') \in \mathbb{R}$ for any $\gamma \in H_1(\widehat{C}_{\underline{s}}, \mathbb{Z})$.

From uniqueness it follows that

$$\Psi(\underline{c} + \underline{c}') = \Psi(\underline{c}) + \Omega(\underline{c}') \tag{6.2}$$

for any such \underline{c} and \underline{c}' , since the right-hand side is a differential on $C_{\underline{s}}$ satisfying the same conditions as the left-hand side. This relation will be used in the next section.

Consider the RN differential $\Psi_X = \Psi_{w,u,\underline{s}}$ on $X_{w,u,\underline{s}}$: it has prescribed singularities at p_ℓ , is holomorphic elsewhere, and its periods over all cycles are real. Therefore, by the uniqueness of the latter, $\Psi_{w,u,\underline{s}}$ is equal to $\Psi(\underline{c})$, for $c_e := \frac{1}{2\pi} \int_{\gamma_e} \Psi_{w,u,\underline{s}}$.

Thus constructing the RN differential Ψ_X is equivalent to determining \underline{c} (depending on w , u , and \underline{s}) such that $\Psi_X = \Psi(\underline{c})$. We will construct \underline{c} recursively, starting with the solution of the *flow* Kirchhoff problem, and then recursively solving the *force* Kirchhoff problem, with the force being the imaginary parts of the periods of the previous term in the series. The difficulty is that the periods over the paths passing through the nodes may diverge. We first show that they have well-controlled logarithmic divergences.

Notation 6.4. For any closed path γ on C , let $\widehat{\gamma}_{\underline{s}}$ be the collection of paths that are the intersections $\gamma \cap \widehat{C}_{\underline{s}}^v$ (conveniently, in our plumbing setup $\widehat{C}_{\underline{s}}^v$ is a subset of C^v for any \underline{s}). These $\widehat{\gamma}_{\underline{s}}$ do not form a closed path on $C_{\underline{s}}$, as the point $\gamma \cap \gamma_e$ may not be the pre-image of $\gamma \cap \gamma_{-e}$ under the identification I_e . We choose a starting point on γ arbitrarily, and then denote by $\gamma_{e_1}, \dots, \gamma_{e_N}$ (possibly with repetitions) the set of seams that γ intersects (oriented so that γ crosses from γ_{e_j} to γ_{-e_j}), and denote by C^{v_j} the component of C that γ lies on after crossing γ_{e_j} . Let then $\widehat{\gamma}_{\underline{s}}^j$ be the segment of γ contained in C^{v_j} and going from $\gamma_{e_{j-1}}$ to γ_{e_j} . Let δ_j be an arc of γ_{e_j} connecting the point $\gamma \cap \gamma_{e_j}$ to $I_{e_j}^{-1}(\gamma \cap \gamma_{-e_j})$. Finally, let $\gamma_{\underline{s}}$ be the closed path on $C_{\underline{s}}$ obtained by traversing $\{\widehat{\gamma}_{\underline{s}}^1, \delta_1, \dots, \widehat{\gamma}_{\underline{s}}^N, \delta_N\}$ in this order.

Lemma 6.5. *For any closed path γ on C and any \underline{c} satisfying conditions (0) and (1) in the Kirchhoff problem, there exist constants $\Pi_\gamma(\underline{c})$ (independent of \underline{s} , but depending on w and u) and $M_5(\underline{c})$ (independent of w , u , and \underline{s}) such that*

$$\left| \operatorname{Im} \int_{\gamma_{\underline{s}}} \Psi(\underline{c}) - \sum_{j=1}^N c_{e_j} \log |s_{e_j}| - \Pi_\gamma(\underline{c}) \right| \leq M_5(\underline{c}) \sqrt{|\underline{s}|} \tag{6.3}$$

for any sufficiently small w , u , and \underline{s} .

Furthermore, the constant $\Pi_\gamma(\underline{c})$ only depends on the class of γ in the cohomology of the dual graph Γ of C .

We note that for each j there are two choices of δ_{e_j} , by going in opposite directions around γ_{e_j} ; however, since $\text{Im} \int_{\gamma_e} \Psi(\underline{c}) = 0$ by definition, the imaginary part of the integral of $\Psi(\underline{c})$ is the same for either choice.

For the second statement, note that there is a map $H_1(C, \mathbb{Z}) \rightarrow H_1(\Gamma, \mathbb{Z})$. This map is obtained by ‘extending’ every node of C to a segment, and then contracting every irreducible component of C to a point. Geometrically, the image of γ in $H_1(\Gamma, \mathbb{Z})$ simply encodes the sequence in which γ passes through the nodes of C .

Proof of Lemma 6.5. We recall from Lemma 6.2 that

$$\Psi(\underline{c}) = \Phi(\underline{c}) - \omega(\underline{c}),$$

and we start by estimating the periods of $\Phi(\underline{c})$. We break each path $\gamma_{\underline{s}}^j$ into three parts: γ_1 contained in $V_{e_{j-1}}$ and connecting a boundary point of $U_{e_{j-1}}$ to the boundary of $V_{e_{j-1}}$, a similar path γ_2 contained in V_{e_j} , and the path γ_3 connecting a boundary point of $V_{e_{j-1}}$ to a boundary point of V_{e_j} . Since the neighbourhoods V_e are independent of \underline{s} , $\int_{\gamma_3} \Phi(\underline{c})$ is independent of \underline{s} . For γ_2 , we write

$$\Phi(\underline{c}) = \frac{ic_{e_j}}{z_{e_j}} dz_{e_j} + \Phi'(\underline{c})$$

in V_{e_j} , where $\Phi'(\underline{c})$ denotes the regular part. Integrating the singular part over γ_2 , which goes from some $z' \in \partial V_{e_j}$ (that is, $|z'| = 1$) to some $z \in \partial U_{e_j}$ (that is, $|z| = \sqrt{|s_{e_j}|}$) yields

$$\text{Im} \int_{\gamma_2} \frac{-ic_e}{z_{e_j}} dz_{e_j} = -c_e \log \left| \frac{z}{z'} \right| = -c_e \log \sqrt{|s_{e_j}|}.$$

To estimate $\int_{\gamma_2} \Phi'(\underline{c})$, we can add to γ_2 a path connecting z to 0. The resulting path does not depend on \underline{s} , and thus the integral of $\Phi'(\underline{c})$ over it is independent of \underline{s} . Since $\Phi'(\underline{c})$ is regular in V_{e_j} , its norm there is bounded, and thus the integral of $\Phi'(\underline{c})$ over a path in V_{e_j} connecting z to 0 is bounded by a constant times $|z| = \sqrt{|s_{e_j}|}$. We thus see that there exist constants α and β independent of \underline{s} (but depending on \underline{c} , and on w, u) such that

$$\left| \text{Im} \int_{\gamma_2} \Phi(\underline{c}) - \alpha - c_{e_j} \log \sqrt{|s_{e_j}|} \right| < \beta \sqrt{|s_{e_j}|}.$$

An analogous estimate holds for $\text{Im} \int_{\gamma_1} \Phi(\underline{c})$, where we note that the path is now oriented from the boundary of $U_{e_{j-1}}$ to the boundary of $V_{e_{j-1}}$, but also that the residue is equal to $ic_{-e_{j-1}} = -ic_{e_{j-1}}$.

Combining $\gamma_1, \gamma_2,$ and γ_3 together, we thus finally see that there exist a constant $\Pi_\gamma^j(\underline{c})$ independent of \underline{s} (and depending smoothly on w and u) and a constant $M_6(\underline{c})$ independent of $w, u,$ and \underline{s} such that

$$\left| \operatorname{Im} \int_{\widehat{\gamma}_{\underline{s}}^j} \Phi^{v_j}(\underline{c}) - \frac{1}{2} c_{e_{j-1}} \log |s_{e_{j-1}}| - \frac{1}{2} c_{e_j} \log |s_{e_j}| - \Pi_\gamma^j(\underline{c}) \right| < M_6(\underline{c}) \sqrt{|\underline{s}|} \quad (6.4)$$

for all w, u, \underline{s} sufficiently small.

We now estimate $\int_{\gamma_{\underline{s}}^j} \omega(\underline{c})$. By Corollary 5.4, $\left| \int_{\gamma_3} \omega(\underline{c}) \right|$ is bounded by a constant times $|f| \sqrt{|\underline{s}|}$, which is the size of the bound we want. Corollary 5.4 does not apply to $\int_{\gamma_1} \omega(\underline{c})$ and $\int_{\gamma_2} \omega(\underline{c})$, as these paths depend on \underline{s} . However, γ_2 is contained in V_{e_j} , and formula (5.38) with the singular integral computed in (5.39) expresses $\omega|_{V_{e_j}}$ as the sum of a holomorphic form on V_{e_j} and $-f_{-e}(s_e z_e^{-1})$. Similarly to the case of $\Phi'(\underline{c})$, the integral of the holomorphic form has the form required, while $-\int_{z'}^z f_{-e}(s_e z_e^{-1}) = \int_{s_e/z}^{s_e/z'}$ $f_{-e}(z_{-e})$ is also the integral of a holomorphic form on V_{-e} from a point s_e/z of absolute value $\sqrt{|s_e|}$ to the point s_e/z' of absolute value $|s_e|$. Thus the same argument applies, and altogether $\int_{\gamma_2} \omega(\underline{c})$ differs from some constant α' by less than $\beta' \sqrt{|s_{e_j}|}$, for some other constant β' . The integral $\int_{\gamma_1} \omega(\underline{c})$ is completely analogous, so in the end we obtain the bound

$$\left| \operatorname{Im} \int_{\widehat{\gamma}_{\underline{s}}^j} \omega^{v_j}(\underline{c}) - E^j(\underline{c}) \right| < M_7(\underline{c}) \sqrt{|\underline{s}|} \quad (6.5)$$

with some constant E^j independent of \underline{s} (but depending on w and u) and M_7 independent of $w, u,$ and \underline{s} .

To estimate the integral of $\Psi(\underline{c})$ over the segments δ_j , we first note that

$$\operatorname{Im} \int_{\delta_j} i c_{e_j} d \log z_{e_j} = 0$$

for the singular part of $\Phi(\underline{c})$. Then $\int_{\delta_j} \Phi'(\underline{c})$ is bounded by a constant independent of \underline{s} times the L^2 -norm of $\Phi'(\underline{c})$ on V_{e_j} (which is also a constant independent of \underline{s}) times the length of δ_j , which is at most $2\pi \sqrt{|s_{e_j}|}$. Similarly, $\operatorname{Im} \int_{\delta_j} \omega(\underline{c})$ is bounded by a constant independent of \underline{s} times $\sqrt{|s_{e_j}|}$, since $\omega(\underline{c})$ restricted to the seam is the sum of a form holomorphic on V_e and a pullback of the form f_{-e} holomorphic on V_{-e} , restricted to the seam.

Thus, altogether the lemma follows by subtracting equation (6.5) from equation (6.4) and summing over all j (notice that each e_j appears twice in the sum and thus the two fractions $1/2$ in front of $c_{e_j} \log |s_{e_j}|$ appearing in (6.4) for $\widehat{\gamma}_j$ and $\widehat{\gamma}_{j+1}$ add up to the coefficient 1 in (6.3)).

To prove that $\Pi_\gamma(\underline{c})$ only depends on the class of the image of γ in $H_1(\Gamma, \mathbb{Z})$, we note that on each C^v the differential $\Psi(\underline{c})$ is real-normalized, so that

$$\text{Im} \int_{\gamma^v} \Psi(\underline{c}) = 0$$

for any closed loop $\gamma^v \in H_1(C^v, \mathbb{Z})$. Therefore, $\Pi_\gamma(\underline{c}) = \Pi_{\gamma'}(\underline{c})$ for any paths γ and γ' that only differ on C^v , which is exactly to say that Π only depends on $\gamma \in H_1(\Gamma, \mathbb{Z})$. \square

Remark 6.6. The above proof uses details of the estimates of the ARN solution. Obtaining similar results using the classical approach to the jump problem, with the Cauchy kernel $K_{C_{\underline{s}}}$ on a varying Riemann surface, appears to be much harder.

We now construct the RN differential $\Psi_{w,u,\underline{s}}$ on any smooth jet curve explicitly, as $\Psi(\underline{c})$, with \underline{c} defined as the sum of a series. We let $\underline{c}^{(0)}(\underline{s}) := \{c_e^{(0)}(\underline{s})\}$ be the solution of the flow Kirchhoff problem on the dual graph of $C_{u,0}$ with inflows ir_ℓ , and resistances $\rho_{|e|}(\underline{s}) = -\log |s_e|$ being the log-plumbing coordinates. While our construction depends on w, u , and \underline{s} —and here we write this out explicitly—note that the Kirchhoff problem and its solution are independent of u . We only work with smooth curves here, that is, all the s_e are non-zero.

A priori, by Lemma 6.5 the imaginary parts of the integrals of $\Psi(\underline{c}^{(0)})$ may have logarithmic divergences. However, the sum of these logarithmic divergences on a closed path

$$\sum c_e^{(0)} \log |s_e| = -\sum c_e^{(0)} \rho_{|e|}$$

is precisely the left-hand side of equation (1.2) (condition (2)) of the Kirchhoff problem. Since $\underline{c}^{(0)}$ is the solution of the flow Kirchhoff problem, this sum is equal to zero, as there is no electromotive force. Hence the corresponding logarithmic divergences cancel, so that we have

$$\left| \text{Im} \int_{\gamma_{\underline{s}}} \Psi(\underline{c}^{(0)}) - \Pi_\gamma(\underline{c}^{(0)}) \right| < M(\underline{c}^{(0)}) \sqrt{|\underline{s}|}, \tag{6.6}$$

with the constant M independent of w, u , and \underline{s} and Π_γ independent of \underline{s} . To deal with those periods of $\Psi(\underline{c}^{(0)})$ that have a non-zero imaginary part, we introduce the correction $\underline{c}^{(1)}(\underline{s})$ to be the solution of the *force* Kirchhoff problem on the dual graph of $C_{u,0}$, with electromotive force

$$\mathcal{E}_{\gamma_{\underline{s}}}^{(0)} := -\text{Im} \int_{\gamma_{\underline{s}}} \Psi(\underline{c}^{(0)}). \tag{6.7}$$

Recall that by (6.2)

$$\Psi(\underline{c}^{(0)}(\underline{s}) + \underline{c}^{(1)}(\underline{s})) - \Psi(\underline{c}^{(0)}(\underline{s})) = \Omega(\underline{c}^{(1)}(\underline{s})),$$

where Ω is defined in Notation 6.3. Lemma 6.5 also applies to bound the periods of $\Omega(\underline{c}^{(1)}(\underline{s}))$, where we change notation to emphasize that the constants here are for the holomorphic differential,

$$\left| \text{Im} \int_{\gamma_{\underline{s}}} \Omega(\underline{c}^{(1)}(\underline{s})) - \sum_{e: q_e \in \gamma} c_e^{(1)} \log |s_e| - \tilde{\Pi}_\gamma(\underline{c}^{(1)}) \right| \leq \tilde{M}(\underline{c}^{(1)}) \sqrt{|\underline{s}|}.$$

Therefore, altogether we can estimate the imaginary parts of periods:

$$\left| \operatorname{Im} \int_{\gamma_{\underline{s}}} \Psi(\underline{c}^{(0)}(\underline{s}) + \underline{c}^{(1)}(\underline{s})) - \tilde{\Pi}_{\gamma}(\underline{c}^{(1)}) \right| \leq \tilde{M}(\underline{c}^{(1)})\sqrt{|\underline{s}|}. \tag{6.8}$$

We therefore proceed recursively, defining for any $l > 1$ the electromotive force

$$\mathcal{E}_{\gamma_{\underline{s}}}^{(l)} := \sum_{e: q_e \in \gamma} c_e^{(l)} \log |s_e| - \operatorname{Im} \int_{\gamma_{\underline{s}}} \Omega(\underline{c}^{(l)}) \tag{6.9}$$

and letting $c_e^{(l+1)}(\underline{s})$ be the solution of the force Kirchhoff problem with electromotive force $\mathcal{E}_{\gamma_{\underline{s}}}^{(l)}$, so that the estimate analogous to (6.8) holds:

$$\left| \operatorname{Im} \int_{\gamma_{\underline{s}}} \Psi \left(\sum_{i=0}^l \underline{c}^{(i)}(\underline{s}) \right) - \tilde{\Pi}_{\gamma}(\underline{c}^{(l)}) \right| \leq \tilde{M}(\underline{c}^{(l)})\sqrt{|\underline{s}|}. \tag{6.10}$$

Thus once we show that $c_e^{(l)}(\underline{s})$ converge to zero as $l \rightarrow \infty$, since $\tilde{\Pi}_{\gamma}(\underline{c}^{(l)})$ and $\tilde{M}(\underline{c}^{(l)})$ depend continuously on $\underline{c}^{(l)}$ and vanish for $\underline{c} = 0$ (the differential $\Omega(\underline{0})$ vanishes as it is the ARN solution of the jump problem with zero jump), it will follow that the imaginary parts of the periods of $\Psi(\sum_{i=0}^{\infty} \underline{c}^{(i)})$ are equal to zero—provided that the sum of the series converges, which we will now prove.

Proposition 6.7. *For any stable jet curve X there exists $t \in \mathbb{R}_+$ such that for any \underline{s} satisfying $|\underline{s}| < t$ the series*

$$c_e(\underline{s}) := \sum_{l=0}^{\infty} c_e^{(l)}(\underline{s}) \tag{6.11}$$

with terms recursively defined above, converge and the differential $\Psi(\underline{c})$ is the RN differential on $X_{w,u,\underline{s}}$.

Proof. Since $\Omega(\underline{c})$ depends linearly on \underline{c} , the map sending $\underline{c}^{(l)}$ to $\mathcal{E}^{(l)}$ is a linear map of finite-dimensional real vector spaces. Denoting by M the norm of this linear map, it follows that $|\mathcal{E}^{(l)}| \leq M|\underline{c}^{(l)}|$, where $|\underline{c}| := \max_e |c_e|$. By construction, $\underline{c}^{(l+1)}(\underline{s})$ is the solution of the force Kirchhoff problem with electromotive force $\mathcal{E}^{(l)}$. The a priori bound for the solutions of the force Kirchhoff problem given by Lemma 1.7 yields

$$|\underline{c}^{(l+1)}(\underline{s})| \leq (-\log |\underline{s}|)N|\mathcal{E}^{(l)}| \leq MN(-\log |\underline{s}|)|\underline{c}^{(l)}(\underline{s})| \tag{6.12}$$

(where we recall that N is the rank of $H^1(\Gamma)$ and is thus some constant). Hence for $|\underline{s}| < t = e^{-MN}$ the terms $\underline{c}^{(l)}$ are bounded by a geometric series with ratio less than 1. Therefore, the individual terms go to zero, while the sum of the series (6.11) converges, and as explained above, this implies that $\Psi(\underline{c})$ constructed from the sum of these series is an RN differential. \square

7. The limit RN differential: proof of Theorem 3.7

We now prove the main result on limits of RN differentials. The proof will crucially use the a priori bounds on solutions of the Kirchhoff problem given by Lemmas 1.6 and 1.7, which in particular imply that the residues of RN differentials are a priori bounded *uniformly* in a neighbourhood of any given stable curve.

Proof of Theorem 3.7. Let $\{X_k\}$ be a sequence of smooth jet curves converging to a stable jet curve X . Let $\underline{c}_k^{(0)}$ be the solution of the flow Kirchhoff problem with resistances $\rho_{|e|,k} = -\log |s_{e,k}|$ and inflows $f_{\ell,k}$, which we assume to converge to some limit $\underline{c}^{(0)}$. Let Ψ_k denote the RN differential on X_k , which by Proposition 6.7 is equal to $\Psi_k(\underline{c}_k)$, where \underline{c}_k is the sum of the series constructed there, of which $\underline{c}_k^{(0)}$ is the first term. As in the proof of uniqueness of the ARN solution, and as in the beginning of the proof of Proposition 5.3, we apply Stokes' theorem as in formula (5.1) for the norm of $\Psi_k - \Psi_k(\underline{c}_k^{(0)})$. As before, we now sum the results over all e , and look at the pairs of terms corresponding to e and $-e$. Since all periods of Ψ_k are real, it follows that $F_k = \text{Im} \int \Psi_k$ is a single-valued global function on $C_{\underline{s}}$, and thus its values on γ_e and the pullback of its values on γ_{-e} under I_e are equal. On the other hand,

$$F_k(\underline{c}_k^{(0)}) = \text{Im} \int \Psi_k(\underline{c}_k^{(0)})$$

is multi-valued on $C_{\underline{s}}$, as the integrals over cycles intersecting the seams may not be real. Thus the difference of the values of $F_k(\underline{c}_k^{(0)})$ on $\widehat{C}^{v(e)}$ restricted to γ_e , and on $\widehat{C}^{v(-e)}$ restricted to γ_{-e} and pulled back under I_e , is equal to the imaginary part of the integral of $\Psi_k(\underline{c}_k^{(0)})$ over some cycle γ_e^\vee on $C_{\underline{s}}$ intersecting γ_e . Therefore, altogether we see that

$$\begin{aligned} \|\Psi_k - \Psi_k(\underline{c}_k^{(0)})\|^2 &= \sum_e \int_{\gamma_e} (F_k - F_k(\underline{c}_k^{(0)})) d(F_k^* - F_k^*(\underline{c}_k^{(0)})) \\ &= \sum_e \left| \text{Im} \int_{\gamma_e^\vee} \Psi_k(\underline{c}_k^{(0)}) \right| \cdot \left| \int_{\gamma_e} d(F_k^* - F_k^*(\underline{c}_k^{(0)})) \right| \\ &= \pi \sum_e |c_{e,k} - c_{e,k}^{(0)}| \cdot \left| \text{Im} \int_{\gamma_e^\vee} \Psi_k(\underline{c}_k^{(0)}) \right|, \end{aligned} \tag{7.1}$$

where we recall that by Lemma 6.5 the period over γ_e^\vee only depends on the class of this cycle in $H_1(\Gamma, \mathbb{Z})$, that is, only on the sequence of nodes that the path passes through. We now take k sufficiently large so that w_k , u_k , and \underline{s}_k (which all converge to zero as $k \rightarrow \infty$) are sufficiently small for all the bounds in all the previous results to apply.

The integral on the right-hand side in the last line of (7.1) is by definition $-\mathcal{E}_{\gamma_e^\vee, k}^{(0)}$ used in the construction of the RN differential in the previous section. The class γ_e^\vee does not depend on \underline{s} , and thus by the proof of Proposition 6.7 we have

$$|\mathcal{E}_{\gamma_e^\vee, k}^{(0)}| < M_{\gamma_e^\vee} |c_k^{(0)}|.$$

We now let $\underline{c}_k = \{c_{e,k}\}$ be those constructed in the previous section, that is, these are given by the sums of the series for X_k , and such that $\Psi_k = \Psi_{X_k}(\underline{c}_k)$. Then by construction and (6.12) we have the estimate

$$|c_{e,k} - c_{e,k}^{(0)}| = \left| \sum_{l=1}^{\infty} c_{e,k}^{(l)} \right| \leq -\frac{|\underline{c}^{(0)}|}{\log |\underline{s}_k|}.$$

Since $|\underline{c}^{(0)}|$ is uniformly bounded by (1.3) for all w, u , and \underline{s} sufficiently small (this is a crucial use of the a priori bounds for solutions of the Kirchhoff problem!) — and thus for all k sufficiently large, altogether (7.1) implies the bound

$$\|\Psi_k - \Psi_k(\underline{c}_k^{(0)})\| < M(-\log |\underline{s}_k|)^{-1/2} \tag{7.2}$$

for some constant M independent of sufficiently large k .

Recalling from Lemma 6.2 that $\Psi_k(\underline{c}_k^{(0)}) - \Phi_k(\underline{c}_k^{(0)})$ is the ARN solution of the jump problem, the norm of which is bounded by Proposition 5.3, we obtain

$$\|\Psi_k(\underline{c}_k^{(0)}) - \Phi_k(\underline{c}_k^{(0)})\|_{C^v_s} < M_1 |\underline{s}_k|^{1/2} \tag{7.3}$$

for some constant M_1 . Thus, finally, the convergence of $\underline{c}_k^{(0)}$ implies the convergence of $\Phi_k(\underline{c}_k^{(0)})$, which depend on them continuously, and then the two bounds above imply the convergence of

$$\Psi_k = \Psi_{X_k}(\underline{c}_k),$$

which is to say that the limit RN differential exists.

Now for the second part of the statement, if the sequence $\{X_k\}$ is admissible, then by Lemma 1.13, $\underline{c}_k^{(0)}$ converge to the solution $\underline{c}^{(0)}$ of the multi-scale Kirchhoff problem with generalized resistance

$$\mathbb{P}_\rho = \lim \mathbb{P}_{\rho_k} \in \mathbb{S}_+^{\#|E|-1}.$$

By the above argument the RN differentials Ψ_k then converge to $\Phi(\underline{c}^{(0)})$. \square

8. Limits of zeros of RN differentials

In this section we finally state and prove our main result describing limits of zeros of RN differentials. We will show that if the limits of zeros of Ψ_{X_k} exist as a collection of points on C with multiplicities, then these limits are the divisor of zeros of a suitable ‘twisted’ collection of RN differentials on the components C^v , which may have higher order poles at some of the nodes where we specify that the residues are given by the series (6.11). This twisted differential Φ arises as the limit of restrictions of Ψ_k to C^v , scaled by some sequence of positive reals μ_k depending on v .

Suppose that X_k is a sequence of smooth jet curves such that C_k converge to a stable curve C , and that in this sequence the limit of the divisors of zeros of Ψ_k exists as a collection of points of C with multiplicities. Since the space of singular parts σ_ℓ , considered up to scaling all of them at once by \mathbb{R}_+ , is compact, there exists a subsequence in which the singular parts $\{\sigma_{\ell,k}\}$, considered up to scaling

by \mathbb{R}_+ , converge. Since the zeros of a differential are preserved under such scaling, we can further rescale all singular parts in the sequence so that the singular parts themselves converge (not just up to scaling). Hence without loss of generality we can assume that the sequence of smooth jet curves X_k converges to a stable jet curve X . Furthermore, since every such convergent sequence contains an admissible subsequence, without loss of generality for the rest of this section we will fix once and for all an admissible sequence $\{X_k\} = \{X_{w_k, u_k, s_k}\}$ converging to X . By Theorem 3.7 there exists a limit RN differential Ψ on C , and our goal is to investigate the limits of zeros of Ψ_k on those components C^v where Ψ is identically zero.

Notation 8.1. By a *subcurve* D of a stable curve C we mean the combinatorial data of a subset of the set of irreducible components C^v of C . Geometrically, we think of D as the union of the corresponding components, which in particular may be disconnected. The data of a subcurve $D \subset C$ then also defines subcurves $D_u \subset C_{u, \underline{0}}$ of nearby stable curves. We will call two subcurves of C disjoint if no irreducible component of C is contained in both of them. In particular, subcurves that we call disjoint may still intersect at the nodes.

Given a subcurve $D \subset C$, we let I_D be the set of internal nodes of D , that is, the set of $e \in E(\Gamma)$ such that $q_e, q_{-e} \in D$. We denote by E_D the set of nodes where D meets its complement, that is, the set of $e \in E(\Gamma)$ such that $q_e \in D$ but $q_{-e} \notin D$.

Definition 8.2. Given a sequence of meromorphic differentials ν_k on smooth curves $\{C_k\}$ converging to C , and a sequence of positive reals μ_k , we say that there exists a *scale- μ limit* of ν_k on a subcurve $D \subset C$ if for any $C^v \subset D$ there exists a not *identically* zero meromorphic differential ν^v on C^v such that for any compact set $K \subset C^v \setminus \bigcup_{e \in E_v} \{q_e\}$, the sequence of differentials $\mu_k \nu_k|_K$ converges to $\nu^v|_K$.

More generally, given a decomposition

$$D = D^{(0)} \cup D^{(1)} \cup \dots \cup D^{(L)}$$

into disjoint subcurves, and given sequences of positive reals $\mu_k^{(0)}, \dots, \mu_k^{(L)}$, we say that there exists a *multi-scale- μ limit* of ν_k on D if for any $0 \leq \lambda \leq L$ there exists a scale- $\mu^{(\lambda)}$ limit of ν_k on $D^{(\lambda)}$. We denote the collection of limits ν^v for all $C^v \subset D$ by $\mathbb{P}_D^\mu \nu$.

Of course there is never a unique choice of a sequence μ_k such that the scale- μ limit exists; any other sequence μ'_k such that there exists a finite non-zero limit of μ_k/μ'_k yields the same notion of existence of scaled limits. For multi-scale limits we will thus always number the subcurves $D^{(\lambda)}$ in such a way that

$$\lim_{k \rightarrow \infty} \frac{\mu_k^{(\lambda')}}{\mu_k^{(\lambda)}} = 0 \quad \text{for any } \lambda' < \lambda$$

(if some such limit is finite, then we consider the union of $D^{(\lambda)}$ and $D^{(\lambda')}$ as one subcurve). We think of the multi-scale limit $\mathbb{P}_D^\mu \nu$ as a collection of meromorphic differentials on all $C^v \subset D$ such that on each subcurve $D^{(\lambda)}$ the collection of differentials $\mathbb{P}_{D^{(\lambda)}}^{\mu^{(\lambda)}} \nu$ is defined up to rescaling all of it by a positive real constant.

In this terminology, the main theorem will consist of arguing that any admissible sequence has a subsequence such that for this subsequence one can define

a stratification $C = C^{(0)} \cup \dots \cup C^{(L)}$ and sequences $\mu_k^{(0)}, \dots, \mu_k^{(L)}$ such that there exists a multi-scale- μ limit $\mathbb{P}_C^\mu \Psi$ of Ψ_k . The proof will use auxiliary RN differentials constructed on *plumbed subcurves*; to define them, we introduce more notation.

Notation 8.3. For our fixed admissible sequence $\{X_k\}$, given a subcurve $D \subset C$, we denote by D_k the (possibly disconnected) smooth curve obtained by taking the union of all irreducible components of $C_{u_k,0}$ that are contained in D , and plumbing them at every internal node $e \in I_D$, with plumbing parameter $s_{e,k}$. Denote by $\Psi_{D_k}^+$ the RN differential on D_k , whose only singularities are:

- (a) $\sigma_{\ell,k}$ at all the points $p_\ell \in D_k$, and
- (b) simple poles at external nodes $e \in E_D$, with residue $ic_{e,k}$

(where we recall that the c_k are such that $\Psi_k = \Psi_{X_k}(c_k)$, which is to say that $2\pi c_{e,k} = \int_{\gamma_e} \Psi_k$).

Given a subcurve $D \subset C$, we will also consider the differential on the complementary subcurve $D' := \overline{C} \setminus D$, defined similarly, except that the singularities at the external nodes in $E_{D'} = -E_D$ will be prescribed by ‘balancing’ the jets of $\Psi_{D_k}^+$ at the nodes $e \in E_D$.

Notation 8.4. Given a meromorphic differential ν^v on C^v and given $e \in E_v$, we write

$$m_e := \text{ord}_{q_e} \nu^v$$

(which is negative if ν^v has a pole at q_e) and denote by $u_{j,e}$ the coefficients of the Laurent expansion of ν^v near q_e , so that

$$\nu^v|_{V_e} =: \sum_{j=m_e}^{\infty} u_{j,e} z_e^j dz_e.$$

We fix once and for all a positive integer m , which will eventually be assumed to be sufficiently large. We then denote by $J_e(\nu^v)$ the sum of the first order polar part and the holomorphic m -jet of the differential ν^v near q_e , that is, we define

$$J_e(\nu^v) := \sum_{j=-1}^{m-1} u_{j,e} z_e^j dz_e$$

(the inclusion of $u_{-1,e} z_e^{-1} dz_e$ in the jet is for convenience, so that the notation below is simplified; by abuse of notation we will keep calling J_e the m -jet).

Notation 8.5. Given a subcurve $D \subset C$, with complementary subcurve D' , the *balancing differential* is the RN differential $\Psi_{D_k}^-$ on D'_k , whose only singularities are:

- (a) $\sigma_{\ell,k}$ at all the points $p_\ell \in D'_k$, and
- (b) $\sigma_{-e,k} := I_{-e}^*(J_e(\Psi_{D_k}^+))$ at each external node $e \in E_{D'} = -E_D$.

Explicitly, this is to say that the singular part $\sigma_{-e,k}$ of $\Psi_{D_k}^-$ at q_{-e} is

$$\sigma_{-e,k} = - \left(s_e \sum_{j=-1}^{m-1} s_e^j u_{j,e,k} z_{-e}^{-j-2} \right) dz_{-e}, \tag{8.1}$$

where $u_{j,e,k}$ are the coefficients of the m -jet $J_e(\Psi_{D_k}^+)$, and we recall that $u_{-1,e,k} = ic_{e,k}$ by the definition of $\Psi_{D_k}^+$.

From now on we will denote by

$$\mathcal{S}_{D,k} := (\{\sigma_{\ell,k}\}_{p_\ell \in D'}; \{\sigma_{-e,k}\}_{e \in E_D}) \tag{8.2}$$

the collection of all prescribed ‘balancing’ singular parts of $\Psi_{D_k}^-$. Thus $\mathcal{S}_{d,k}$ is a point in the vector space

$$\mathbb{C}^{(m+1)\#E_D + \sum_{p_\ell \in D'}(m_\ell+1)},$$

and we denote by $\mathbb{P}\mathcal{S}_{D,k}$ the corresponding point in the sphere, which is its quotient by \mathbb{R}_+ . In particular, $\mathbb{P}\mathcal{S}_{D,k}$ only makes sense if at least one prescribed singular part of $\Psi_{D_k}^-$ is non-zero.

Definition 8.6. Given a subcurve $D \subset C$, we call the admissible sequence $\{X_k\}$ *jet-admissible on D* if at least one singular part in \mathcal{S}_D is non-zero, and there exists a limit $\mathbb{P}\mathcal{S}_D := \lim_{k \rightarrow \infty} \mathbb{P}\mathcal{S}_{D,k}$ in the sphere.

We now define recursively a stratification of C and the corresponding multi-scale; first we simplify the notation.

Notation 8.7. Given some disjoint subcurves $C^{(0)}, \dots, C^{(\lambda)}$ of C , we write

$$C^{(\leq \lambda)} := C^{(0)} \cup \dots \cup C^{(\lambda)},$$

and denote by

$$C^{(> \lambda)} := \overline{C \setminus C^{(\leq \lambda)}}$$

the complementary subcurve. We further write for brevity

$$\mathcal{S}_k^{(\lambda)} = \mathcal{S}_{C_k^{(\leq \lambda-1)}} \quad \text{and} \quad E^{(\lambda)} := E_{C^{(\leq \lambda)}},$$

and we write

$$\Psi_k^{(\leq \lambda)} := \Psi_{C_k^{(\leq \lambda)}}^+ \quad \text{and} \quad \Psi_k^{(> \lambda)} := \Psi_{C_k^{(\leq \lambda)}}^-.$$

Definition 8.8. Suppose that for some $\lambda \geq 0$ the disjoint subcurves $C^{(0)}, \dots, C^{(\lambda)}$ are already given. Suppose moreover that for some given multi-scale $\mu_k^{(0)}, \dots, \mu_k^{(\lambda)}$ there exists a multi-scale- μ limit $\mathbb{P}_{C^{(\leq \lambda)}}^\mu \Psi$ of the differentials Ψ_k on $C^{(\leq \lambda)}$. Suppose furthermore that

$$\lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \Psi_k|_{C_k^{(> \lambda)}} = 0.$$

We then say that $\{X_k\}$ is *jet-convergent at step λ* if it is jet-admissible on $C^{(\leq \lambda')}$ for any $\lambda' < \lambda$.

We call an admissible sequence *jet-convergent* if $C^{(\leq L)} = C$ for some $\lambda = L$. In this case we call the decomposition $C = C^{(0)} \cup \dots \cup C^{(L)}$ the *order of vanishing stratification*.

Remark 8.9. The notion of order of vanishing stratification is closely related to the notion of a weak full order induced by a level function, as defined in [2].

The definition of jet-convergent sequences is motivated by the fact (which will be proven below) that for any sequence jet-convergent at level λ , both the scale $\mu^{(\lambda)}$ and the scale- $\mu^{(\lambda)}$ limit of the sequence of differentials on $C^{(\lambda)}$ are determined uniquely by the behaviour at level $\lambda - 1$.

Indeed, by the definition of jet-admissibility, for a sequence that is jet-convergent at step λ there must exist a sequence of positive reals $\mu_k^{(\lambda)}$ such that there exists a non-zero limit

$$\mathcal{J}^{(\lambda)} := \lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \mathcal{J}_k^{(\lambda)}. \tag{8.3}$$

The fact that X_k is an admissible sequence means that $\mathbb{P}\rho_k$ converge in $\mathbb{S}_+^{\#|E|-1}$, which implies that any subset of coordinates of $\mathbb{P}\rho_k$ also converges in the corresponding blowup of the sphere. Therefore, it follows that the sequence of smooth curves $C_k^{(>\lambda-1)}$ is also admissible. Denoting by $X_k^{(>\lambda-1)}$ the smooth jet curve with underlying curve $C_k^{(>\lambda-1)}$ and with prescribed singular parts $\mu_k^{(\lambda)} \mathcal{J}_k^{(\lambda-1)}$, we see that the sequence converges to a stable jet curve $X^{(>\lambda-1)}$ with singular parts $\mathcal{J}^{(\lambda)}$. Thus Theorem 3.7 implies that there exists a limit RN differential

$$\Phi^{(\lambda)} := \lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \Psi_k^{(>\lambda)}. \tag{8.4}$$

In our recursive construction of jet-convergent sequences the subcurve $C^{(\lambda)} \subset C^{(>\lambda-1)}$ will be defined as the subcurve consisting of all irreducible components of C on which the differential $\Phi^{(\lambda)}$ is not identically zero.

Then the proof of the main theorem on limits of zeros of RN differentials essentially reduces to proving the equality

$$\Phi^{(\lambda)} = \lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \Psi_k \tag{8.5}$$

on every irreducible component of the subcurve $C^{(\lambda)}$.

Assuming that this equality holds, we make the following definition.

Definition 8.10. For any component $C^v \subset C^{(\lambda)}$ we write

$$\mathbb{P}\Phi^v := \mathbb{P}\Phi^{(\lambda)}|_{C^v}$$

and call the collection of all such $\{\mathbb{P}\Phi^v\}$ the *twisted limit differential* on C . The *divisor of zeros* of the twisted limit differential is defined to be the set of zeros of all $\mathbb{P}\Phi^v$ with multiplicities, away from all the nodes, together with every node $q_{|e|}$ of C counted with multiplicity

$$\text{ord}_{q_e} \mathbb{P}\Phi^{v(e)} + \text{ord}_{q_{-e}} \mathbb{P}\Phi^{v(-e)} + 2,$$

and together with every marked point p_ℓ counted with multiplicity

$$m_\ell + 1 - \text{ord}_{p_\ell} \mathbb{P}\Phi^v,$$

where C^v is the component containing p_ℓ .

Remark 8.11. By Theorem 3.7 the differential $\mathbb{P}\Phi^v$ is an RN differential on C^v whose singular parts are the scaled limits of the singular parts $\sigma_{-e,k}$ given by (8.1) and of the singularities $\sigma_{\ell,k}$ at the marked points p_ℓ that lie on C^v . Hence $\mathbb{P}\Phi^v$ may have higher order poles at the nodes.

We are now ready to state the main theorem on the limits of zeros of RN differentials. Until now the integer m in the definition of the balancing differential was arbitrary. Now we will choose it to be sufficiently large, in order to guarantee existence of jet-convergent sequences (and to ensure that the inequality (8.19) holds). We write

$$m_0 := 2g - 2 + \sum_{\ell} (m_{\ell} + 1).$$

Theorem 8.12. *For any fixed $m > 2m_0$, any admissible sequence X_k contains a jet-convergent subsequence. For any jet-convergent sequence of smooth jet curves equality (8.5) holds, where $\Phi^{(\lambda)}$ is defined by (8.4). For any jet-convergent sequence the limits of zeros of Ψ_k on C_k exist and form the divisor of zeros of the twisted limit differential, counted with multiplicities.*

We will prove the theorem by induction on the number of levels of the order of vanishing stratification. The base case of induction is $L = 0$, in which case $C = C^{(0)}$, $\Psi_k = \Psi_k^{(\leq \lambda)}$, and the limit RN differential Ψ does not vanish identically on any irreducible component C^v of C . Therefore, the theorem in this case reduces to showing that the limit RN differential Ψ is given by $\Phi = \Phi^{(0)}$, which is precisely the statement of the theorem on limit RN differentials, Theorem 3.7, in this case.

Inductive assumption at step $\lambda \geq 0$: assume that for a sequence $\{X_k\}$ that is jet-convergent at step λ , equality (8.5) holds, where $\Phi^{(\lambda)}$ is defined by (8.4); assume moreover that if $C^{(\leq \lambda)} \subsetneq C$, then the limit $\lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \Psi_k|_{C^{(> \lambda)}}$ is identically zero.

To deduce the inductive assumption at step $\lambda + 1$ from the inductive assumption at step λ , we will need the following two lemmas. First we prove a lemma showing that multi-scale- $\mu_k^{(0)}, \dots, \mu_k^{(\lambda)}$ limits of $\Psi_k|_{C^{(\leq \lambda)}}$ and $\Psi_k^{(\leq \lambda)}$ are equal.

Lemma 8.13. *If the inductive assumption at step λ holds, then for any $\lambda' \leq \lambda$ the following equality holds:*

$$\Phi^{(\lambda')} = \lim_{k \rightarrow \infty} \mu_k^{(\lambda')} \Psi_k|_{C^v} = \lim_{k \rightarrow \infty} \mu_k^{(\lambda')} \Psi_k^{(\leq \lambda)}|_{C^v}.$$

Proof. By the inductive assumption, the multi-scale limit of $\Psi_k|_{C^{(\leq \lambda)}}$ is equal to Φ , and thus we need to show that the multi-scale limit of $\Psi_k^{(\leq \lambda)}$ is the same. Let $v_k^{(\lambda)}$ be the ARN solution of the following jump problem on $C^{(\leq \lambda)}$: $v_k^{(\lambda)}$ has zero jumps on the seams γ_e with $e \in I_{C^{(\leq \lambda)}}$, and on γ_e with $e \in E^{(\lambda)}$ it has a jump equal to

$$(\Psi_k - ic_{e,k} z_e^{-1} dz_e)|_{\gamma_e}. \tag{8.6}$$

Formally it is a new type of jump problem since the collection of the initial data are set not only on the seams at nodes of $C^{(\leq \lambda)}$ but also on the seams γ_e , $e \in E^{(\lambda)}$, which are boundaries of the neighbourhoods $U_e^{s_{e,k}}$ of the points $q_e \in C^{(\leq \lambda)}$. Since

$$\int_{\gamma_e} (\Psi_k - ic_{e,k} d \log z_e) = 0,$$

the solution of this jump problem is verbatim the same and is given by the Cauchy integrals which now contain integration over γ_e , $e \in E^{(\lambda)}$. The same bounds hold, that is, the L^2 -norm of $v_k^{(\lambda)}$ is bounded by the L^∞ -norm of the initial data, which

is the L^∞ -norm of $\Psi_k - ic_{e,k} d \log z_e$ on γ_e . Then by the assumption of the lemma there exists a constant M such that

$$\|v_k^{(\lambda)}\|_{C_k^{(\leq \lambda)}} < M(\mu_k^{(\lambda)})^{-1} \sqrt{|s_k|}. \tag{8.7}$$

Consider the differential $\tilde{\Psi}_k^{(\leq \lambda)}$ which is equal to $\Psi_k - v_k^{(\lambda)}$ on $C^{(\leq \lambda)} \setminus \bigcup_{e \in E^{(\lambda)}} U_e^{s_{e,k}}$ and equals $ic_{e,k} d \log z_e - v_k^{(\lambda)}$ inside $U_e^{s_{e,k}}$. By the definition of $v_k^{(\lambda)}$ it has zero jumps on all the seams including γ_e , $e \in E^{(\lambda)}$, that is, it is a meromorphic differential on $C^{(\leq \lambda)}$, has the same singularities as $\Psi_k^{(\leq \lambda)}$, and is real-normalized. Hence $\tilde{\Psi}_k^{(\leq \lambda)} = \Psi_k^{(\leq \lambda)}$. Then (8.7) implies that on any compact set $K \subset C^v \subset C^{(\lambda')}$ not containing nodes

$$\lim_{k \rightarrow \infty} \mu_k^{(\lambda')} (\Psi_k - \Psi_k^{(\leq \lambda)})|_K = 0. \quad \square$$

We now obtain some bounds for the orders of zeros and poles of $\Phi^{(\lambda)}$.

Lemma 8.14. *Suppose the inductive assumption holds at step λ . For any $e \in E^{(\lambda)}$ let $m_e := \text{ord}_{q_e} \Phi^{(\lambda)}$. Then the following inequality holds:*

$$\sum_{e \in E^{(\lambda)}} m_e \leq m_0. \tag{8.8}$$

Proof. Recall that $\Psi_k^{(\leq \lambda)}$ is a meromorphic differential on $C_k^{(\leq \lambda)}$ whose only singularities are poles of orders $m_\ell + 1$ at the marked points p_ℓ , and possibly simple poles at $E^{(\lambda)}$. Thus the total number of zeros of $\Psi_k^{(\leq \lambda)}$, counted with multiplicity, is at most

$$2g(C_k^{(\leq \lambda)}) - 2 + \sum_{\ell} (m_\ell + 1) + \#E^{(\lambda)}.$$

Furthermore, from Lemma 8.13 we know that

$$\lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \Psi_k^{(\leq \lambda)}|_{C^{(\lambda)}} = \Phi^{(\lambda)},$$

which is a differential that is regular at all q_e for $e \in E^{(\lambda)}$. Since for k sufficiently large the total number of zeros and poles of $\Psi_k^{(\leq \lambda)}$ within V_e , counted with multiplicities, is independent of k , and it is equal to m_e for the limit differential $\Phi^{(\lambda)}$, it follows that for any k sufficiently large $\Psi_k^{(\leq \lambda)}$ has $m_e + 1$ zeros in V_e , and one simple pole there. Therefore, altogether we obtain for the number of zeroes of $\Psi_k^{(\leq \lambda)}$ the inequality

$$\sum_{e \in E^{(\lambda)}} (m_e + 1) \leq 2g(C_k^{(\leq \lambda)}) - 2 + \sum_{\ell} (m_\ell + 1) + \#E^{(\lambda)},$$

which gives the statement of the lemma upon canceling $\#E^{(\lambda)}$ that appears on both sides and noticing that $g(C_k^{(\leq \lambda)}) \leq g(C_k)$. \square

We are now ready to prove the main result of this section.

Proof of Theorem 8.12. Assume that for an admissible sequence $\{X_k\}$ the inductive assumption is satisfied at step λ . Our first goal is to show that there exists a subsequence for which we can choose a scale $\mu^{(\lambda+1)}$, a differential $\Phi^{(\lambda+1)}$, and a subcurve $C^{(\lambda+1)}$.

Indeed, consider the set of singular parts $\mathcal{S}_k^{(\lambda)}$ of differentials $\Psi_k^{(>\lambda)}$. At the points q_{-e} , $e \in E^{(\lambda)}$, these singular parts are defined by the balancing condition

$$\sigma_{-e,k} = I^*(J_e(\Psi_k^{(\leq\lambda)})).$$

By the inductive assumption, the differentials $\Psi_k^{(\leq\lambda)}$ multiplied by $\mu_k^{(\lambda)}$ converge to $\Phi^{(\lambda)}$. Lemma 8.14 then implies that the m -jet of $\Phi^{(\lambda)}$ is non-zero, and thus not all singular parts $\mathcal{S}_k^{(\lambda)}$ are zero. Thus the projectivization $\mathbb{P}\mathcal{S}_k^{(\lambda)}$ of this set of singular parts is well defined. This projectivization is a point on a sphere, and since the sphere is compact, there exists a subsequence of $\{X_k\}$ such that $\mathbb{P}\mathcal{S}_k^{(\lambda)}$ converge on the sphere. Then for this subsequence there exists a sequence of positive real numbers $\mu_k^{(\lambda+1)}$ such that

$$\mathcal{S}^{(\lambda+1)} := \lim_{k \rightarrow \infty} \mu_k^{(\lambda+1)} \mathcal{S}_k^{(\lambda+1)}. \tag{8.9}$$

Since X_k is an admissible sequence, the sequence $\mathbb{P}\rho_k$ converges in $\mathbb{S}_+^{\#|E|-1}$, and thus every subset of components of $\mathbb{P}\rho_k$ also converges in the iterated real oriented blowup of the corresponding sphere. Therefore, the sequence of jet curves $C_k^{(>\lambda)}$ is also admissible. Denote by $X_k^{(>\lambda)}$ the smooth jet curve with underlying smooth curve $C_k^{(>\lambda)}$ and with prescribed singular parts $\mu_k^{(\lambda+1)} \mathcal{S}_k^{(\lambda+1)}$. This sequence of jet curves must then converge to a stable jet curve $X^{(>\lambda)}$ with prescribed singular parts $\mathcal{S}^{(\lambda+1)}$. By Theorem 3.7 there exists a limit RN differential in this sequence, and thus we can define $\Phi^{(\lambda+1)}$ by equation (8.4), with λ replaced by $\lambda + 1$. We can then finally define the subcurve $C^{(\lambda+1)}$ to be the union of all irreducible components of $C \setminus C^{(\leq\lambda)}$ on which $\Phi^{(\lambda+1)}$ is not identically zero.

Recall now that the equality $\lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \Psi_k|_{C_k^{(>\lambda)}} = 0$ is also a part of the inductive assumption; it immediately follows that

$$\lim_{k \rightarrow \infty} \mu_k^{(\lambda)} \mathcal{S}_k^{(\lambda+1)} = 0.$$

Since there exists a finite $\mu_k^{(\lambda+1)}$ -scaled limit $\mathcal{S}^{(\lambda+1)}$ of the singular parts $\mathcal{S}_k^{(\lambda+1)}$, as defined above, it follows that

$$\lim_{k \rightarrow \infty} \frac{\mu_k^{(\lambda)}}{\mu_k^{(\lambda+1)}} = 0.$$

Thus, upon passing to a subsequence, we will from now on assume that $\{X_k\}$ is jet-admissible at step λ , that is, that equality (8.9) holds.

The following lemma proves the crucial part of the step of induction.

Lemma 8.15. *If the inductive assumption at step λ holds, then on any compact subset $K \subset C^{(\lambda+1)}$ that does not contain any nodes the differentials $\mu_k^{(\lambda+1)} \Psi_k$ restricted to K converge to $\Phi^{(\lambda+1)}|_K$.*

Before proving this essential lemma, we will analyze the behaviour of $\Phi^{(\lambda+1)}$ in a neighbourhood of a point q_{-e} for $e \in E^{(\lambda)}$.

Lemma 8.16. *Suppose the inductive assumption at step λ holds. Then there exists a constant M such that for any $e \in E^{(\lambda)}$ and any k the inequality*

$$\mu_k^{(\lambda+1)} |s_{e,k}|^{m_e+1} < M \mu_k^{(\lambda)} \tag{8.10}$$

holds.

Proof. Let $u_{j,e} := \lim_{k \rightarrow \infty} \mu_k^{(\lambda)} u_{j,e,k}$ be the limits of the scaled coefficients of the Taylor expansions of $\Psi_k^{(\leq \lambda)}$ at q_e if these exist. Since $\text{ord}_{q_e} \Phi^{(\lambda)} = m_e$ by definition, the limit $u_{m_e,e}$ exists and is non-zero, while $u_{j,e} = 0$ for any $0 \leq j < m_e$. Jet-convergence at step λ means that there exists the scale- $\mu^{(\lambda+1)}$ limit of the singular part $\sigma_{-e,k}$. Hence for all $e \in E^{(\ell)}$ and $-1 \leq j \leq m-1$ (where we write $u_{-1,e,k} := -i c_{e,k}$) there exist limits

$$\lim \mu_k^{(\lambda+1)} u_{-j-2,-e,k} = - \lim \mu_k^{(\lambda+1)} s_{e,k}^{j+1} u_{j,e,k}$$

— where we have used the balancing condition. In particular, for $j = m_e$ there exists a finite limit

$$a := \lim_{k \rightarrow \infty} \mu_k^{(\lambda+1)} s_{e,k}^{m_e+1} u_{m_e,e,k} \tag{8.11}$$

of this sequence. Since the limit $u_{m_e,e} = \lim \mu_k^{(\lambda)} u_{m_e,e,k}$ also exists and is finite and non-zero, the ratio of these two sequences, which is $\mu_k^{(\lambda+1)} s_{e,k}^{m_e+1} (\mu_k^{(\lambda)})^{-1}$, tends to the finite non-zero limit $a/u_{m_e,e}$ and, in particular, is bounded above by some constant M independent of k . \square

We can extend this analysis to bound the pole order of $\Phi^{(\lambda+1)}$ at q_{-e} , which will be used below. Recall that we have written $m_e = \text{ord}_{q_e} \Phi^{(\lambda)}$.

Lemma 8.17. *Suppose the inductive assumption at step λ holds. Then for any $e \in E^{(\lambda)}$*

$$\text{ord}_{q_{-e}} \Phi^{(\lambda+1)} \geq -m_e - 2. \tag{8.12}$$

Proof. Continuing in the setup of the proof of the previous lemma, note that since there exists a scale- $\mu^{(\lambda)}$ limit of $\Psi_k^{(\leq \lambda)}$ on $C^{(\lambda)}$, which by Lemma 8.13 is equal to $\Phi^{(\lambda)}$, it follows that for any integer j with $m_e < j < m$ the coefficients $\mu_k^{(\lambda)} u_{j,e,k}$ are bounded independent of k . Hence there exist k_0 and a constant M such that for all $k > k_0$ the inequality

$$\left| \frac{u_{m_e,e,k}}{u_{j,e,k}} \right| > M_1 \tag{8.13}$$

holds.

Suppose now for contradiction that the pole order of $\Phi^{(\lambda+1)}$ at q_{-e} is higher than $m_e + 2$, that is, for some j with $m_e < j < m$ the scale- $\mu^{(\lambda+1)}$ limit of $u_{-j-2,e,k}$ is non-zero. By the balancing condition this is equivalent to the limit

$$b := \lim_{k \rightarrow \infty} \mu_k^{(\lambda+1)} s_{e,k}^{j+1} u_{j,e,k}$$

being non-zero. Dividing equation (8.11) by this limit and using the bound (8.13) yields for the absolute value

$$\left| \frac{a}{b} \right| = \lim \left| \frac{\mu_k^{(\lambda+1)} s_{e,k}^{m_e+1} u_{m_e,e,k}}{\mu_k^{(\lambda+1)} s_{e,k}^{j+1} u_{j,e,k}} \right| = \lim \left| \frac{u_{m_e,e,k} s_{e,k}^{m_e-j}}{u_{j,e,k}} \right| > M_1 |s_{e,k}|^{m_e-j} = \infty,$$

which is a contradiction. \square

We can now show that the divisor of zeros of Φ indeed does not include nodes with negative coefficients.

Corollary 8.18. *The divisor of zeros of the twisted limit differential is a linear combination of points of the nodal curve with non-negative coefficients.*

Proof. By definition the statement means that we need to prove that

$$\text{ord}_{q_e} \Phi^{v(e)} + \text{ord}_{q_{-e}} \Phi^{v(-e)} \geq -2$$

for every node e . For a node e such that there exists λ such that $C^{v(e)}, C^{v(-e)} \subset C^{(\lambda)}$, the twisted differential $\Phi^{(\lambda)}$ is a limit RN differential and thus has at most simple poles at the nodes, so the inequality is immediate. For the other nodes there exists λ such that $q_e \in C^{(\lambda)}$ while $q_{-e} \notin C^{(\leq \lambda)}$. In this case Lemma 8.17 gives precisely the required inequality. \square

We can now give the proof of the main technical lemma.

Proof of Lemma 8.15. Together, the differentials $\Psi_k^{(\leq \lambda)}$ and $\Psi_k^{(> \lambda)}$ define a differential on the smoothing of $C_{u_k, \underline{0}}$ at all the nodes in $E \setminus E^{(\lambda)}$, which is then real-normalized on all cycles not passing through nodes in $E^{(\lambda)}$. We thus consider the jump problem with *zero jumps* at all seams corresponding to internal nodes $e \in I_{C^{(\leq \lambda)}}$ and $e \in I_{C^{(> \lambda)}}$ and with the jump on the seam γ_e for any $e \in E^{(\lambda)}$ given by

$$\Psi_k^{(\leq \lambda)}|_{\gamma_e} - I_e^* (\Psi_k^{(> \lambda)}|_{\gamma_{-e}}) \tag{8.14}$$

and let $\omega_k^{(\lambda)}$ be the ARN solution of this jump problem on C_k . Then the difference

$$\nu := (\Psi_k^{(\leq \lambda)} \sqcup \Psi_k^{(> \lambda)}) - \omega_k^{(\lambda)}$$

is a differential on $C_k^{(\leq \lambda)} \sqcup C_k^{(> \lambda)}$ that also has no jumps on any seam γ_e for any $e \in E^{(\lambda)}$. Therefore, ν is a well-defined differential on the plumbed curve C_k , satisfying the following properties:

- (a) its only singularities are at p_ℓ , with the singular parts prescribed by the coordinates w_k (the singularities at q_e for $e \in E^{(\lambda)}$ are cut out by plumbing);
- (b) ν has a real integral over any cycle on C_k not intersecting the seams γ_e for $e \in E^{(\lambda)}$;

(c) by the definition of the residues of $\Psi_k^{(\leq \lambda)}$ at q_e the integral of ν over any seam γ_e for $e \in E^{(\lambda)}$ is equal to $2\pi c_{e,k}$, which is the integral of Ψ_k over that seam (the integral of the ARN solution over the seam is zero).

Therefore, $\Psi_k - \nu$ is a holomorphic differential on C_k such that all its periods over the cycles not intersecting the seams γ_e with $e \in E^{(\lambda)}$ are real, and the

periods over these seams are zero. Verbatim, the same argument that proves the uniqueness of $\Psi_X(\underline{c})$ by applying Stokes' theorem shows that such a differential is zero, and thus it follows that ν is equal to Ψ_k .

The advantage of this construction of Ψ_k over the one used in § 6 is that the balancing condition (8.1) gives a much better upper bound for the initial data of the jump problem whose solution is $\omega_k^{(\lambda)}$. Indeed, for any $e \in E^{(\lambda)}$ let us define holomorphic differentials $f_{e,k}$ on V_e and $f_{-e,k}$ on V_{-e} by

$$f_{e,k} := \Psi_k^{(\leq \lambda)}|_{V_e} - J_e(\Psi_k^{(\leq \lambda)}) = \sum_{j=m}^{\infty} u_{j,e,k} z_e^j dz_e, \tag{8.15}$$

where $u_{j,e,k}$ denote the coefficients of the Taylor series expansion of $\Psi_k^{(\leq \lambda)}$ at q_e , and respectively

$$f_{-e,k} := \Psi_k^{(> \lambda)} - \sigma_{-e,k} = \sum_{j=0}^{\infty} u_{j,-e,k} z_{-e}^j dz_{-e}, \tag{8.16}$$

where $u_{j,-e,k}$ denote the coefficients of the Laurent series expansion of $\Psi_k^{(> \lambda)}$ at q_{-e} .

From the balancing condition (8.1) it follows that

$$\Psi_k^{(\leq \lambda)}|_{\gamma_e} - I_e^*(\Psi_k^{(> \lambda)}|_{\gamma_{-e}}) = f_{e,k}|_{\gamma_e} - I_e^*(f_{-e,k}|_{\gamma_{-e}}).$$

Since $f_{e,k}$ is holomorphic in V_e and $f_{-e,k}$ is holomorphic in V_{-e} , we can apply Proposition 5.3 to bound the ARN solution $\omega_k^{(\lambda)}$ of the jump problem posed above.

Since $C^v \setminus \{V_e\}_{e \in E_v}$ is a compact subset of $C^v \setminus \{q_e\}_{e \in E_v}$, the sequence of differentials $\mu_k^{(\lambda)} \Psi_k^{(\leq \lambda)}$ on it converges. Thus each individual term of their Taylor expansions also converges, and hence $\mu_k^{(\lambda)} f_{e,k}$, being the sum of all the terms of the Taylor expansion starting from the m th term, converges on $C^v \setminus \{V_e\}_{e \in E_v}$. In particular, $\mu_k^{(\lambda)} f_{e,k}$ converge on the circle $\{|z_e| = 1\} = \partial V_e$ — which is where we need to take the norm to apply the bound of Proposition 5.3. From (8.15) we see that for any $e \in E^{(\lambda)}$ the differential $\mu_k^{(\lambda)} f_{e,k}$ has a zero of order at least m at q_e . Therefore, there exists a constant M independent of sufficiently small w , u , and \underline{s} (so that it works for all sufficiently large k) such that

$$|f_{e,k}|_{\underline{s}} < M (\mu_k^{(\lambda)})^{-1} \max_{e \in E^{(\lambda)}} |s_{e,k}|^{(m+1)/2}.$$

We now recall that $m_e \leq m_0 < m/2$ for any e by Lemma 8.14, so that $|s_{e,k}|^{(m+1)/2} < |s_{e,k}|^{m_e+1/2}$ for any sufficiently large k . Using this inequality for the right-hand side above and multiplying this bound by $|\mu_k^{(\lambda+1)}|$ yields

$$\mu_k^{(\lambda+1)} |f_{e,k}|_{\underline{s}} < M \max_{e \in E^{(\lambda)}} |s_{e,k}|^{1/2}, \tag{8.17}$$

by applying the bound for $\mu_k^{(\lambda+1)}/\mu_k^{(\lambda)}$ given by Lemma 8.16.

We now bound $f_{-e,k}$. By the assumption of jet-convergence at step λ the differentials $\mu_k^{(\lambda+1)} \Psi_k^{(> \lambda)}$ converge to $\Phi^{(\lambda+1)}$. Since $\mu_k^{(\lambda+1)} f_{-e,k}$ is the regular part

of $\mu_k^{(\lambda+1)}\Psi_k^{(>\lambda)}$, these also converge, and thus there exists a constant M_2 independent of sufficiently small \underline{s} and sufficiently large k such that

$$\mu_k^{(\lambda+1)}|f_{-e,k}|_{\underline{s}} < M_2 \max_{e \in E^{(\lambda)}} |s_{e,k}|^{1/2}. \tag{8.18}$$

Using for the initial data $\mu_k^{(\lambda+1)}(f_{e,k} - I_e^*(f_{-e,k}))$ of the jump problem the upper bounds provided by (8.17) and (8.18), we finally get from Proposition 5.3 that there is a constant M_3 such that

$$\mu_k^{(\lambda+1)}\|\omega^{(\lambda)}\|_{C_k} < M_3|\underline{s}_k|^{1/2}. \tag{8.19}$$

Since the limit of $\mu_k^{(\lambda+1)}\Psi_k^{(>\lambda)}$ is finite and non-zero on any component $C^v \subset C^{(\lambda+1)}$, it follows that for any compact set $K \subset C_k^{(>\lambda)}$ we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \mu_k^{(\lambda+1)}\Psi_k^k|_K &= \lim_{k \rightarrow \infty} \mu_k^{(\lambda+1)}(\Psi_k^{(>\lambda)} - \omega_k^{(\lambda)})|_K \\ &= \lim_{k \rightarrow \infty} \mu_k^{(\lambda+1)}\Psi_k^{(>\lambda)}|_K = \Phi^{(\lambda+1)}|_K. \quad \square \end{aligned}$$

This lemma completes the proof of the inductive step. Indeed, Lemma 8.15 shows that if the inductive assumption holds at step λ , then there exists a scale $\mu_k^{(\lambda+1)}$ such that the inductive assumption holds, for a suitable subcurve, also at step $\lambda + 1$.

In particular we have proven the following.

Corollary 8.19. *Any sequence $\{X_k\}$ that is jet-convergent at step λ has a subsequence that is jet-convergent at step $\lambda + 1$.*

Since the order of vanishing stratification is finite, to complete the proof of the theorem it remains to determine the limits of zeros of Ψ_k . Away from the nodes of C , the limits of zeros of Ψ_k are clearly the same as the limits of zeros of $\mu_k^{(\lambda)}\Psi_k$, which are simply the zeros of $\mathbb{P}\Phi$ counted with multiplicities. We thus need to show that a node $q_{|e|}$ is the limit of $\text{ord}_{q_e} \Phi^{v(e)} + \text{ord}_{q_{-e}} \Phi^{v(-e)} + 2$ zeros of Ψ_k , counted with multiplicities.

Denote by $K_{e,k}$ the compact set $(V_e \setminus U_e^{s_{e,k}}) \sqcup (V_{-e} \setminus U_{-e}^{s_{e,k}})/(\gamma_e \sim \gamma_{-e})$, where the seams are identified via I_e , as usual. The differentials dz_e in $V_{-e} \setminus U_{-e}^{s_{e,k}}$ and $s_{e,k}z_{-e}^{-2}dz_{-e}$ in $V_{-e} \setminus U_{-e}^{s_{e,k}}$ match each other on the seam γ_e and hence define a holomorphic differential $\zeta_{e,k}$ on $K_{e,k}$ that is nowhere zero. Hence the number of zeros of the differential Ψ_k in $K_{e,k}$ is equal to the number of zeros in $K_{e,k}$ of the function $\Psi_k/\zeta_{e,k}$. By the argument principle the latter is equal to the integral of $(2\pi i)^{-1} d \log(\Psi_k/\zeta_{e,k})$ over the boundary $\partial K_{e,k}$, which is the union of the circles $|z_e| = 1$ and $|z_{-e}| = 1$, with opposite orientations. On these circles $\mathbb{P}\Psi_k$ converges to $\mathbb{P}\Phi^{v(e)}$ and $\mathbb{P}\Phi^{v(-e)}$, respectively, and thus the integrals of $d \log(\Psi_k/\zeta_{e,k})$ over them converge to the integrals of $d \log(\Phi^{v(e)}/dz_e)$ and $d \log(z_{-e}^2 \Phi^{v(-e)}/dz_{-e})$, respectively. Thus the total number of zeros of Ψ_k within $K_{e,k}$, for k sufficiently large, is equal to the total number of zeros and poles of $\Phi^{v(e)}$ in V_e plus the total number of zeros and poles of $z_{-e}^2 \Phi^{v(-e)}$ in V_{-e} , all counted with multiplicity. Since the only zeros or poles of Φ in these neighbourhoods are at the origins, the statement about the multiplicity of the zero at $q_{|e|}$ follows.

The computation of the multiplicity of p_ℓ as a limit of zeros of Ψ_k is straightforward—the point p_ℓ is a smooth point of the component C^v that contains it, and thus the multiplicity of it as the limit of zeros of Ψ_k is precisely the difference of the pole orders of Ψ_k and Φ at p_ℓ .

Our main theorem, Theorem 8.12 on limits of zeros of RN differentials, is thus proven. \square

Corollary 8.20. *Suppose that for a sequence of smooth jet curves $\{X_k\}$ converging to a stable jet curve X the limit of the zeros of the RN differentials Ψ_k exists. Then the limits of zeros are the divisor of zeros of the twisted limit differential for some jet-convergent sequence.*

Proof. As argued in the proof of Proposition 3.14, the compactness of $\mathbb{S}_+^{\#E-1}$ implies that any sequence $\{X_k\}$ has an admissible subsequence. Furthermore, Corollary 8.19 shows that there exists a jet-convergent subsequence of this admissible subsequence. By Theorem 8.12 the limit of zeros of the RN differentials corresponding to this jet-convergent subsequence is the divisor of zeros of the twisted limit differential in this subsequence. Since the limit of zeros exists for the whole sequence, it must then be equal to the limit of zeros for this subsequence. \square

9. Appendix: m -balanced approximation

As we see in the proof of the main theorem, Theorem 8.12 on limits of zeros of RN differentials, the main motivation for introducing the balancing condition (8.1) for the jet of the differential at q_e and the singular part at q_{-e} is to ensure that the jump is sufficiently small, so that Proposition 5.3 applies, and yields a bound for the ARN solution. The bound shows that for \underline{s} sufficiently small this ARN solution is smaller than the differentials themselves. In this appendix we develop this idea into a general notion of m -balanced approximations. The balancing construction used in the previous section will correspond to the case $m = 0$ of this more general construction. While not used for the proof of our main results, the notion of m -balanced approximations for $m > 0$ gives a general framework for future works aimed at understanding more precisely the asymptotic behaviour of meromorphic differentials under degeneration.

For a fixed $m \in \mathbb{Z}_{\geq 0}$, we continue with the notation of the previous section. Denote by W the set of all collections of RN meromorphic differentials $\Phi = \{\Phi^v\}$ on C^v that have poles of order up to $m_\ell + 1$ at each marked point p_ℓ , are holomorphic away from the marked points and the pre-images of the nodes, and have poles of order at most $m + 1$ at the pre-images of all nodes, with opposite residues at q_e and q_{-e} for any node e . We note that W is a finite-dimensional vector space, and for further use we denote by $W_0 \subset W$ the vector subspace of those differentials that are regular at all p_ℓ and have zero residue at any node (while still allowed to have higher order poles there). For $\Phi \in W$ we denote by $u_{j,e}$ the coefficients of the Laurent series of $\Phi^{v(e)}$ at q_e , by σ_e the singular part, and by J_e the m -jet of the holomorphic part of $\Phi^{v(e)}$ plus the polar term of order -1 .

For a jet curve $X = X_{w,u,s}$ we continue to denote by $\underline{c} = \underline{c}(X)$ the sum of the series as constructed in Proposition 6.7, that is, the values such that $\Psi_X = \Psi_X(\underline{c}(X))$. The m -balanced approximation is then defined as follows.

Definition 9.1. For a given X and for a fixed integer $m \geq 0$, an element

$$\Phi_X[m] = \{\Phi[m]^v\} \in W$$

is called an m -balanced approximation if the following conditions hold.

- (a) At each point p_ℓ the singular part of $\Phi[m]$ is as prescribed by the coordinates w .
- (b) At any pre-image q_e of any node, the residue of $\Phi[m]^{v(e)}$ is equal to $\text{Res}_{q_e} \Phi[m]^{v(e)} = ic_e$.
- (c) At any pre-image q_{-e} of any node, the singular part σ_{-e} of $\Phi[m]^{v(-e)}$ is equal to the pullback under I_e^* of the m -jet J_e of $\Phi[m]^{v(e)}$ at q_e :

$$\sigma_{-e} = I_e^*(J_e(\Phi[m]^{v(e)})),$$

which is the balancing condition explicitly written in equation (8.1).

We note that condition (c) prescribes both σ_{-e} in terms of the jet of $\Phi[m]^{v(e)}$ and, by choosing q_{-e} instead, also the singular part σ_e in terms of the jet of $\Phi[m]^{v(-e)}$. Therefore, the existence of m -balanced approximations cannot be argued by constructing them starting from some component, and then proceeding to define them explicitly on the adjoining component. We thus first need to prove that the m -balanced approximations exist. To prove this, one could first argue uniqueness as we do below, and then deduce the existence by noting that the conditions imposed on an m -balanced approximation are a system of non-homogeneous linear equations on the singular parts, which must then have a solution. However, for possible applications it is important to be able to compute the m -balanced approximation, and we thus give a proof by an explicit construction of the approximation as the sum of a recursively defined series.

Proposition 9.2. *For any fixed m there exists a constant t_m such that for any $|w|, |u|, |\underline{s}| < t_m$ there exists a unique m -balanced approximation $\Phi_X[m]$.*

Proof. Similarly to the construction of the RN differential in plumbing coordinates, we will prove the existence by constructing the approximation as the sum of a series $\Phi_X[m] = \sum_{l=0}^\infty \Phi^{(l)}$, now with the first term $\Phi^{(0)} := \Phi(\underline{c}(X))$.

The further terms $\Phi^{(l)}$ for $l > 1$ will lie in W_0 , so that adding them to $\Phi^{(0)}$ does not change the singular parts of Φ at p_ℓ or the residues at the nodes. To define $\Phi^{(l)}$ we introduce a linear operator $R: W \rightarrow W_0$, which we think of as ‘balancing’ the singular parts. An element of W_0 is prescribed by its singular parts at each q_e , and we define $R(\Phi)$ by prescribing its singular parts at each q_{-e} to be

$$s_e \sum_{j=0}^{m-1} s_e^j u_{j,e} z_{-e}^{-j-2} dz_{-e}, \tag{9.1}$$

where the $u_{j,e}$ are the coefficients of the jet J_e of Φ (note that this formula prescribes a singular part without residue, as required for an element of W_0).

We introduce a norm on W by taking the maximum of norms of the singular parts. Let the linear operator $R': W \rightarrow W_0$ be defined by prescribing the singular parts to be

$$R'(\sigma)_{-e} = s_e^{-1} R(\sigma)_{-e}.$$

Since it depends smoothly on u and the collection of singular parts $R(\sigma)$ is obtained by multiplying the collection of singular parts $R'(\sigma)_{-e}$ by the diagonal matrix of s_e , it follows that there exists a constant M_m such that for any $\Phi \in W$ we have

$$|R(\Phi)| < |\underline{s}| |R'(\Phi)| \leq M_m |\underline{s}| |\Phi|. \tag{9.2}$$

We now define the terms of the series by setting $\Phi^{(l+1)} := R(\Phi^{(l)})$ for any $l \geq 0$, so that the above bound (9.2) shows that

$$|\Phi^{(l+1)}| \leq M_m |\underline{s}| |\Phi^{(l)}| \tag{9.3}$$

for any l . Thus, for $|\underline{s}| < M_m^{-1}$ the norms of the terms are bounded by a geometric sequence with ratio less than 1, and thus the series converges.

To prove uniqueness of the approximation, suppose that Φ' is the difference of any two m -balanced approximations. Then $\Phi' \in W_0$ is a collection of RN differentials holomorphic at all the marked points and satisfying all the balancing conditions (9.1). But then $R(\Phi') = \Phi'$ by definition, and thus it follows that $|\Phi'| \leq M_m |\underline{s}| |\Phi'|$, which for $|\underline{s}| < M_m^{-1}$ is a contradiction unless Φ' is zero. \square

By using techniques similar to the proof of Lemma 8.14 it can be shown that for $m > 2m_0$ each meromorphic differential $\Phi[m]^v$ of an m -balanced approximation is not identically zero. To prove this, one proceeds inductively by the number of components of C and uses estimates similar to those in Lemma 8.14 to show that each m -jet $J_e(\Phi)$ is not identically zero — which then implies that no singular part σ_{-e} is identically zero, and thus no RN differential $\Phi[m]^v$ is identically zero.

Furthermore, consider the jump problem with initial data

$$\Phi[m]^{v(e)}|_{\gamma_e} - I_e^* (\Phi[m]^{v(-e)}|_{\gamma_{-e}}).$$

Let $\omega_X[m]$ be the ARN solution of this jump problem, so that $\Phi_X[m] - \omega_X[m]$ then glues to define a differential on C , which is then easily seen to be equal to Ψ_X . The balancing condition then shows that the first m terms of the initial data of the jump problem cancel (as in equations (8.15) and (8.16) in the proof of Theorem 8.12), and thus gives a bound for the ARN solution. As a result, one obtains the following.

Lemma 9.3. *For any w, u , and \underline{s} sufficiently small there exists a constant M such that for any compact set $K \subset C^v \setminus \bigcup_{e \in E_v} \{q_e\}$ the inequality*

$$\|\Psi_X - \Phi_X[m]\|_K < M |\underline{s}|^{(m+1)/2} \tag{9.4}$$

holds.

This lemma says that on those components C^v where the limit RN differential Ψ is not identically zero (that is, on $C^{(0)}$), $\Phi_X[m]$ approximates Ψ_X up to order $|\underline{s}|^{(m+1)/2}$. In the same spirit as before, we can also approximate $\Psi_X|_{C^{(>0)}}$ with the same precision, by using the order of vanishing stratification.

More precisely, suppose that X_k is a jet-convergent sequence, as defined in Definition 8.8, with multi-scale $\mu^{(0)}, \dots, \mu^{(L)}$. We define a new sequence of differentials as follows.

Notation 9.4. For any $0 \leq \lambda \leq L$ we denote by $\Psi_k[m]^{(\leq \lambda)}$ and $\Psi_k[m]^{(> \lambda)}$ the RN differentials on $C_k^{(\leq \lambda)}$ and $C_k^{(> \lambda)}$, respectively, whose only singular parts are as follows:

- at every point $p_\ell \in C^{(\leq \lambda)}$, the differential $\Psi_k[m]^{(\leq \lambda)}$ has singular part $\sigma_{\ell,k}$;
- at every $q_e, e \in E^{(\lambda)}$, the differential $\Psi_k[m]^{(\leq \lambda)}$ has singular part $\sigma_{e,k}[m]$ of order at most $m + 1$, with residue $ic_{e,k}$;
- at every $q_{-e}, e \in E^{(\lambda)}$, the differential $\Psi_k[m]^{(> \lambda)}$ has singular part $\sigma_{-e,k}[2m]$ of order at most $2m + 1$;
- the following enhanced balancing condition holds:

$$\sigma_{e,k}[m] + J_e^{[2m]}(\Psi_k[m]^{(\leq \lambda)}) = I_e^*(\sigma_{-e,k}[2m] + J_{-e}^{[m]}(\Psi_k[m]^{(> \lambda)})), \quad (9.5)$$

where $J_e^{[m]}(\cdot)$ is the polar term of order -1 plus the m -jet of the regular part of the corresponding differential.

What the enhanced balancing condition says is that $\sigma_{e,k}[m]$ is the singular part that is m -balanced with the m -jet of $\Psi_k[m]^{(> \lambda)}$ at q_{-e} , while $\sigma_{-e,k}[2m]$ is the singular part that is $2m$ -balanced with the $2m$ -jet of $\Psi_k[m]^{(> \lambda)}$ at q_e . Viewed this way, the original balancing condition (8.1) requires 0-balancing in one direction, and m -balancing in the other direction.

The existence of such a pair of differentials $\Psi_k[m]^{(\leq \lambda)}$ and $\Psi_k[m]^{(> \lambda)}$ requires a proof, since unlike the case of $\Psi_k^{(\leq \lambda)}$, which is defined directly, and unlike the case of $\Psi_k^{(> \lambda)}$, which is defined by prescribing its singular parts, the differentials $\Psi_k[m]^{(\leq \lambda)}$ and $\Psi_k[m]^{(> \lambda)}$ must satisfy the enhanced balancing condition, which restricts the singularities of both of them. However, arguing the same way as in the proof of existence of m -balanced approximations, one can prove that such differentials exist. Essentially the argument again boils down to noticing the extra powers of $s_{e,k}$ appearing in front of the singular parts $\sigma_{e,k}[m]$ and $\sigma_{-e,k}[2m]$, which imply the uniqueness of solution. Then since the enhanced balancing condition is a non-homogeneous system of \mathbb{R} -linear equations, it follows that this system is non-degenerate and has a solution. Similarly to our proof of Proposition 9.2, one can construct these differentials explicitly as sums of recursively defined series — which essentially amounts to inverting a linear operator as a recursively defined series with terms decaying as powers of s .

Definition 9.5. For a given jet-convergent sequence X_k , we let $\Phi_k[m]^{(\lambda+1)}$ be the m -balanced approximation of $\Psi_k[m]^{(> \lambda)}$ on $C_k^{(> \lambda)}$. We then call the collection of differentials $\Phi_k[m]^{(\lambda+1)}$ on $C^{(\lambda+1)}$ the m -balanced approximation on X_k .

The use of the enhanced balancing condition is to guarantee a still better bound on the initial data in the suitable jump problem, and the name of m -balanced approximation is justified by the following main result about it.

Proposition 9.6. For any jet-convergent sequence X_k , any $\lambda \geq 0$, and any $C^v \subset C^{(\lambda+1)}$, the following inequality holds:

$$\mu_k^{(\lambda+1)} \|\Psi_k - \Phi_k[m]^{(\lambda+1)}\|_{C^v} < M_1 |s_k|^{(m+1)/2}.$$

The proof is completely parallel to the proof of Theorem 8.12, and since we do not require Proposition 9.6 for the proof of our main result, Theorem 8.12, we do not give the full details of the proof, just indicating the outline, for possible future applications.

Idea of the proof. Mimicking the proof of Theorem 8.12, one first needs to prove the statement analogous to Lemma 8.13, showing that the multi-scale- μ limits of Ψ_k and $\Psi_k[m]^{(\leq \lambda)}$ are the same. The proof is by using the ARN solution of the jump problem and noticing that the presence of positive powers of $s_{e,k}$ in each singular part $\sigma_{e,k}$ of $\Psi_k[m]^{(\leq \lambda)}$ at q_e for $e \in E^{(\lambda)}$, as given by the enhanced balancing condition, ensures that in the limit as $k \rightarrow \infty$ these singular parts go to zero. One then shows, similarly to the main part of the proof of Theorem 8.12, that the RN differential Ψ_k is equal to

$$\Psi_k = (\Psi_k[m]^{(\leq \lambda)} \sqcup \Psi_k[m]^{(> \lambda)}) - \omega_k[m]^{(\lambda)},$$

where $\omega[m]^{(\lambda)}$ is the ARN solution of the jump problem with zero jumps at all seams corresponding to internal nodes $e \in I_{C^{(\leq \lambda)}}$ and $e \in I_{C^{(> \lambda)}}$, and with the jump on seam γ_e for any $e \in E^{(\lambda)}$ given by

$$\Psi_k[m]^{(\leq \lambda)}|_{\gamma_e} - I_e^*(\Psi[m]_k^{(> \lambda)}|_{\gamma_{-e}}). \tag{9.6}$$

Identically to the arguments in the proof of (8.19), one can verify that the enhanced balancing condition (9.5) indeed gives the extra m th power of $s_{e,k}$ in the bounds for the initial data for this jump problem, so that both upper bounds (8.17) and (8.18) are improved by an extra factor of $|s_{e,k}|^m$. Thus another application of the bound for the ARN solution of the jump problem, given by Proposition 5.3, guarantees that there exists a constant M such that for any $C^v \subset C^{(\lambda+1)}$ the inequality

$$\mu_k^{(\lambda+1)} \|\omega[m]^{(\lambda)}\|_{C^v} < M |s_k|^{(m+1)/2}$$

holds. The proposition thus follows. \square

By taking the limit as $k \rightarrow \infty$, the proposition of course implies that in any jet-convergent sequence the scale- $\mu^{(\lambda)}$ limit of $\Phi_k[m]^{(\lambda)}$ on $C^{(\lambda)}$ is equal to the twisted differential $\Phi^{(\lambda)}$ that appears in Theorem 8.12. Moreover, the bound in Proposition 9.6 then shows that the collection of differentials $\Phi_k[m]^{(\lambda)}$ for all $0 \leq \lambda \leq L$ gives an approximation to Ψ_k which, after scaling by the corresponding scale $\mu^{(\lambda)}$, is still within $|s|^m$. This information allows the study of differentials Ψ_k in a degenerating sequence of jet curves with arbitrary precision; in more generality, the method of considering m -balanced approximations can also be applied to studying degenerations of other kinds of differentials on sequences of degenerating Riemann surfaces.

Bibliography

[1] E. Arbarello, M. Cornalba, and P. A. Griffiths, *Geometry of algebraic curves*, vol. II, With a contribution by J. D. Harris, Grundlehren Math. Wiss., vol. 268, Springer, Heidelberg 2011, xxx+963 pp.

- [2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Moeller, “Compactification of strata of Abelian differentials”, *Duke Math. J.* **167**:12 (2018), 2347–2416.
- [3] L. Bers, “Spaces of degenerating Riemann surfaces”, *Discontinuous groups and Riemann surfaces* (Univ. Maryland, College Park, MD 1973), Ann. of Math. Stud., vol. 79, Princeton Univ. Press, Princeton, NJ 1974, pp. 43–55.
- [4] D. Chen, “Degenerations of abelian differentials”, *J. Differential Geom.* **107**:3 (2017), 395–453.
- [5] G. Farkas and R. Pandharipande, “The moduli space of twisted canonical differentials”, *J. Inst. Math. Jussieu* **17**:3 (2018), 615–672.
- [6] Q. Gendron, “The Deligne–Mumford and the incidence variety compactifications of the strata of $\Omega\mathcal{M}_g$ ”, *Ann. Inst. Fourier (Grenoble)* **68**:3 (2018), 1169–1240.
- [7] W. D. Gillam, *Oriented real blowup*, preprint, 21 pp., <http://www.math.boun.edu.tr/instructors/wdgillam/manuscripts.html>.
- [8] S. Grushevsky and I. Krichever, “The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces”, *Surveys in differential geometry*, vol. XIV: *Geometry of Riemann surfaces and their moduli spaces*, Surv. Differ. Geom., vol. 14, Int. Press, Somerville, MA 2009, pp. 111–129.
- [9] S. Grushevsky and I. Krichever, “Real-normalized differentials and the elliptic Calogero–Moser system”, *Complex geometry and dynamics. The Abel symposium 2013*, Abel Symp., vol. 10, Springer, Cham 2015, pp. 123–137.
- [10] S. Grushevsky and I. Krichever, *Real-normalized differentials and cusps of plane curves*, in preparation.
- [11] X. Hu and C. Norton, “General variational formulas for Abelian differentials”, *Int. Math. Res. Notices*, 2018, rny106, Publ. online; 2018 (v1 – 2017), 40 pp., arXiv: 1705.05366.
- [12] И. М. Кричевер, “Спектральная теория ‘конечнозонных’ нестационарных операторов Шрёдингера. Нестационарная модель Пайерлса”, *Функц. анализ и его прил.* **20**:3 (1986), 42–54; English transl., I. M. Krichever, “Spectral theory of finite-zone nonstationary Schrödinger operators. A nonstationary Peierls model”, *Funct. Anal. Appl.* **20**:3 (1986), 203–214.
- [13] И. М. Кричевер, “Метод усреднения для двумерных ‘интегрируемых’ уравнений”, *Функц. анализ и его прил.* **22**:3 (1988), 37–52; English transl., I. M. Krichever, “Method of averaging for two-dimensional ‘integrable’ equations”, *Funct. Anal. Appl.* **22**:3 (1988), 200–213.
- [14] И. М. Кричевер, “Вещественно-нормированные дифференциалы и гипотеза Арбарелло”, *Функц. анализ и его прил.* **46**:2 (2012), 37–51; English transl., I. M. Krichever, “Real normalized differentials and Arbarello’s conjecture”, *Funct. Anal. Appl.* **46**:2 (2012), 110–120.
- [15] И. М. Кричевер, С. П. Новиков, “Алгебры типа Вирасоро, римановы поверхности и струны в пространстве Минковского”, *Функц. анализ и его прил.* **21**:4 (1987), 47–61; English transl., I. M. Krichever and S. P. Novikov, “Virasoro-type algebras, Riemann surfaces and strings in Minkowski space”, *Funct. Anal. Appl.* **21**:4 (1987), 294–307.
- [16] L. Lang, *Harmonic tropical curves*, 2015, 46 pp., arXiv: 1501.07121.
- [17] C. R. Norton, *Limits of real-normalized differentials on stable curves*, Ph.D. Thesis, Stony Brook Univ. 2014, 115 pp.
- [18] B. Osserman, “Limit linear series for curves not of compact type”, *J. Reine Angew. Math.*, 2017, Publ. online; 2014, 34 pp., arXiv: 1406.6699.
- [19] Yu. L. Rodin, *The Riemann boundary problem on Riemann surfaces*, Math. Appl. (Soviet Ser.), vol. 16, D. Reidel Publishing Co., Dordrecht 1988, xiv+199 pp.

- [20] M. Schiffer and D. C. Spencer, *Functionals of finite Riemann surfaces*, Princeton Univ. Press, Princeton, NJ 1954, x+451 pp.
- [21] S. A. Wolpert, “Infinitesimal deformations of nodal stable curves”, *Adv. Math.* **244** (2013), 413–440.
- [22] Э. И. Зверович, “Краевые задачи теории аналитических функций в гёльдеровских классах на римановых поверхностях”, *УМН* **26**:1(157) (1971), 113–179; English transl., È. I. Zverovich, “Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces”, *Russian Math. Surveys* **26**:1 (1971), 117–192.

Samuel Grushevsky

Stony Brook University, Stony Brook, NY, USA

E-mail: sam@math.stonybrook.edu

Received 12/DEC/18

Igor M. Krichever

Columbia University, New York, USA;

Skolkovo Institute for Science and Technology;

National Research University

Higher School of Economics;

Institute for Information Transmission Problems

of the Russian Academy of Sciences

(Kharkevich Institute);

L. D. Landau Institute for Theoretical Physics

RAS

E-mail: krichev@math.columbia.edu

Chaya Norton

Concordia University, Montreal, QC, Canada;

Centre de Recherches Mathématiques (CRM),

Université de Montréal, Montreal, QC, Canada

E-mail: nortonch@crm.umontreal.ca