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Commuting Difference Operators and the Combinatorial Gale Transform*

I. M. Krichever

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ABSTRACT. We develop the spectral theory of n-periodic strictly triangular difference operators $L = T^{-k-1} + \sum_{j=1}^{k} a_i^j T^{-j}$ and the spectral theory of the "superperiodic" operators for which all solutions of the equation $(L+1)\psi = 0$ are (anti)periodic. We show that, for a superperiodic operator L of order $k+1$, there exists a unique superperiodic operator $\mathscr{L}$ of order $n-k-1$ which commutes with L and show that the duality $L \leftrightarrow \mathscr{L}$ coincides, up to a certain involution, with the combinatorial Gale transform recently introduced in [21].

KEY WORDS: spectral theory of linear difference operators, commuting difference operators, frieze patterns, moduli spaces of n-gons, Gale transform.

1. Introduction

In recent years, the theory of linear difference equations of the form

$$V_i = a_i^1 V_{i-1} - a_i^2 V_{i-2} + \cdots + (-1)^{k-1} a_i^k V_{i-k} + (-1)^k V_{n-k-1} \tag{1}$$

with real periodic coefficients

$$a_i^j = a_{i+n}^j \in \mathbb{R}, \qquad i \in \mathbb{Z},\ 1 \leqslant j \leqslant k\,,$$

all of whose solutions are n-(anti)periodic, that is, such that

$$V_{i+n} = (-1)^k V_i, \tag{2}$$

has seen a burst of interest due to connections with the theory of n-gons in projective space (extensively studied in [5], [7], [9], and [11]) and the theory of SL_{k+1}-frieze patterns (created by Coxeter and Conway in [3] and [4] and subsequently developed in [1], [2], [19], [20], [22], [26], and [27]). Frieze patterns are a particular case of more general patterns arising in the theory of *bilinear discrete Hirota* equations (see [31] and the references therein).

Our study was motivated by the recent work [21], where it was shown that the moduli space $\mathscr{E}_{k+1,n}$ of Eqs. (1) satisfying the constraint (2) is isomorphic, as an algebraic variety, to the moduli space $\mathscr{F}_{k+1,n}$ of SL_{k+1}-frieze patterns of width $w = n-k-2$ and that, if n and $k+1$ are coprime, then both spaces are isomorphic to the moduli space $\mathscr{C}_{k+1,n}$ of n-gons in $\mathbb{RP}^k$. The main result of [21] is a description of the *combinatorial Gale* transform, which is the duality between the spaces $\mathscr{E}_{k+1,n}$ and $\mathscr{E}_{n-k-1,n}$ analogous to the classical Gale duality on the moduli spaces of point configurations. The former and the latter are induced by the duality of the Grassmannians $\mathrm{Gr}(k+1,n)$ and $\mathrm{Gr}(n-k-1,n)$.

Recall that if $\widehat{V}$ is an $n\times(k+1)$ matrix of rank $k+1$ representing a point of $Gr(k+1,n)$, then the dual point is represented by an $n\times(n-k-1)$ matrix $\widehat{W}$ such that $\widehat{W}^T\widehat{V}$=0. The space of n-gons in $\mathbb{P}^k$ is the quotient of the Grassmannian by the torus action: $\mathscr{C}_{k+1,n} \simeq \mathrm{Gr}(k+1,n)/\mathbb{T}^{n-1}$ [9]. The points of Grassmannians representing dual n-gons are related by $\widehat{W}^T D\widehat{V} = 0$, where D is a nonsingular diagonal $n\times n$ matrix.

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If vectors $V^{(j)}$ form a basis of the solutions of Eq. (1), then the matrix $\widehat{V}_{(i)} := (V^{(j)}_{i+1}, V^{(j)}_{i+2}, \dots, V^{(j)}_{i+n})$ determines an embedding (depending on the choice of i) of $\mathscr{E}_{k+1,n}$ into the Grassmannian $\mathrm{Gr}(k+1,n)$. As shown in [21] (Proposition 4.3.1), the defining property of the combinatorial Gale dual equations is the relation

$$\widehat{W}^T_{(i)} \widehat{\varepsilon} \widehat{V}_{(j)} = 0, \qquad i - j = n - k - 1 \pmod{n}, \tag{3}$$

where $\widehat{\varepsilon}$ is the diagonal matrix $(1, -1, 1, \dots)$.

The simplest example of a combinatorial Gale transform is related to Gauss' *pentagramma mirificum* [8]. This is a duality between 5-periodic difference equations of the third and the second order. More precisely, duality means that if all solutions of the equation

$$V_i = a_i V_{i-1} - b_i V_{i-2} + V_{i-3}, \qquad a_i = a_{i+5},\ b_i = b_{i+5}, \tag{4}$$

are 5-periodic, then all solutions of the equation

$$W_i = a_i W_{i-1} - W_{i-2}$$

are 5-antiperiodic: $W_i = -W_{i+5}$. In terms of n-gons, this transformation sends projective equivalence classes of pentagons in $\mathbb{P}^2$ to those in $\mathbb{P}^1$. In terms of frieze patterns, this is a duality between 5-periodic Coxeter friezes and 5-periodic SL_3-friezes.

The first goal of this work is to introduce a *spectral parameter* into Eq. (1) and develop the Block–Flouque spectral theory of the corresponding difference operators. We emphasize that the idea to introduce a spectral parameter into (1) is not new by itself. In [24] the spectral parameter s was introduced into Eq. (4) as

$$V_i = s a_i V_{i-1} - s^{-1} b_i V_{i-2} + V_{i-3}, \qquad a_i = a_{i+5},\ b_i = b_{i+5}, \tag{5}$$

and used for constructing a complete set of integrals in involution for the pentagram map defined in [29]. The algebraic-geometric integrability of the pentagram map was proved in [30] by using the Lax operator with spectral parameter gauge equivalent to (5) (for Liouville–Arnold integrability, see [25]). In a similar way the spectral parameter was introduced into higher-order linear equations for proving the algebraic-geometric integrability of higher pentagram maps ([12], [13]; see also [18]). An approach to introduce the spectral parameter into Eq. (1) by using cluster algebras was proposed in [6] and [10]; for the SL_3-case, it turned out to lead to an equation equivalent to Eq. (5).

The spectral parameter s in (5) reflects the scaling invariance of the pentagram map (see [24]). In this paper we consider a *different* spectral problem. Namely, let $\mathscr{D}_{k+1,n}$ be the (affine) space of monic "strictly" triangular n-periodic difference operators with complex (or real) coefficients:

$$\left\{ L \in \mathscr{D}_{k+1,n} \;\middle|\; L = T^{-k-1} + \sum_{j=1}^{k} a_i^j T^{-j},\ a_i^j = a_{i+n}^j \right\}. \tag{6}$$

Here and below, T is the shift operator acting on infinite sequences $\psi = \{\psi_i\}$, $i \in \mathbb{Z}$, i.e., $(T\psi)_i = \psi_{i+1}$. It will also be assumed, unless stated otherwise, that the leading coefficient of L does not vanish:

$$a_i^1 \neq 0.$$

Note that, for $E = -1$, the gauge transformation

$$V = \varepsilon\psi, \qquad \varepsilon := \{\varepsilon_i = (-1)^i\} \tag{7}$$

transforms the equation

$$L\psi = E\psi \tag{8}$$

for the eigenvectors of L into Eq. (1).

Remark 1.1. Similarly, the spectral parameter can be introduced into Eq. (1) as the coefficient of the last term on the right-hand side of the equation, but the two ways of introducing the spectral parameter are in fact equivalent due to the involution on $\mathscr{D}_{k+1,n}$:

$$L \longmapsto L^\sigma := T^{-k-1}(L^* + 1) - 1, \tag{9}$$

where L^* is the formal adjoint operator

$$L^* = T^{k+1} + \sum_{j=1}^{k} T^j a_i^j = T^{k+1} + \sum_{j=1}^{k} a_{i+j}^j T^j.$$

Under the gauge transformation (7) the constraint (2) defining the special triangular operators $L \in \mathscr{E}_{k+1,n} \subset \mathscr{D}_{k+1,n}$ takes the form: *All solutions Ψ of the equation $(L+1)\Psi = 0$ have the monodromy property*

$$\Psi_i = (-1)^{n+k}\Psi_{i-n}. \tag{10}$$

For brevity, throughout the rest of the paper, the operators $L \in \mathscr{E}_{k+1,n}$ will be called *superperiodic.*

Remark 1.2. Note that the choice of the multiplier $\mu = (-1)^{n+k}$ in (10) can be seen as the choice of a normalization in the following slightly more general setting. For a pair of complex numbers (e, μ), consider the (affine) space of triangular n-periodic operators of order $k+1$ such that *all solutions* of the equation $(L - e)\Psi = 0$ are Bloch solutions with the same multiplier μ:

$$\mathscr{E}_{k+1,n}^{e,\mu} = \{L \in \mathscr{D}_{k+1,n} \mid (L-e)\Psi = 0 \Rightarrow \mu\Psi_i = \Psi_{i-n}\}. \tag{11}$$

The same argument as in [21] shows that this space is nonempty if and only if

$$\mu^{k+1} = (-1)^{kn} e^n. \tag{12}$$

Indeed, the operator L is monic. Therefore, if $\Psi^{(j)}$ is a basis in the space of solutions of (11), then $\det \Phi_{(i)} = (-1)^k e^{-1} \det \Phi_{(i-1)}$, where $\Phi_{(i)} := \{\Psi_{i-s}^{(j)}\}_{s=0,\dots,k}^{j=0,\dots,k}$. The monodromy operator T^{-n} restricted to the space spanned by $\Psi^{(j)}$ is the scalar μ. Therefore, the last relation implies (12).

Obviously, over the complex numbers, all admissible pairs (e, μ) are in the same orbit of the scaling transformation $(e, \mu) \to (c^{k+1}e, c^n\mu)$, $c \in \mathbb{C}$. Over the reals, it can be directly verified that if n and $k+1$ are coprime, then the parity of nk coincides with the parity of k and *any real admissible pair can be obtained by a real scaling from the pair* $(-1, (-1)^{n+k})$ determining superperiodic operators.

The scaling transformation introduced above reflects the scaling transformation of operators

$$\tau_c \colon \mathscr{D}_{k+1,n} \longmapsto \mathscr{D}_{k+1,n}, \qquad \tau_c(L) = c^{k+1} C L C^{-1}, \tag{13}$$

where C is the diagonal operator and $(C\psi)_i = c^i \psi_i$. Note that the multiplication by c^{k+1} of the gauge transformed operator CLC^{-1} is introduced in order to make the operator $\tau_c(L)$ monic.

In the theory of ordinary linear differential operators it is well known that if the kernel of an operator B is a subspace of the kernel of another operator A, $\ker B \subset \ker A$, then the latter is divisible by the former, i.e., there exists an operator C such that $A = BC$. This is also true for difference operators. It is easy to show that if an operator is superperiodic, then so is its adjoint. As a consequence, we obtain that the original defining characterization of superperiodic operators is equivalent to the following operator condition.

Lemma 1.3. *Let $L \in \mathscr{D}_{k+1,n}$ be a n-periodic triangular operator of order $k+1$. Then it is superperiodic if and only if the operator $T^{-n} - (-1)^{n+k}$ is right and left divisible by $L+1$, i.e., there exist operators L_r and L_l in $\mathscr{D}_{n-k-1,n}$ such that*

$$T^{-n} - (-1)^{k+n} = (L_l - (-1)^{k+n})(L+1) = (L+1)(L_r - (-1)^{k+n}). \tag{14}$$

In Section 3 we show that if the period n and the order $k+1$ of L are coprime, then the Bloch solutions of (8), i.e., solutions that are eigenvectors for the monodromy operator T^n,

$$w\psi_i = \psi_{i-n}, \tag{15}$$

coincide with the discrete analogue of the Baker–Akhiezer function defined on the spectral curve of L. This Bloch spectral curve is different from the spectral curves arising in the theory of the pentagram map but is relevant to the combinatorial Gale transform, as will be seen in what follows.

More precisely, using the identification of the Bloch solutions of (6) with the discrete Baker–Akhiezer function, we establish a connection of the spectral theory of triangular operators with the theory of rank-1 commuting difference operators developed in [15] and [23], and through this connection we clarify the defining constraint on the operators $L \in \mathscr{E}_{k+1,n}$ and the combinatorial Gale duality.

The appearance of commuting operators in the theory of superperiodic difference operators is a direct corollary of our first main result.

Theorem 1.4. *If n and $k+1$ coprime, then*

(i) *the operators L_r and L_l defined by relations* (14) *for a superperiodic operator L coincide, i.e., $L_r = L_l$;*

(ii) *the involution* (9) *transforms the operator $\mathscr{L} := L_r = L_l$ into an operator gauge equivalent to the operator $\mathscr{G}(L)$ Gale dual to L:*

$$\mathscr{G}(L) = \varepsilon \mathscr{L}^{\sigma} \varepsilon^{-1}. \tag{16}$$

The statement (i) of the theorem and Eqs. (14) imply the following assertion.

Corollary 1.5. *The operators L and $\mathscr{L}$ commute with each other*:

$$[L, \mathscr{L}] = 0.$$

As we see, the existence of a commuting operator is a necessary condition for a periodic triangular operator to be superperiodic. It turns out that this condition is necessary and sufficient for operators that can be obtained from superperiodic operators by the scaling transformation (13).

Theorem 1.6. *Suppose that, for an operator $L \in \mathscr{D}_{k+1,n}$ with $2(k+1) < n$, $(n, k+1) = 1$, there exists a commuting operator $K \in \mathscr{D}_{n-k-1,n}$. Then, in general position,*

(i) *the operator L is superperiodic up to the scaling transformation* (13) *with some constant c, i.e., $\tau_c(L) \in \mathscr{E}_{k+1,n}$;*

(ii) *there is a unique polynomial P such that the operator $\mathscr{L} = K + P(L)$ is superperiodic up to the scaling transformation* (13).

Here "general position" means "for a Zariski open set of parameters." The precise meaning will be given in the proof of the theorem in Section 4.

2. The Discrete Baker–Akhiezer Function

To begin with, let us recall basic facts of the theory of commuting rang-1 difference operators.

Let Γ be a smooth algebraic curve of genus g. Fix two points $p_\pm \in \Gamma$ and let $D = \gamma_1 + \dots + \gamma_g$ be a generic effective divisor on Γ of degree g. By the Riemann–Roch theorem one computes $h^0(D + i(p_+ - p_-)) = 1$ for any $i \in \mathbb{Z}$ and generic D. Let $\psi_i(p)$, $p \in \Gamma$, denote the unique section of this bundle. This means that ψ_i is the unique, up to a constant factor, meromorphic function such that (away from the marked points $p_\pm$) it has poles only at γ_s, and the multiplicity of each such pole is not greater than the multiplicity of γ_s in D, while at the point p_+ (p_-) the function ψ_i has a zero (respectively, a pole) of order i.

If we fix local coordinates z in neighborhoods of the marked points, then the Laurent series for $\psi_i(p)$ at these points have the form

$$\psi_i = z^{\pm i} \biggl(\sum_{s=0}^{\infty} \xi_s^{\pm}(i) z^s \biggr), \qquad z = z(p),\ p \to p_\pm.$$

The function $\psi := \psi_i(p)$ of the discrete variable i and the point p of the curve Γ is called the discrete Baker–Akhiezer function [15]. Throughout the paper it will be assumed that ψ is normalized by the condition $\xi_0^+ = 1$. Notice that, under the change of local coordinate $z \to \tilde{z} = cz + O(z^2)$ near the marked point p_+, the normalized Baker–Akhiezer function transforms into

$$\tilde{\psi}_i(p) = c^i \psi_i(p) \tag{17}$$

(compare with the scaling transformation (13)).

Let $\mathscr{A}(p_+, p_-)$ be the ring of meromorphic functions on Γ that are holomorphic away from the marked points $p_\pm$. The uniqueness of the Baker–Akhiezer function easily implies the following assertion.

Lemma 2.1 [15]. *For each $A \in \mathscr{A}(p_+, p_-)$, there exists a unique difference operator*

$$L_A = \sum_{j=-k_-}^{k_+} a_i^j T^{-j} \iff (L_A\psi)_i = \sum_{j=k_-}^{k_+} a_i^j \psi_{i-j}$$

such that

$$L_A \psi = A\psi. \tag{18}$$

Here $k_\pm$ are the orders of the poles of A at $p_\pm$.

Corollary 2.2. *The operators L_A commute with each other, i.e.,* $[L_A, L_B] = 0$.

The coefficients of the operator L_A are determined recursively from the system of equations obtained by substituting the Laurent expansion of ψ and A near the marked points. For example, if $A = c_0 z^{-k_+} + c_1 z^{-k_+ + 1} + \dots$ is the expansion of A near the points p_+, then

$$a_i^{k_+} = c_0, \quad a_i^{k_+ - 1} = c_0(\xi_1^+(i - k_+) - \xi_1^+(i)) - c_1, \quad \dots .$$

Similarly, if $A = c_0^- z^{-k_-} + \dots$ is the expansion of A near p_-, then

$$a_i^{-k_-} = \frac{c_0^- \xi_0^-(i)}{\xi_0^-(i + k_-)}, \quad \dots . \tag{19}$$

Remark 2.3. For further use note that if A is in the subring $\mathscr{A}_+(p_+, p_-) \subset \mathscr{A}(p_+, p_-)$ of meromorphic functions on Γ that have a pole only at p_+ and vanish at p_-, then the operator L_A is strictly lower triangular. In this case, the formula (19) for the leading coefficient a_i^1 takes the form

$$a_i^1 = e^{x_i - x_{i-1} + l}, \qquad x_n := \ln \xi_0^-(i),\ l := \ln c_0^-.$$

Remark 2.4. The correspondence described above extends to the case of singular curves. More precisely ([15], [23]), there is a natural correspondence

$$\mathscr{A} \longleftrightarrow \{\Gamma, p_\pm, \mathscr{F}\} \tag{20}$$

between commutative rings $\mathscr{A}$ of ordinary linear difference operators containing a pair of monic operators of coprime orders and sets of algebraic-geometric data $\{\Gamma, p_\pm, \mathscr{F}\}$, where Γ is an algebraic curve with a pair of distinguished smooth points and $\mathscr{F}$ is a torsion-free rank-1 sheaf on Γ such that

$$h^0(\Gamma, \mathscr{F}(ip_+ - ip_-)) = h^1(\Gamma, \mathscr{F}(ip_+ - ip_-)) = 0.$$

The correspondence becomes one-to-one if the rings $\mathscr{A}$ are considered modulo conjugation $\mathscr{A}' = g\mathscr{A}g^{-1}$, $g = \{g_i\}$ (in its final form the correspondence (20) is due to Mumford).

The discrete Baker–Akhiezer function and, therefore, the coefficients of the corresponding difference operators can be expressed in terms of the Riemann theta-function

$$\theta(z) := \theta(z|B) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i(m,z) + \pi i(Bm,m)}, \qquad z \in \mathbb{C}^g,$$

determined by the matrix B of b-periods of the normalized holomorphic differentials $d\omega_j$ on Γ:

$$B_{j,k} = \oint_{b_k} d\omega_j, \qquad \delta_{j,k} = \oint_{a_k} d\omega_j.$$

Lemma 2.5 [15]. *The Baker–Akhiezer function is given by the formula*

$$\psi_i(p) = r_i \frac{\theta(A(p) + iU + Z)\theta(A(p_+) + Z)}{\theta(A(p_+) + iU + Z)\theta(A(p) + Z)} e^{i\Omega_0(p)}. \tag{21}$$

Here

(a) $\Omega_0(p)$ *is the Abelian integral* $\int^p d\Omega_0$ *corresponding to the unique normalized* (*by* $\oint_{a_k} d\Omega_0 = 0$) *meromorphic differential on* Γ *which has simple poles at the marked point* $p_\pm$ *with residues* ± 1, *respectively*;

(b) $A(p)$ *is the Abel transform, i.e., a vector with coordinates* $A(p) = \int^p d\omega_k$;

(c) U *is the vector* $A(p_+) - A_(p_-)$;

(d) Z *is an arbitrary vector* (*it reflects the arbitrariness in the choice of the divisor of poles of the Baker–Akhiezer function*).

From (21) it easily follows that the coefficients of the commuting difference operators L_A are *quasi-periodic* functions of the variable i. The operators with periodic coefficients are singled out by the following constraints on the algebraic-geometric data.

Lemma 2.6. *Suppose that on* Γ *there exists a meromorphic function* $w = w(p)$ *which is holomorphic away of the marked point* p_+, *where it has a pole of order* n, *and has a zero of order* n *at the marked point* p_-. *Then the coefficients of the operators* L_A *are* n*-periodic.*

Proof. The function w, if exists, is unique up to multiplication by a constant. If the first jet of a local coordinate near the point p_+ is fixed, then w can be normalized by the condition $w = z^{-n} + O(z^{-n+1})$. The uniqueness of ψ_i implies then that relation (15) holds. Hence the L_A are n-periodic.

Real operators. The formula (21) for generic $i \in C$ defines $\psi_i(p)$ as a multivalued function on Γ. A single-valued branch of it can be defined on Γ with a cut between the marked points which does not intersect the chosen basis of a- and b-cycles. The coefficients a_i^j of the corresponding operators can be seen as *meromorphic* functions of the complex variable i. The spectral data corresponding to operators with real (for $i \in \mathbb{R}$) coefficients are singled out as follows.

Lemma 2.7. *Suppose that on an algebraic curve* Γ *with two marked points* $p_\pm$ *there exists an antiholomorphic involution* τ *for which the marked points are fixed, i.e.,* $\tau(p_\pm) = p_\pm$. *Then the Baker–Akhiezer function* ψ *corresponding to a real divisor* $D = \tau(D)$ *satisfies the relation*

$$\psi(p) = \bar{\psi}(\tau(p)) \tag{22}$$

(*provided that the coordinate* z *in a neighborhood of* p_+ *used for the normalization of* ψ *is also real*: $\bar{z} = \tau^* z$).

Equation (22) implies that if the function A is real, i.e., $A = \bar{A}(\tau(p))$, then the corresponding operator L_A has real coefficients. Still these coefficients may be singular functions of the variable $i \in \mathbb{R}$. In the framework of the finite-gap theory there are two basic types of conditions sufficient for the regularity of the corresponding operators. Let us present one of them, which is relevant to the case under consideration.

Recall that an antiholomorphic involution τ of a genus-g smooth algebraic curve Γ has at most $g + 1$ fixed ovals. The curves having the maximum number of fixed ovals, $A_0, \ldots, A_g$, are called *M-curves*.

Lemma 2.8. *Let* Γ *be an* M*-curve with two marked points* $p_\pm \in A_0$, *and let* D *be a set of* g *points* $\gamma_s \in A_s$, $s = 1, \ldots, g$. *Then, for any real function* $A = \bar{A}(\tau(p)) \in \mathscr{A}(p_+, p_-)$, *the coefficients of the operators* L_A *corresponding to the Baker–Akhiezer function constructed from* D *are real and* *nonsingular. Moreover, their leading coefficients* $a_i^{-k_-}$ *are sign definite*:

$$a_i^{k_+} = c_0, \qquad \operatorname{sgn}(a_i^{-k_-}) = \operatorname{sgn} c_0^-.$$

The proof of the lemma is standard in finite-gap theory (see, for example, the proof of a similar statement for the difference Schrödinger operator in [16]).

In what follows, we shall use one more basic concept of algebraic-geometric integration theory.

The dual Baker–Akhiezer function. The concept of the dual Baker–Akhiezer function ψ^+ is universal and is at the heart of the notion of Hirota's bilinear form of soliton equations. In the discrete case, the dual Baker–Akhiezer function corresponds to the dual divisor $D^+ = \gamma_1^+ + \dots + \gamma_g^+$, which is defined by the equation

$$D + D^+ = K + p_- + p_+ \in J(\Gamma),$$

where K is the canonical class. In other words, the points of D and D^+ are zeros of the meromorphic differential $d\Omega$ with simple poles at $p_\pm$ with residues ± 1, respectively. The dual Baker–Akhiezer function is then defined by the following analytic properties:

(i) the function ψ_i^+ (as a function of the variable $p \in \Gamma$) is meromorphic everywhere except at the points $p_\pm$ and has at most simple poles at the points $\gamma_1^+, \dots, \gamma_g^+$ (if all of them are distinct);

(ii) in a neighborhood of the points $p_\pm$ the function ψ has the form

$$\psi_i^+ = z^{\mp i}\left(\sum_{s=0}^{\infty} \chi_s^\pm(i) z^s\right), \qquad z = z(p),\ p \to p_\pm,\ \chi_0^+ = 1.$$

In fact, this is the same Baker–Akhiezer-type function, and therefore it admits an explicit theta-function expression of the same type:

$$\psi_i^+(p) = r_i^+ \frac{\theta(A(p) - iU - Z - A(p_+) - A(p_-))\,\theta(A(p_-) + Z)}{\theta(A(p_-) + iU + Z)\,\theta(A(p) - Z - A(p_+) - A(p_-))}\, e^{-i\Omega_0(p)}.$$

Lemma 2.9. *Let* ψ *and* ψ^+ *be the Baker–Akhiezer function and its dual. Then*

$$\operatorname{res}_{p_+}(\psi_i^+ \psi_j)\, d\Omega = \delta_{i,j}, \qquad i, j \in \mathbb{Z}. \tag{23}$$

By the definition of duality, the differential on the left-hand side of (23) for $i \neq j$ has a pole only at *one* of the marked points $p_\pm$. Hence, its residue vanishes. The differential $\psi_i^+ \psi_i\, d\Omega$ has poles at p_+ and p_- with residues ± 1, respectively. This completes the proof of the lemma.

Corollary 2.10. *Let* ψ *be the Baker–Akhiezer function, and let* L_A *be the linear operator determined by* (18) *with this* ψ. *Then the dual Baker–Akhiezer function is a solution of the formal adjoint equation*

$$\psi^+ L_A = A\psi^+.$$

Recall that the right action of a difference operator is defined as the formal adjoint action, i.e., $f^+ T = T^{-1} f^+$.

3. The Bloch–Floquet Theory of Triangular Periodic Difference Operators

Let us briefly recall the conventional setting of the spectral theory of periodic difference operators. Consider an n-periodic linear difference operator L. The monodromy operator T^{-n} preserves the $(k+1)$-dimensional space $\mathscr{L}(E)$ of solutions of the equation $L\psi = E\psi$. Let $T^{-n}(E)$ be the restriction of T^{-n} to $\mathscr{L}(E)$. The common eigenvalues of L and the monodromy operator satisfy the algebraic equation

$$R(w, E) = \det(w \cdot 1 - T^{-n}(E)) = 0.$$

The same equation can be obtained by considering the n-dimensional space $\mathscr{T}(w)$ of solutions of Eq. (15) and denoting the restriction of L to $\mathscr{T}(w)$ by $L(w)$. Then

$$R(w, E) = \det(L(w) - E \cdot 1) = 0.$$

Our first observation, which turns out to be crucial for the further considerations, is as follows.

Lemma 3.1. *If n and $k+1$ are coprime, then the spectral curve of an operator $L \in \mathscr{D}_{k+1,n}$ is defined by an equation of the form*

$$R(w,E) = w^{k+1} - E^n + \sum_{i>0,\, j\geqslant 0,\, ni+(k+1)j<n(k+1)} r_{ij} w^i E^j = 0 \tag{24}$$

with

$$r := r_{1,0} = \prod_{j=1}^{n} a_j^1. \tag{25}$$

Remark 3.2. Lemma 3.1 asserts that the Newton polygon of R is the triangle with vertices $(0,n)$, $(k+1,0)$, and $(1,0)$ and gives the explicit form of the coefficients corresponding to the vertices. Before presenting a proof of the lemma, let us elaborate on the specifics of the algebraic curves defined by Eq. (24).

From (24) it follows that the affine curve defined by the equation $R(w,E)=0$ at infinity is compactified by one *smooth* point p_+. At this point the function $w=w(p)$, $p\in\Gamma$, has a pole of order n and $E=E(p)$ has a pole of order $k+1$. Notice that (24) implies also that if $w=0$, then $E=0$. Moreover, if $r\neq 0$, then one branch of the multivalued function $w(E)$ defined by (24) near the origin is of the form

$$w = r^{-1}E^n\left(1+\sum_{s=1}^{\infty} v_s E^s\right).$$

Hence the curve Γ has another marked *smooth* point p_- at which w has a zero of order n and E has a simple zero. The degrees of the zero and pole divisors are equal; therefore, w does not vanish on $\Gamma\setminus p_-$.

Let $\mathscr{S}_{k+1,n}$ be the family of curves Γ defined by equations of the form (24). Its dimension (the number of parameters, that is, of the coefficients r_{ij}) equals $k(n+1)/2$. For generic values of r_{ij}, the curve Γ is smooth and has genus $g=k(n-1)/2$. Let $\mathscr{P}_{k+1,n}$ be the Jacobian bundle over an open subspace of smooth curves in $\mathscr{S}_{k+1,n}$. A generic point $\{\Gamma, D\}\in\mathscr{P}_{k+1,n}$ of this bundle defines a unique Baker–Akhiezer function ψ and, hence (by Lemma 2.1), the operator L_E corresponding to the function $E=E(p)$. This function has a pole of order $k+1$ at p_+ and vanishes at p_-. Hence L_E is strictly lower triangular. Moreover, Corollary 2.6 implies that L_E is n-periodic. The established correspondence

$$\mathscr{P}^0_{k+1,n} \subset \mathscr{P}_{k+1,n} \longmapsto \mathscr{D}_{k+1,n} \tag{26}$$

of an open domain will be referred to as *the inverse spectral transform.*

Example. For $k=1$ and $n=2m+1$, the family $\mathscr{S}$ is the family of hyperelliptic curves defined by the equation

$$w^2 + Q_m(E)w - E^{2m+1} = 0,$$

where Q_m is a polynomial of degree m.

Proof of Lemma 3.1. By definition, the polynomial R is of degree $k+1$ in w and of degree n in E. The matrix $L(w)$ has $k+1$ nonzero diagonals below the main diagonal and $k+1$ diagonals in the upper right corner. The entries of the latter contain w as a multiplier. Hence $R(0,E)=E^n$, and the summation in (24) is over $i>0$. The coefficient $r_{1,0}$ is the product of the entries of the first above the main diagonal and the coefficient of w in the lower left corner of the matrix. This proves (25).

It remains only to prove that the summation in (24) is over the pairs of indices (i,j) such that $ni+j(k+1)<n(k+1)$. For further use, we present the proof of this statement, which goes along the same lines as in the theory of commuting differential operators (see details in [17]).

Lemma 3.3. *Let L be in $\mathscr{D}_{k+1,n}$. If n and $(k+1)$ are coprime, then there exists a unique formal series*

$$E(z) = z^{-(k+1)}\left(1+\sum_{s=1}^{\infty} e_s z^s\right) \tag{27}$$

such that the equation $L\psi = E\psi$ has the unique formal solution

$$\psi_i(z) = z^i\left(1 + \sum_{s=1}^{\infty}\xi_s^+(i)z^s\right) \tag{28}$$

with periodic coefficients

$$\xi_s^+(i) = \xi_s^+(i+n) \tag{29}$$

normalized by the condition $\xi_s^+(0) = 0$.

Proof. The substitution of (28) and (27) into the equation $L\psi = E\psi$ gives a system of difference equations for the unknown coefficients e_s and ξ_s of the series. The first of them is the equation

$$e_1 + \xi_1^+(i) - \xi_1^+(i-k-1) = a_i^k. \tag{30}$$

The periodicity constraint (29) uniquely defines $e_1 = n^{-1}\sum_{i=1}^n a_i^k$ and reduces the difference equation (30) of order $k+1$ to a difference equation of order 1, namely,

$$me_s + \xi_1^+(i) - \xi_1^+(i-1) = \sum_{j=0}^{m-1} a_{i-j(k+1)}^k, \tag{31}$$

where m is the positive integer smaller than n such that $m(k+1) = 1 \ (\operatorname{mod} n)$. Equation (31) and the initial condition $\xi_1^+(0) = 0$ uniquely define $\xi_1^+(i)$.

For arbitrary s, the defining equation for e_s and ξ_s^+ has the form

$$e_s + \xi_s^+(i) - \xi_s^+(i-k-1) = Q_s(e_1, \ldots, e_{s-1}; \xi_1, \ldots, \xi_{s-1}),$$

where Q_s is an explicit function linear in $e_{s'}$ and $\xi_{s'}$, $s' < s$, and polynomial in a_i^j. The same argument as above shows that this equation has a unique periodic solution. This proves the lemma.

By definition, the series $E(z)$ is formal. It turns out that it is a convergent series and coincides with the Puiseuz series of the function E regarded as an analytic function on the algebraic curve determined by Eq. (24) near infinity. More precisely, an argument identical to that in [14] (see details in [17]) proves the following lemma.

Lemma 3.4. *Let $E(z)$ be the formal series defined in Lemma* 3.3; *then the characteristic polynomial $R(w, E)$ is equal to*

$$R(w, E) = \prod_{j=1}^{n}(E - E(z_j)), \qquad z_j^{-n} = w. \tag{32}$$

The right-hand side of (32) is a symmetric function of the variables z_j. Hence, a priori, it is a polynomial in the variable E and a (formal) Laurent series in w^{-1}. Equation (32) states that this is a polynomial in w. It also implies that the degree in z of all terms in $R(w, E)$ except w^{k+1} and E^n is strictly less than $n(k+1)$. This proves Lemma 3.1.

In a similar way we describe the Bloch solution near the second marked point.

Lemma 3.5. *If L is a strictly lower triangular operator* (*not necessarily periodic*), *then the equation $L\psi = E\psi$ has a unique formal solution of the form*

$$\psi_i(E) = e^{x_i}E^i\left(1 + \sum_{s=1}^{\infty}\xi_s^-(i)E^s\right), \qquad e^{x_i - x_{i-1}} := a_i^1, \tag{33}$$

normalized by the condition $\xi_s^-(0) = 0$.

Proof. The substitution of (33) into (8) gives a system of equations for the unknown coefficients ξ_s^-. They are nonhomogeneous first-order linear difference equations

$$\xi_s^-(i) - \xi_s^-(i-1) = q_s(\xi_1^-, \ldots, \xi_{s-1}^-),$$

which, together with the initial conditions, recursively define the $\xi_s^-(i)$ for all i.

The uniqueness of the formal solution (33) implies the following assertion.

Corollary 3.6. *If $L \in \mathscr{D}_{k+1,n}$, then the formal series* (33) *is the Bloch solution, i.e., it satisfies* (15) *with*

$$w(E) = \psi_n(E) = E^n \sum_{s=0}^{\infty} w_s E^s.$$

The coordinates of the Bloch solution $\psi_i = \psi_i(p)$, $p = (w, E) \in \Gamma$, normalized by the condition $\psi_0 \equiv 0$ are rational functions of w and E. A standard argument of the theory of finite-gap integration shows that the poles of ψ_i do not depend on i. In order to find the degree of the pole divisor D, it is enough, as usual, to consider the matrix $F(E)$ with entries $F^{ij} = \psi_i(p_j(E))$, $0 \leqslant i \leqslant k$, where the $p_j(E) = (w_j, E)$, $1 \leqslant j \leqslant k+1$, are the preimages of E on Γ under the projection map $p \in \Gamma \to E(p) \in \mathbb{C}$. The matrix F depends on the ordering of the preimages, but the function $f(E) = \det^2 F(E)$ does not, i.e., f is a rational function of E. It has double poles at the projections of the poles of ψ (if they are distinct) and a pole of order $2k$ at $E = 0$. If Γ is smooth, then f has zeros at the images of the finite branching points of the cover $E\colon \Gamma \to \mathbb{C}$. The multiplicity of the zero of f then equals the multiplicity of the branching. The infinity point p_+ is a branching point of the cover $\Gamma \to \mathbb{C}$ of multiplicity k. From (28) it follows that at $E = \infty$ the function f has a zero of order k. The degrees of the pole and zero divisors of a rational function coincide. Hence $2 \deg D + 2k = \nu$, where ν is the total multiplicity of the branch points. The cover $\Gamma \to \mathbb{C}$ is of degree $k+1$. Hence, by the Hurwitz formula, the genus of Γ (if it is smooth) is equal to $2g = \nu - 2k$. Therefore, $\deg D = g$, and we have proved that the Bloch solutions on smooth spectral curves coincide with the discrete Baker–Akhiezer function.

Remark 3.7. The coefficients r_{ij} of the characteristic equation (24) are polynomial functions of the coefficients of the operator $L \in \mathscr{D}_{k+1,n}$. The straightforward proof that they are independent is highly nontrivial, and the author is not aware of any universal approach to this problem except the combined use of the direct and inverse spectral transforms. Under this approach, in order to prove their independence, it is enough to show that the image of the correspondence $\mathscr{D}_{k+1,n} \to \mathscr{S}_{k+1,n}$ contains at least one smooth spectral curve. This is established by constructing the inverse map (26).

Summarizing the results presented above, we obtain the following statement.

Theorem 3.8. *If n and $k+1$ are coprime, then the map* (26) *is a one-to-one correspondence between a Zariski open subset of the Jacobian bundle over the family of spectral curves given by* (24) *and an open subset in the space $\mathscr{D}_{k+1,n}$ of n-periodic lower triangular difference operators of order $k+1$.*

4. Superperiodic Difference Operators

In this section we consider the spectral theory of superperiodic operators. The spectral curve Γ_{spec} of an operator $L \in \mathscr{E}_{k+1,n}$ is never smooth. The definition of $\mathscr{E}_{k+1,n}$ implies that the point $p_0 = (w = (-1)^{n+k}, E = -1) \in \Gamma_{\mathrm{spec}}$ is a multiple point of order $k+1$ (at which all sheets of the cover $E\colon \Gamma_{\mathrm{spec}} \to \mathbb{C}$ intersect transversally). This condition requires the vanishing of the coefficients of the Tailor expansion of R at p_0 of degree less than k, which gives $(k+1)(k+2)/2$ linear equations on the coefficients of R. There is one relation between these equations, because if $w = (-1)^{n+k}$ is a root of the equation $R(w, -1) = 0$ of multiplicity *at least* k, then it is of multiplicity $k+1$. Hence the space $\mathscr{S}_{\mathrm{spec}}$ of the spectral curves of operators $L \in \mathscr{E}_{k+1,n}$ is of dimension $k(n+1)/2 - (k+1)(k+2)/2 + 1 = k(n-k-2)/2$.

Let $\Sigma_{k+1,n}$ be the space of algebraic curves Γ obtained by a partial normalization $\pi\colon \Gamma \to \Gamma_{\mathrm{spec}}$ of spectral curves resolving the multiple point p_0; this means that the map π is one-to-one except at $k+1$ smooth point $p^j \in \Gamma$, $j = 1, \dots, k+1$, that are the preimages of p_0, i.e., $\pi(p^j) = p_0$. In other words, a smooth curve Γ in $\Sigma_{k+1,n}$ is characterized by the following properties:

(i) *on Γ there exists a meromorphic function w having one pole at p_+ of order n and having a zero of order n at the other distinguished point p_-;*

(ii) *on Γ there exists a meromorphic function E having one pole at p_+ of order $k+1$ and such that the zeros p^j of the function $E+1$ are distinct and, moreover, at p^j the function $w - (-1)^{k+n}$*

also vanishes:

$$E(p^j) = -1, \quad w(p^j) = (-1)^{k+n}, \qquad j = 1, \dots, k+1.$$

Under the normalization π the arithmetic genus of $\Gamma_{\rm spec}$ decreases by $k(k+1)/2$. Hence, for an open set in $\Sigma_{k+1,n}$ corresponding to spectral curves $\Gamma_{\rm spec}$ for which p_0 is the only singularity, the corresponding curve Γ is smooth and of genus $g = k(n-1)/2 - k(k+1)/2 = k(n-k-2)/2$.

Consider the preimage $\pi^*(\psi)$ (for which we will keep the same notation ψ) of the Bloch function under the normalization map. It has the same expansions (28) and (33) at the marked points $p_\pm \in \Gamma$. The function $E(p)$, $p \in \Gamma$, defines a cover $\Gamma \to \mathbb{C}$ which has $k+1$ sheets over the point $E = -1$. The same counting as above gives that ψ has g poles on Γ. Hence we obtain the following statement.

Theorem 4.1. *For a Zariski open set of operators $L \in \mathscr{E}_{k+1,n}$, the corresponding Bloch solutions ψ of the equation $L\psi = E\psi$ are parameterized by the points p of a smooth algebraic curve $\Gamma \in \Sigma_{k+1,n}$. Moreover, the function $\psi_i(p)$ is the Baker–Akhiezer function.*

Notice that the evaluation of (21) at the points $p^j \in \Gamma$ gives explicit theta-functional expressions for the basis $\Psi^{(j)} := \psi(p^j)$ of the solutions of the equation $(L+1)\Psi = 0$.

Corollary 4.2. *The inverse and direct spectral transforms establish a one-to-one correspondence*

$$\mathscr{P} \iff \mathscr{E}_{k+1,n}$$

between an open set in the Jacobian bundle $\mathscr{P}$ over the subspace of smooth curves in $\Sigma_{k+1,n}$ and an open set of superperiodic operators.

Now we are ready to prove Theorem 1.4.

Proof. Consider the function

$$G = \frac{w - (-1)^{n+k}}{E+1} + (-1)^{n+k}$$

on $\Gamma \in \Sigma_{k+1,n}$. At the zeros p^j of the denominator the numerator vanishes as well. Hence $G(p)$ is holomorphic on $\Gamma \setminus p_+$. At the marked point p_+ the function G has a pole of order $n-k-1$. It vanishes at p_-. By Lemma 2.1 there exists an operator $\mathscr{L} = L_G \in \mathscr{D}_{n-k-1,n}$ such that $\mathscr{L}\psi = G\psi$, where ψ is the Bloch solutions of the equation $L\psi = E\psi$. From the definition of G it follows that the operator equation

$$T^{-n} - (-1)^{k+n} = (\mathscr{L} - (-1)^{k+n})(L+1) = (L+1)(\mathscr{L} - (-1)^{k+n})$$

holds. This proves the part (i) of the theorem.

The function $w - (-1)^{k+n}$ has n zeros on Γ, $k+1$ of which are the points p^j. Let q^m, $m = 1, \dots, n-k-1$, be the set of the remaining zeros. At these points $G(q^m) = (-1)^{k+n}$. Hence $\mathscr{L} \in \mathscr{E}_{n-k-1,n}$.

In order to prove the last statement of the theorem, it remains only to prove relation (16). The latter is a direct corollary of the following statement.

Lemma 4.3. *Let $\widehat{\Psi}_{(i)}$ be the $n \times (k+1)$ matrix with columns*

$$(\psi_{i+1}(p^j), \dots, \psi_{i+n}(p^j)),$$

and let $\widehat{\Psi}^+_{(i)}$ be the $(n-k-1) \times n$ matrix with rows

$$(\psi^+_{i+1}(q^m), \dots, \psi^+_{i+n}(q^m)),$$

where ψ and ψ^+ are the Baker–Akhiezer function and its dual on $\Gamma \in \Sigma_{k+1,n}$ and the points $\{p^j, q^m\}$ are zeros of the function $w - (-1)^{k+n}$. Then the orthogonality relation

$$\widehat{\Psi}^+_{(i)} \widehat{\Psi}_{(i)} = 0 \tag{34}$$

holds for all i.

Proof. The relation (34) is a corollary of the orthogonality of the eigenvectors and covectors of a linear operator which correspond to different eigenvalues:

$$E(q^m)\langle\psi^+(q^m),\psi(p^j)\rangle = \langle\psi^+(q^m), L\psi(p^j)\rangle = E(p^j)\langle\psi^+(q^m),\psi(p^j)\rangle, \tag{35}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the pairing between vectors and covectors. By definition $E(p^j) = -1$. The function E is of degree $k+1$. Hence it does not equal -1 at any other point of Γ. This implies $E(p^j) \neq E(q^m)$. Thus, (35) holds only if

$$\langle\psi^+(q^m)\psi(p^j)\rangle = \sum_{s=1}^{n}\psi^+_{i+s}(q^m)\psi_{i+s}(p^j) = 0,$$

which is equivalent to (34).

For $L \in \mathscr{E}_{k+1,n}$, Eq. (34) coincides with the duality relation (3) between solutions of the equations corresponding to the operators $L = L_E$ and $\varepsilon L^\sigma_G \varepsilon^{-1}$. This proves the part (ii) of Theorem 1.4.

Proof of Theorem 1.6. The direct spectral transform (20) for commuting difference operators is constructed along the same lines as in the case of the direct spectral transform for periodic operators. Let us briefly recall some necessary steps of the construction.

Let $\mathscr{D}_{k+1}$ be the affine space of monic strictly triangular operators of order $k+1$ (not necessarily periodic). Consider a pair $L \in \mathscr{D}_{k+1}$ and $K \in \mathscr{D}_{n-k-1}$ with $[L, K] = 0$. The restriction of the operator K to the $(k+1)$-dimensional space of solutions of the equation $(L - E)\psi = 0$ defines a finite-dimensional linear operator $K(E)$.

Lemma 4.4 [15]**.** *If n and $k+1$ are coprime, then the characteristic equation of the operator $K(E)$ has the form*

$$\mathscr{R}(\kappa, E) = \kappa^{k+1} - E^{n-k-1} + \sum_{(i,j)\in I}\rho_{ij}\kappa^i E^j = 0, \tag{36}$$

where the summation is over the set of pairs of nonnegative integers (i, j) such that $0 < i(n-k-1)+j(k+1) < (n-k-1)(k+1)$. For commuting pairs in general position (when the spectral curve Γ defined by (36) is smooth), the common eigenfunction ψ of the commuting operators L and K, for which $L\psi = E\psi$ and $K\psi = \kappa\psi$, $(\kappa, E) \in \Gamma$, is the discrete Baker–Akhiezer function.

The number of coefficients in (36) is $(k+2)(n-k)/2 - 2$. For the generic values of these coefficients, the corresponding curve Γ is smooth and has genus $g = k(n-k-2)/2$. The two marked points on Γ are the infinity point p_+, where E and κ have poles of orders $k+1$ and $n-k-1$, respectively, and the point p_-, where these two functions vanish. If Γ is smooth, then the ring $\mathscr{A}_+$ of meromorphic functions on Γ having pole only at p_+ and vanishing at p_- is generated by E and κ. If $2k+2 < n$, then the function E can be characterized as a generator of $\mathscr{A}_+$ with the lowest order of the pole at p_+. This condition determines it uniquely up to multiplication by a constant: $E' = c^{k+1}E$. The second generator κ of the ring is unique up to the transformations

$$\kappa' = c^{n-k}\kappa - \sum_{s=1}^{[n/(k+1)]-1}\alpha_s E^s,$$

which preserve the order of the pole at p_+ and the form of Eq. (36).

As mentioned above, the generic pairs of commuting operators have quasi-periodic coefficients. By the assumption of the theorem the operator L is n-periodic, i.e., $[L, T^{-n}] = 0$. This implies that the ring $\mathscr{A}_+$ contains a function w which has a pole of order n at p_+ and a zero of order n at p_-. Because A_+ is generated by E and κ, this function can be represented in the form

$$w = \kappa E - e\kappa + Q(E), \tag{37}$$

where e is a constant and Q is a polynomial of degree d such that $d(k+1) < n$ (recall that, by assumption, $2(k+1) < n$) and vanishing at $E = 0$.

Relation (37) implies that the function w takes the same value $\mu := Q(e)$ at the zeros p^j of the function $E - e$. In general position these zeros are simple. Therefore, the evaluation of the

Baker–Akhiezer at these points gives a basis of solutions of the equation $(L-e)\Psi = 0$ having the same monodromy multiplier μ.

Remark 4.5. In the statement of the theorem, "general position" means precisely that $\{p^j\}$ is a set of $k+1$ distinct points.

The scaling of E and κ by a constant c corresponds to the scaling transformation (13) of operators. By a proper choice of the scaling constant c the constant e in (37) can be transformed into $e = -1$. Hence the operator L is superperiodic up to the scaling transformation. This proves the first statement of the theorem.

To prove the second statement it is enough to note that the polynomial $Q(E) - Q(e)$ is always divisible by $E - e$, i.e., there is a unique polynomial $P(E)$ without constant term, $P(0) = 0$, such that $Q(E) - \mu = (E-e)(P(E) + e^{-1}\mu)$ holds. Then from (37) we obtain the equation

$$w(p) - \mu = (E(p) - e)(\lambda(p) + e^{-1}\mu), \qquad \lambda(p) = \kappa + P(E).$$

This implies that the operator $\mathscr{L} = K + P(L)$ is superperiodic up to the scaling transformation, which completes the proof of the theorem.

Remark 4.6. Throughout the paper we considered mostly operators with complex coefficients, but all constructions and results admit a simple real reduction. If the coefficients of the operator L are real, then so are the coefficients of the characteristic polynomials. Therefore, complex conjugation defines an antiholomorphic involution τ of the corresponding spectral curves. Equation (22) implies that the divisor D is invariant under the involution $\tau(D) = D$. Hence the direct and the inverse spectral transform establish one-to-one correspondence between the space of generic operators with real coefficients and an open part of the space of a bundle over the space of *real* spectral curves whose fiber is the real locus of the corresponding Jacobian. Note that the bundle over the space of M-curves whose fiber is the real locus of the Jacobian (the divisors D described in Lemma 2.8) corresponds to operators L with sign-definite coefficients a_j^1.

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Department of Mathematics, Columbia University
Laboratory of Representation Theory and Mathematical Physics,
National Research University Higher School of Economics
Institute for Information Transmission Problems,
Russian Academy of Sciences
Landau Institute for Theoretical Physics,
Russian Academy of Sciences
e-mail: krichev@math.columbia.edu

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