INTEGRABLE DISCRETE SCHRÖDINGER EQUATIONS AND A CHARACTERIZATION OF PRYM VARIETIES BY A PAIR OF QUADRISECANTS

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Abstract

We prove that Prym varieties are characterized geometrically by the existence of a symmetric pair of quadrisecant planes of the associated Kummer variety. We also show that Prym varieties are characterized by certain (new) theta-functional equations. For this purpose we construct and study a difference-differential analog of the Novikov-Veselov hierarchy.

Contents

1. Introduction

The problem of characterizing the locus $\mathcal{P}_g$ of Prym varieties in the moduli space $\mathcal{A}_g$ of all principally polarized abelian varieties (ppav's) is well known and has attracted a lot of interest over the years. Geometrically, Prym varieties may in some sense be the easiest ppav to understand beyond Jacobians, and one could hope that studying them would be a first step toward understanding the geometry of more general abelian varieties as well.

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Recall that Fay's trisecant formula [10] is the statement that the Kummer image of a Jacobian variety of a curve admits a 4-dimensional family of trisecant lines. In [12], Gunning obtained a solution to the classical Riemann-Schottky problem of characterizing Jacobians among all ppav's by showing that the existence of a 1-dimensional family of trisecant lines of the Kummer variety characterizes Jacobians. Gunning's characterization of the Jacobian locus was extended by Welters, who proved that the Jacobian locus can be characterized by the existence of a formal 1-parametric family of flexes of the Kummer variety (see [32], [33]) (recall that a flex of a variety is a line tangent to it with multiplicity 3). The flexes arise as the limiting case of the trisecant when the three intersection points converge.

In [1], Arbarello and De Concini showed that Welters's characterization is equivalent to an infinite system of partial differential equations representing the so-called Kadomtsev-Petvishvili (KP) hierarchy, and they proved that only a finite number of these equations is sufficient. They thus established a relation of the geometric approach to the Schottky problem with the integrable systems approach, in which a much stronger characterization of the Jacobian locus was earlier conjectured by Novikov in the framework of the soliton theory, providing that Jacobians are characterized by the property of their theta functions to provide explicit solutions of the KP equation. Novikov's conjecture is equivalent to the statement that the Jacobians are characterized by the existence of a length 3 formal jet of flexes to the Kummer variety.

Welters, inspired by Gunning's theorem and Novikov's conjecture, proved later by Shiota [28], formulated in [33] the following still stronger conjecture: that the existence of *one* trisecant (or one flex, or one semidegenerate trisecant) in fact already characterizes Jacobians. Welters's conjecture was recently proved by Krichever in [18] and [19].

Prym varieties possess generalizations of some properties of Jacobians. Beauville and Debarre [3] and Fay [11] showed that the Kummer images of Prym varieties admit a 4-dimensional family of quadrisecant planes (as opposed to a 4-dimensional family of trisecant lines for Jacobians). Similarly to the case of Jacobians, it was then shown by Debarre in [6] that the existence of a 1-dimensional family of quadrisecants characterizes Prym varieties among all ppav's. However, Beauville and Debarre in [3] constructed a ppav that is *not* a Prym but such that its Kummer image *has* a quadrisecant plane. Thus, no analog of the trisecant conjecture for Prym varieties was conjectured, and the question of characterizing Prym varieties by a finite amount of geometric data (i.e., by polynomial equations for theta functions at a finite number of points) remained completely open.

From the point of view of integrable systems, attempts to prove the analog of Novikov's conjecture for the case of Prym varieties of algebraic curves with two smooth fixed points of involution were made in [30], [29], and [3]. In [30] it was shown that the Novikov-Veselov (NV) equation provides a solution of the characterization problem up to the possible existence of additional irreducible components. In [29] and [3], the characterizations of Prym varieties in terms of BKP and NV equations were proved only under certain additional assumptions. In [3], moreover, an example of a ppav that is not a Prym, but for which the theta function gives a solution to the BKP equation, was constructed, and thus no analog of Novikov's conjecture for Prym varieties was made. Recently, Krichever [17] proved that Prym varieties of algebraic curves with two smooth fixed points of involution are characterized among all ppav's by the property of their theta functions to provide explicit formulas for solutions of the integrable 2-dimensional Schrödinger equation.

The main goal of this article is to prove that Prym varieties are characterized among all ppav's by the property of their Kummer images admitting a *symmetric pair of quadrisecant* 2*-planes* (see the statement of Corollary 1.2 for a precise formulation). That there exists such a symmetric pair of quadrisecant planes for the Kummer image of a Prym variety can be deduced from the description of the 4-dimensional family of quadrisecants, using the natural involution on the Abel-Prym curve. However, the statement that a symmetric pair of quadrisecants in fact characterizes Pryms seems completely unexpected.

Our geometric characterization of Prym varieties follows from a characterization of Prym varieties among all ppav's by some theta-functional equations (see the statement of Theorem 1.1 for a precise formulation), which, by using Riemann's bilinear addition theorem, can be shown to be equivalent to the existence of a symmetric pair of quadrisecant planes. In order to obtain such a characterization of Prym varieties, we introduce, develop, and study a *new* hierarchy of difference equations, starting from a discrete version of the Schrödinger equation. The hierarchy we construct can be thought of as a discrete analog of the Novikov-Veselov hierarchy.

The structure of this article is as follows. In Section 2, we give a brief overview of notation and of the main results. In Section 3, we give an analytic proof (not using the results of Beauville-Debarre and Fay) of the fact that Kummer images of Prym varieties admit a symmetric pair of quadrisecant planes. This is done by constructing (using algebro-geometric techniques) new difference potential Schrödinger operators that play a crucial role in all our further considerations. Our construction is a discrete version of the well-known Novikov-Veselov construction [31] of potential 2-dimensional Schrödinger operators (the latter is a reduction of a more general construction of Schrödinger operators in a magnetic field first proposed in [9]). In Section 4, we introduce a discrete analog of the Novikov-Veselov hierarchy and study its properties. It is a set of difference-differential equations describing integrable deformations of potential difference Schrödinger operators.

In Section 5, we construct a wave solution of the discrete hierarchy constructed in Section 4. While the idea of constructing such a solution is in a way inspired by the success of constructions in [17], [18], and [19], there is no a priori reason why such a solution of the discrete Schrödinger hierarchy can be constructed assuming only the existence of a symmetric pair of quadrisecant planes as the first step. The hierarchy we consider involves essentially a pair of functions and is thus essentially a matrix hierarchy, unlike the scalar hierarchy arising for the trisecant case. The argument is very delicate, and involves using the pair of quadrisecant conditions to recursively construct a pair of auxiliary solutions (essentially corresponding to the two components of the kernel, only one of which is the Prym).

In Section 6, we finish the proof of our main result on the characterization of the Prym varieties associated to unramified double covers. One very important detail here is that, unlike the Jacobian case, the Prym variety remains compact under certain degenerations of the curve. No characterization of Prym varieties given in terms of the period matrix of the Prym differentials can single out the possibility of such degenerations, and thus our characterization is of the closure $\overline{\mathcal{P}_g}$ in $\mathcal{A}_g$ of the locus $\mathcal{P}_g$ of Pryms, not only of $\mathcal{P}_g$ itself (see Remark 3.10 for more details).

2. Statement of results

Let B be an indecomposable complex symmetric matrix with positive definite imaginary part. It defines an indecomposable ppav $X := \mathbb{C}^g/\Lambda$, where $\Lambda := \mathbb{Z}^g + B\mathbb{Z}^g \subset \mathbb{C}^g$. The Riemann theta function is given by the formula

$$\theta(B, z) := \sum_{m\in\mathbb{Z}^g} e^{2\pi i(z,m)+\pi i(Bm,m)}, \quad (z, m) = m_1z_1 + \cdots + m_gz_g$$

for $z \in \mathbb{C}^g$. The theta functions of the second order are defined by the formula

$$\Theta[\varepsilon](B, z) := \sum_{m\in\mathbb{Z}^g} e^{2\pi i(2m+\varepsilon,z)+\pi i(2m+\varepsilon,B(m+\varepsilon/2))}$$

for $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^g$. The Kummer variety $K(X)$ is then defined as the image of the Kummer map

$$K : z \longmapsto \big\{\Theta[\varepsilon](z)\big\}_{\text{all }\varepsilon\in(\mathbb{Z}/2\mathbb{Z})^g} \in \mathbb{P}^{2^g-1}.$$

A projective $(m-2)$-dimensional plane $\mathbb{P}^{m-2} \subset \mathbb{P}^{2^g-1}$ intersecting $K(X)$ in at least m points is called an *m-secant of the Kummer variety*.

We now recall the definition of Prym varieties. Indeed, an involution $\sigma : \Gamma \longrightarrow \Gamma$ of a smooth algebraic curve Γ induces an involution $\sigma^* : J(\Gamma) \longrightarrow J(\Gamma)$ of the Jacobian. The kernel of the map $1 + \sigma^*$ on $J(\Gamma)$ is the sum of a lower-dimensional abelian variety, called the *Prym variety* (the connected component of zero in the kernel), and a finite group. The Prym variety naturally has a polarization induced by the principal polarization on $J(\Gamma)$. However, this polarization is not principal, and the Prym variety admits a natural principal polarization if and only if σ has at most two fixed points on Γ. This is the case we concentrate on.

THEOREM 2.1 (Main theorem)
An indecomposable principally polarized abelian variety $(X,\theta) \in \mathcal{A}_g$ lies in the closure of the locus $\mathcal{P}_g$ of Prym varieties of unramified double covers if and only if there exist vectors $A, U, V, W \in \mathbb{C}^g$ representing distinct points in X, none of them points of order 2, and constants $c_1, c_2, c_3, w_1, w_2, w_3 \in \mathbb{C}$ such that one of the following equivalent conditions holds.

(A) *The difference 2-dimensional Schrödinger equation*

$$\psi_{n+1,m+1} - u_{n,m}(\psi_{n+1,m} - \psi_{n,m+1}) - \psi_{n,m} = 0 \tag{2.1}$$

with

$$u_{n,m} := C_{nm}\frac{\theta((n+1)U+mV+\nu_{nm}W+Z)\,\theta(nU+(m+1)V+\nu_{nm}W+Z)}{\theta((n+1)U+(m+1)V+(1-\nu_{nm})W+Z)\,\theta(nU+mV+(1-\nu_{nm})W+Z)}, \tag{2.2}$$

where

$$2\nu_{nm} := 1 + (-1)^{n+m+1}, \qquad C_{nm} := c_3\left(c_2^{2n+1}c_1^{2m+1}\right)^{1-2\nu_{nm}}, \tag{2.3}$$

and where

$$\psi_{n,m} := \frac{\theta(A + nU + mV + \nu_{nm}W + Z)}{\theta(nU + mV + (1-\nu_{nm})W + Z)}\, w_1^n w_2^m w_3^{\nu_{nm}} \left(c_1^m c_2^n\right)^{1-2\nu_{nm}} \tag{2.4}$$

is satisfied for all $Z \in X$.

(B) *We have the identity*

$$w_1w_2(c_1c_2)^{\pm1}\widetilde{K}\Big(\frac{A+U+V\mp W}{2}\Big) - w_1c_3(w_3c_1)^{\pm1}\widetilde{K}\Big(\frac{A+U-V\pm W}{2}\Big) + w_2c_3(w_3c_2)^{\pm1}\widetilde{K}\Big(\frac{A+V-U\pm W}{2}\Big) - \widetilde{K}\Big(\frac{A-U-V\mp W}{2}\Big) = 0, \tag{2.5}$$

where $\widetilde{K} : \mathbb{C}^g \to \mathbb{C}^{2^g}$ is the lifting of the Kummer map to the universal cover.

(C) *The two equations (one for the top choice of signs everywhere and one for the bottom)*

$$
\begin{aligned}
c_1^{\mp 2} c_3^2\, &\theta(Z+U-V)\theta(Z-U\pm W)\,\theta(Z+V\pm W) \\
&+ c_2^{\mp 2} c_3^2\, \theta(Z-U+V)\theta(Z+U\pm W)\,\theta(Z-V\pm W) \\
&= c_1^{\mp 2} c_2^{\mp 2}\, \theta(Z-U-V)\theta(Z+U\pm W)\,\theta(Z+V\pm W) \\
&+ \theta(Z+U+V)\theta(Z-U\pm W)\,\theta(Z-V\pm W) \qquad (2.6)
\end{aligned}
$$

are valid on the theta divisor $\{Z \in X : \theta(Z) = 0\}$.

A purely geometric restatement of part (B) of this result is as follows.

COROLLARY 2.2 (Geometric characterization of Pryms)
A ppav $(X, \theta) \in \mathcal{A}_g$ *lies in the closure of the locus of Prym varieties of unramified (étale) double covers if and only there exist four distinct points* $p_1, p_2, p_3, p_4 \in X$, *none of them points of order* 2, *such that the following two quadruples of points on the Kummer variety of* X,

$$K(p_1+p_2+p_3+p_4),\ K(p_1+p_2-p_3-p_4),\ K(p_1+p_3-p_2-p_4),\ K(p_1+p_4-p_2-p_3)$$

and

$$K(p_1-p_2-p_3-p_4),\ K(p_1-p_2+p_3+p_4),\ K(p_1-p_3+p_2+p_4),\ K(p_1-p_4+p_2+p_3),$$

are linearly dependent.

Equivalently, this can be stated as saying that (X, θ) *lies in the closure of the Prym locus if and only if there exists a pair of symmetric (under the* $z \mapsto 2p_1 - z$ *involution) quadrisecants of* $K(X)$.

Proof
Indeed, statement (B) gives the two linear dependencies for the Kummer images of the two quadruples of point. The six coefficients of linear dependence appearing in these two equations depend on six parameters c_i, w_i and are independent (since all c_i, w_i can be recovered from the six coefficients); thus (B) says that any ppav admitting a symmetric pair of quadrisecants is in the closure of the Prym locus. □

The equivalence of (A) and (B) is a direct corollary of the addition formula for the theta function. The *only if* part of (A) is what we prove in section 2. The statement (C) is actually what we use for the proof of the *if* part of the theorem. The characterization of Pryms by (C) is stronger than the characterization by (A). The implication

(A) $\Rightarrow$ (C) does not require the explicit theta-functional formula for ψ. It is enough to require only that equation (2.1) with u as in (2.2) has *local meromorphic* solutions which are holomorphic outside the divisor $\theta(Un + Vm + Z) = 0$ (see Lemma 5.1).

It would be interesting to try to apply our geometric characterization of Pryms to studying other aspects of Prym geometry and of the geometry of the Prym locus, including the Torelli problem for Pryms, higher-dimensional secancy conditions, representability of homology classes in Pryms, and so on. It is also tempting to ask whether a similar characterization of Prym-Tyurin varieties of higher order may be obtained, or whether one could use secancy conditions to geometrically stratify the moduli space of ppav's. We hope to pursue these questions in the future.

3. Potential reduction of the algebro-geometric 2-dimensional difference Schrödinger operators

To begin with, let us recall a construction of algebro-geometric difference Schrödinger operators proposed in [16] (see details in [22]).

General notation, Baker-Akhiezer functions

Let Γ be a smooth algebraic curve of genus $\hat{g}$. Fix four points $P_1^{\pm}, P_2^{\pm} \in \Gamma$, and let $\hat{D} = \gamma_1 + \cdots + \gamma_{\hat{g}}$ be a generic effective divisor on Γ of degree $\widehat{g}$. We denote by B the period matrix of the curve Γ (the integrals of a basis of the space of abelian differentials on Γ over the b-cycles, once the integrals over the a-cycles are normalized), we denote by $J(\Gamma) = \mathbb{C}^{\widehat{g}}/\mathbb{Z}^{\widehat{g}} + B\mathbb{Z}^{\widehat{g}}$ the Jacobian variety of Γ, and we denote by $\widehat{A} : \Gamma \hookrightarrow J(\Gamma)$ the Abel-Jacobi embedding of the curve into its Jacobian. We further denote by

$$\widehat{\theta}(z) := \theta(B, z)$$

the Riemann theta function of the variable $z \in \mathbb{C}^{\widehat{g}}$.

By the Riemann-Roch theorem, one computes $h^0(\hat{D} + n(P_1^+ - P_1^-) + m(P_2^+ - P_2^-)) = 1$, for any $n, m \in \mathbb{Z}$, and for $\hat{D}$ generic. We denote by $\widehat{\psi}_{n,m}(P),\ P \in \Gamma$ the unique section of this bundle. This means that $\widehat{\psi}_{n,m}$ is the unique (up to a constant factor) meromorphic function such that (away from the marked points $P_i^{\pm}$) it has poles only at γ_s, of multiplicity not greater than the multiplicity of γ_s in $\widehat{D}$, while at points P_1^+, P_2^+ (resp., P_1^-, P_2^-) the function $\widehat{\psi}_{n,m}$ has poles (resp., zeros) of orders n and m.

If we fix local coordinates k^{-1} in the neighborhoods of marked points (it is customary in the subject to think of marked points as punctures, and thus it is common to use coordinates such that k at the marked point is infinite rather than zero), then the

Laurent series for $\psi_{n,m}(P)$, for $P \in \Gamma$ near a marked point, has the form

$$\widehat{\psi}_{n,m} = k^{\pm n}\Big(\sum_{s=0}^{\infty} \xi_s^{\pm}(n,m)k^{-s}\Big), \quad k = k(P), \; P \to P_1^{\pm}, \tag{3.1}$$

$$\widehat{\psi}_{n,m} = k^{\pm m}\Big(\sum_{s=0}^{\infty} \chi_s^{\pm}(n,m)k^{-s}\Big), \quad k = k(P), \; P \to P_2^{\pm}. \tag{3.2}$$

Any meromorphic function on a Riemann surface can be expressed in terms of theta functions, but it is easier to write an expression for $\widehat{\psi}_{n,m}$ using both theta functions and differentials of the third kind. Indeed, for $i = 1, 2$ let $d\widehat{\Omega}^i \in H^0(K_\Gamma + P_i^+ + P_i^-)$ be the differential of the third kind, normalized to have residues ∓ 1 at $P_i^{\pm}$ and with zero integrals over all the a-cycles, and let $\widehat{\Omega}^i$ be the corresponding abelian integral (i.e., the function on the Riemann surface obtained by integrating $d\widehat{\Omega}^i$ from some fixed starting point to the variable point). Then we have the expression

$$\widehat{\psi}_{n,m}(P) = r_{nm}\frac{\widehat{\theta}(\widehat{A}(P) + n\widehat{U} + m\widehat{V} + \widehat{Z})}{\widehat{\theta}(\widehat{A}(P) + \widehat{Z})}\, e^{n\widehat{\Omega}_1(P) + m\widehat{\Omega}_2(P)}, \tag{3.3}$$

where r_{nm} is some constant, $\widehat{U} = \widehat{A}(P_1^-) - \widehat{A}(P_1^+)$, $\widehat{V} = \widehat{A}(P_2^-) - \widehat{A}(P_2^+)$, and

$$\widehat{Z} = -\sum_s \widehat{A}(\gamma_s) + \widehat{\kappa}, \tag{3.4}$$

where $\widehat{\kappa}$ is the vector of Riemann constants. Indeed, to prove that such an expression for $\widehat{\psi}_{n,m}$ is valid, one only needs to verify that both sides have the same zeros and poles, which is clear by construction.

Notation

For the remainder of this article, it is useful to think of n and m as discrete variables, which are shifted by the shift operators that we denote $T_1 : n \mapsto n+1$ and $T_2 : m \mapsto m+1$, respectively. To emphasize the difference between the operator and its action, for a function $f = f(n,m)$ we write $\mathbf{t}_\mu f := T_\mu \circ f$, so that, for example, $T_1(f \cdot g) = \mathbf{t}_1 f \cdot \mathbf{t}_1 g$. We also denote by $H := T_1T_2 - u(T_1 - T_2) - 1$ (where u is a function of the same variables as f, g) the difference operator that is very important for what follows.

THEOREM 3.1 (see [16])

The Baker-Akhiezer function $\widehat{\psi}_{n,m}$ given by formula (3.3) satisfies the following difference equation:

$$\widehat{\psi}_{n+1,m+1} - a_{n,m}\widehat{\psi}_{n+1,m} - b_{n,m}\widehat{\psi}_{n,m+1} + c_{n,m}\widehat{\psi}_{n,m} = 0, \tag{3.5}$$

where we let

$$a_{n,m} := \frac{\xi_0^+(n+1,m+1)}{\xi_0^+(n+1,m)}, \qquad b_{n,m} := \frac{\chi_0^+(n+1,m+1)}{\chi_0^+(n,m+1)}, \tag{3.6}$$

$$c_{n,m} := b_{n,m}\frac{\xi^-(n,m+1)}{\xi_0^-(n,m)} = \frac{\xi^-(n,m+1)\,\chi_0^+(n+1,m+1)}{\xi_0^-(n,m)\,\chi_0^+(n,m+1)}. \tag{3.7}$$

Explicit theta-functional formulas for the coefficients follow from equation (3.3), which implies that

$$\xi_0^{\pm} = r_{nm}\frac{\widehat{\theta}(\widehat{A}(P_1^{\pm}) + n\widehat{U} + m\widehat{V} + \widehat{Z})}{\widehat{\theta}(\widehat{A}(P_1^{\pm}) + \widehat{Z})}\, e^{n\alpha_1^{\pm}+m\alpha_2^{\pm}}, \tag{3.8}$$

$$\chi_0^{\pm} = r_{nm}\frac{\widehat{\theta}(\widehat{A}(P_2^{\pm}) + n\widehat{U} + m\widehat{V} + \widehat{Z})}{\widehat{\theta}(\widehat{A}(P_2^{\pm}) + \widehat{Z})} e^{n\beta_1^{\pm}+m\beta_2^{\pm}}. \tag{3.9}$$

The constants $\alpha_i^{\pm}$, $\beta_i^{\pm}$ are defined by the formulas

$$\alpha_2^{\pm} = \Omega_2(P_1^{\pm});\ \ \Omega_1 = \pm\ln k + \alpha_1^{\pm} + O(k^{-1}),\ \ P \to P_1^{\pm}, \tag{3.10}$$

$$\beta_1^{\pm} = \Omega_1(P_2^{\pm});\ \ \Omega_2 = \pm\ln k + \beta_2^{\pm} + O(k^{-1}),\ \ P \to P_2^{\pm}. \tag{3.11}$$

Setup for the Prym construction

We now assume that the curve Γ is an algebraic curve endowed with an involution σ without fixed points; then Γ is an unramified double cover $\Gamma \longrightarrow \Gamma_0$, where $\Gamma_0 = \Gamma/\sigma$. If Γ is of genus $\widehat{g} = 2g+1$, then by Riemann-Hurwitz the genus of Γ_0 is $g+1$. For the remainder of this article, we assume that $g > 0$ and thus that $\widehat{g} > 1$. On Γ one can choose a basis of cycles a_i, b_i with the canonical matrix of intersections $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$, $0 \le i, j \le 2g$, such that under the involution σ we have $\sigma(a_0) = a_0, \sigma(b_0) = b_0, \sigma(a_j) = a_{g+j}, \sigma(b_j) = b_{g+j}$, $1 \le j \le g$. If $d\omega_i$ are normalized holomorphic differentials on Γ dual to this choice of a-cycles, then the differentials $du_j = d\omega_j - d\omega_{g+j}$, for $j = 1, \ldots, g$ are odd; that is, they satisfy $\sigma^*(du_k) = -du_k$, and we call them *normalized holomorphic Prym differentials*. The matrix of their b-periods

$$\Pi_{kj} = \oint_{b_k} du_j,\ \ 1 \le k, j \le g \tag{3.12}$$

is symmetric, has positive definite imaginary part, and defines the Prym variety

$$\mathcal{P}(\Gamma) := \mathbb{C}^g/\mathbb{Z}^g + \Pi\mathbb{Z}^g$$

and the corresponding Prym theta function

$$\theta(z) := \theta(\Pi, z)$$

for $z \in \mathbb{C}^g$. We assume that the marked points $P_1^\pm$, $P_2^\pm$ on Γ are permuted by the involution; that is, $P_i^+ = \sigma(P_i^-)$. For further use, let us fix in addition a third pair of points $P_3^\pm$ such that $P_3^- = \sigma(P_3^+)$.

The Abel-Jacobi map $\Gamma \hookrightarrow J(\Gamma)$ induces the Abel-Prym map $A : \Gamma \longrightarrow \mathcal{P}(\Gamma)$ (this is the composition of the Abel-Jacobi map $\widehat{A} : \gamma \hookrightarrow J(\Gamma)$ with the projection $J(\Gamma) \to \mathcal{P}(\Gamma)$). Since one may choose the base point involved in defining the Abel-Jacobi map, and thus the Abel-Prym map, let us choose this base point (unique up to a point of order 2 in $\mathcal{P}(\Gamma)$) in such a way that

$$A(P) = -A\big(\sigma(P)\big). \tag{3.13}$$

Admissible divisors. An effective divisor on Γ of degree $\hat{g} - 1 = 2g$, $D = \gamma_1 + \cdots + \gamma_{2g}$, is called *admissible* if it satisfies

$$[D] + [\sigma(D)] = K_\Gamma \in J(\Gamma) \tag{3.14}$$

(where K_Γ is the canonical class of Γ), and if, moreover, $H^0(D + \sigma(D))$ is generated by an even holomorphic differential $d\Omega$; that is,

$$d\Omega(\gamma_s) = d\Omega\big(\sigma(\gamma_s)\big) = 0, \quad d\Omega = \sigma^*(d\Omega). \tag{3.15}$$

Algebraically, what we are saying is the following. The divisors D satisfying (3.14) are the preimage of the point K_Γ under the map $1 + \sigma$ and thus are a translate of the subgroup $\mathrm{Ker}(1 + \sigma) \subset J(\Gamma)$ by some vector. As shown by Mumford [23], this kernel has two components, one of them being the Prym and the other being the translate of the Prym variety by the point of order 2 corresponding to the cover $\Gamma \to \Gamma_0$ as an element in $\pi_1(\Gamma_0)$. The existence of an even differential as in (3.15) picks out one of the two components, and the other one is obtained by adding $A - \sigma(A)$ to the divisor of such a differential, for some A. In Mumford's notation, the component we pick is in fact P^- (when we choose the base point according to (3.13) to identify Pic^0 and $\mathrm{Pic}^{\hat{g}-1}$), but throughout this article we deal with both components, using some point (called P_3^+) and the corresponding shift by $P_3^+ - P_3^-$ to pass from one component to the other. We prove the following statement.

PROPOSITION 3.2
For a generic vector $Z \in \mathbb{C}^g$, the zero divisor D of the function $\theta(A(P) + Z)$ on Γ is of degree $2g$ and satisfies the constraints (3.14) and (3.15); that is, it is admissible.

Remark
We have been unable to find a complete proof of precisely this statement in the literature. However, both Izadi and Smith have independently supplied us with simple proofs of this result, based on Mumford's description and results on Prym varieties. As pointed out to us by a referee, this result can also be easily obtained by applying [10, Proposition 4.1]. The reason we choose to give the longer analytic proof below is because we need some of the intermediate results later on, and also to give an independent analytic proof of some of Mumford's results.

Note that the function $\theta(A(P)+Z)$ is multivalued on Γ, but its zero divisor is well defined. Arguments identical to those used in the standard proof of the inversion formula (3.4) show that the zero divisor $D(Z) := \theta(A(P)+Z)$ is of degree $\hat{g}-1 = 2g$.

LEMMA 3.3
For any pair of points $P_j^\pm$ conjugate under the involution σ, there exists a unique differential $d\Omega_j$ of the third kind (i.e., a dipole differential with simple poles at these points and holomorphic elsewhere) such that it has residues ∓ 1 at these points, is odd under σ, (i.e., satisfies $d\Omega_j = -\sigma^*(d\Omega_j)$), and such that all of its a-periods are integral multiples of πi; that is, such a differential $d\Omega_i$ exists for a unique set of numbers $l_0, \ldots, l_g \in \mathbb{Z}$ satisfying

$$\oint_{a_k} d\Omega_j = \pi i\, l_k, \quad k = 0, \ldots, g. \tag{3.16}$$

Proof
Indeed, by Riemann's bilinear relations, there exists a unique differential $d\Omega$ of the third kind with residues as required, and satisfying $\oint_{a_k} d\Omega = 0$ for all k. Note, however, that then $\oint_{a_k} \sigma^*(d\Omega)$ is not necessarily zero, as the image $\sigma(a_k)$ of the loop a_k, while homologic to a_{g+k} on Γ, is not necessarily homologic to a_{g+k} (resp., to a_0 for $\sigma(a_0)$) on $\Gamma\setminus\{P_j^\pm\}$. Thus each integral $\oint_{a_k} \sigma^*(d\Omega)$, being equal to $2\pi i$ times the winding number of $\sigma(a_k)$ around P_j^+ minus that around P_j^-, is equal to $2\pi i l_k$ for some $l_k \in \mathbb{Z}$. We now subtract from $d\Omega$ the linear combination $\pi i\,\big(l_0 d\omega_0 + \sum_{k=1}^g l_k(d\omega_k + d\omega_{g+k})\big)$ of even abelian differentials to get the desired $d\Omega_j$. $\square$

LEMMA 3.4
For a generic $D = D(Z)$ and for each set of integers (n, m, r) such that

$$n + m + r = 0 \bmod 2, \tag{3.17}$$

the space

$$H^0\big(D + n(P_1^+ - P_1^-) + m(P_2^+ - P_2^-) + r(P_3^+ - P_3^-)\big)$$

is 1*-dimensional. A basis element of this space is given by*

$$\psi_{n,m,r}(P) := h_{n,m,r} \frac{\theta(A(P) + nU + mV + rW + Z)}{\theta(A(P) + Z)} e^{n\Omega_1(P) + m\Omega_2(P) + r\Omega_3(P)}, \tag{3.18}$$

where Ω_j *is the abelian integral corresponding to the differential* $d\Omega_j$ *defined by Lemma 3.3, and* U, V, W *are the vectors of* b*-periods of these differentials; that is*

$$2\pi i U_k = \oint_{b_k} d\Omega_1, \qquad 2\pi i V_k = \oint_{b_k} d\Omega_2, \qquad 2\pi i W_k = \oint_{b_k} d\Omega_3. \tag{3.19}$$

Proof

It is easy to check that the right-hand side of (3.18) is a single-valued function on Γ having all the desired properties, and thus it gives a section of the desired bundle. Note that the constraint (3.17) is required due to (3.16), and the uniqueness of ψ up to a constant factor, that is, the 1-dimensionality of the H^0 above, is a direct corollary of the Riemann-Roch theorem. □

For further use, let us note that bilinear Riemann identities imply that

$$2U = A(P_1^-) - A(P_1^+), \;\; 2V = A(P_2^-) - A(P_2^+), \;\; 2W = A(P_3^-) - A(P_3^+). \tag{3.20}$$

Let us compare the definition of $\widehat{\psi}_{n,m}$ defined for any curve Γ with that of $\psi_{n,m,r}$, which is only defined for a curve with an involution satisfying a number of conditions. To make such a comparison, consider the divisor $\widehat{D} = D + P_3^+$ of degree $\hat{g} = 2g + 1$, and let $\widehat{\psi}_{n,m}$ be the corresponding Baker-Akhiezer function.

COROLLARY 3.5

For the Baker-Akhiezer function $\widehat{\psi}_{nm}$ *corresponding to the divisor* $\widehat{D} = D + P_3^+$*, we have*

$$\widehat{\psi}_{nm} = \psi_{n,m,\nu}, \tag{3.21}$$

where $\nu = \nu_{nm}$ *is defined in (2.3); that is, it is zero or one so that* $n + m + \nu$ *is even.*

COROLLARY 3.6

If $n + m$ *is even, then by formulas (3.3), (3.18), we get*

$$\begin{aligned} &\frac{\widehat{\theta}(\widehat{A}(P) + n\widehat{U} + m\widehat{V} + \widehat{Z})\,\widehat{\theta}(\widehat{A}(P_0) + \widehat{Z})}{\widehat{\theta}(\widehat{A}(P) + \widehat{Z})\,\widehat{\theta}(\widehat{A}(P_0) + n\widehat{U} + m\widehat{V} + \widehat{Z})} = \\ &\frac{\theta(A(P) + nU + mV + Z)\,\theta(A(P_0) + Z)}{\theta(A(P) + Z)\,\theta(A(P_0) + nU + mV + Z)} e^{nr_1 + mr_2}, \end{aligned} \tag{3.22}$$

where $r_i = \int_{P_0}^{P}(d\widehat{\Omega}_i - d\Omega_i)$, *and we recall that* $\widehat{Z} = \widehat{A}(\widehat{D}) + \widehat{\kappa}$ *and that* Z *is its image.*

Remark 3.7
This equality, valid for any pair of points $P, P_0 \in \Gamma$ is a nontrivial identity between theta functions. Thus far we have been unable to derive it directly from the Schottky-Jung relations.

Notation
For brevity, throughout the remainder of the article we use the notation: $\psi_{n,m} := \psi_{n,m,\,\nu_{nm}}$.

LEMMA 3.8
The Baker-Akhiezer function $\psi_{n,\,m}$, *given by*

$$\psi_{n,\,m} = \frac{\theta(A(P) + Un + Vm + \nu_{nm}W + Z)}{\theta(Un + Vm + (1-\nu_{nm})W + Z)\,\theta(A(P) + Z)} \cdot \frac{e^{n\Omega_1(P)+m\Omega_2(P)+\nu_{nm}\Omega_3(P)}}{e^{(2\nu_{nm}-1)(n\Omega_1(P_3^+)+m\Omega_2(P_3^+))}}, \tag{3.23}$$

satisfies equation (2.1); that is,

$$\psi_{n+1,\,m+1} - u_{n,m}(\psi_{n+1,m} - \psi_{n,\,m+1}) - \psi_{n,m} = 0,$$

with $u_{n,m}$ *as in (2.2), (2.3), where*

$$c_1 = e^{\Omega_2(P_3^+)}, \qquad c_2 = e^{\Omega_1(P_3^+)}, \qquad c_3 = e^{\Omega_1(P_2^+)}. \tag{3.24}$$

Proof
Note that the first and last factors in the denominator of (3.23) correspond to a special choice of the normalization constants $h_{n,m,\,\nu}$ in (3.18):

$$\begin{aligned} \psi_{nm}(P_3^-) &= (\theta(Z+W))^{-1}, \quad \nu_{nm} = 0, \\ \psi_{nm}e^{-\Omega_3}|_{P=P_3^+} &= (\theta(Z-W))^{-1}, \quad \nu_{nm} = 1. \end{aligned} \tag{3.25}$$

This normalization implies that for $n+m$ the difference $(\psi_{n+1,m+1} - \psi_{n,m})$ equals zero at P_3^-. At the same time, as a corollary of the normalization, we see that $(\psi_{n+1,m} - \psi_{n,m+1})$ has no pole at P_3^+. Hence, these two differences have the same analytical properties on Γ and thus are proportional to each other (the relevant H^0 is 1-dimensional by Riemann-Roch). The coefficient of proportionality u_{nm} can be found by comparing the singularities of the two functions at P_1^+. □

The second factor in the denominator of formula (3.23) does not affect equation (2.1). Hence, the lemma proves the *only if* part of statement (A) of the main theorem for the case of smooth curves. It remains valid under degenerations to singular curves which are smooth outside of fixed points Q_k which are simple double points, that is, to the curves of type $\{\Gamma, \sigma, Q_k\}$.

Remark 3.9
Equation (2.1) as a special reduction of (3.5) was introduced in [8]. It was shown that equation (3.5) implies a five-term equation

$$\psi_{n+1,m+1} - \tilde{a}_{nm}\psi_{n+1,m-1} - \tilde{b}_{n,m}\psi_{n-1,m+1} + \tilde{c}_{nm}\psi_{n-1,m-1} = \tilde{d}_{n,m}\psi_{n,m} \tag{3.26}$$

if and only if it is of the form (2.1). A reduction of the algebro-geometric construction proposed in [16] was found in the case of algebraic curves with involution having two fixed points. It was shown that the corresponding Baker-Akhiezer functions do satisfy an equation of the form (2.1). Explicit formulas were obtained for the coefficients of the equations in terms of Riemann theta functions. The fact that the Baker-Akhiezer functions and the coefficients of the equations can be expressed in terms of Prym theta functions is new.

We are now ready to complete the proof of Proposition 3.2. Let $\psi_{n,m}$ be the Baker-Akhiezer function given by (3.23). According to Lemma 2.3, it satisfies equation (2.1). The differential $d\psi_{n,m}$ is also a solution of the same equation, and thus using the shift operator notation, we get

$$(T_1 - 1)(\psi^\sigma_{n,m} d\psi_{n,m+1} - \psi^\sigma_{n,m+1} d\psi_{n,m}) = (T_2 - 1)(\psi^\sigma_{n,m} d\psi_{n+1,m} - \psi^\sigma_{n+1,m} d\psi_{n,m}). \tag{3.27}$$

For a generic set of algebro-geometric spectral data, the products $\psi^\sigma_{n,m}\psi_{n,m+1}$ and $\psi^\sigma_{n,m}\psi_{n+1,m}$ are quasi-periodic functions of the variables n and m. The data for which they are periodic is characterized as follows.

Let dp_j, $i = 1, 2$ be abelian differentials of the third kind with residues ∓ 1 at punctures $P_j^\pm$, respectively, and normalized by the condition that all of their periods are *purely imaginary*. We then have

$$\Re \oint_c dp_j = 0, \quad \forall c \in H^1(\Gamma, Z). \tag{3.28}$$

Nondegeneracy of the imaginary part of the period matrix of holomorphic differentials implies that such dp_j exists and is unique. If the periods of dp_j are of the form

$$\oint_c dp_j = \frac{\pi i n_c^j}{N_j}, \quad n_c^j \in \mathbb{Z}, \tag{3.29}$$

then the function $\mu_j(Q) = e^{N_j \int^Q dp_j}$ is single-valued on Γ and has pole of order N_j at P_j^+ and zero of order N_j at P_j^-. From the uniqueness of the Baker-Akhiezer function, it then follows that

$$\psi_{n+2N_1,m} = \frac{\mu_1}{\mu_1(P_3^-)}\,\psi_{n,m}, \qquad \psi_{n,m+2N_2} = \frac{\mu_2}{\mu_2(P_3^-)}\,\psi_{n,m}, \quad \nu = 0$$
$$\psi_{n+2N_1,m} = \frac{\mu_1}{\mu_1(P_3^+)}\,\psi_{n,m}, \qquad \psi_{n,m+2N_2} = \frac{\mu_2}{\mu_2(P_3^+)}\,\psi_{n,m}, \quad \nu = 1. \tag{3.30}$$

These imply that

$$\psi^\sigma_{n+2N_1,m} d\psi_{n+1+2N_1,m} = \psi^\sigma_{n,m} d\psi_{n+1,m} + (\psi^\sigma_{n,m}\psi_{n+1,m})dp_1 \tag{3.31}$$

and that similar monodromy properties obtain for the other terms in (3.27). In this case, the averaging of equation (3.27) in the variables n, m gives the equation

$$\langle \psi^\sigma(\mathbf{t}_2\psi - \mathbf{t}_2\psi^\sigma)\psi\rangle_2 dp_1 = \langle \psi^\sigma(\mathbf{t}_1\psi) - (\mathbf{t}_1\psi^\sigma)\psi\rangle_1\, dp_2. \tag{3.32}$$

Here $\langle\cdot\rangle_1$ stands for the mean value in n and $\langle\cdot\rangle_2$ stands for the mean value in m. For a generic curve, differentials dp_j have no common zeros. Hence, for such curves the differential

$$d\Omega = \frac{dp_1}{\langle \psi^\sigma \mathbf{t}_1\psi - \mathbf{t}_1\psi^\sigma\psi\rangle_1} = \frac{dp_2}{\langle \psi^\sigma \mathbf{t}_2\psi - \mathbf{t}_2\psi^\sigma\psi\rangle_2} \tag{3.33}$$

is holomorphic on Γ. It has zeros at the poles of ψ and ψ^σ. The curves for which (3.29) holds for some N_j are dense in the moduli space of all smooth genus g curves. This proves that equation (3.33) holds for any curve. Proposition 3.2 is proved.

Remark 3.10

We have thus proved that for any Prym variety, part (A) of the main theorem is satisfied. Note, however, that the statement of the main theorem is for all abelian varieties in the closure of the locus $\mathcal{P}_g$ in $\mathcal{A}_g$. To show that condition (A) holds for ppav's in the closure of $\mathcal{P}_g$, it is enough to note that (A) is an algebraic condition, and thus is valid on the closure of $\mathcal{P}_g$. The only thing left to verify is that in the closure, all of the points A, U, V, W can be chosen not to be points of order 2. Since we have shown that for Prym varieties (in the open part $\mathcal{P}_g$) the quadrisecancy occurs for any choice of points on the Abel-Prym curve, and this curve in the limit degenerates to a curve (which is thus not contained in the finite set of points of order 2), the limit points can be chosen not to be of order 2 as well.

When we prove the characterization (the *only if* part of the main theorem) in Section 5, there are no problems with the closure, as we are able to show explicitly that condition (C) (implied by (A)) exhibits the abelian variety as the Prym for a possibly nodal curve. In particular, the case of the nodal curve corresponds to double covers branched in two points, considered, for example, in [17].

4. A discrete analog of Novikov-Veselov hierarchy

In this section, we introduce *multiparametric* deformations of the Baker-Akhiezer functions and prove that they satisfy a system of difference-differential equations. The compatibility conditions of these equations can be regarded as a discrete analog of the Novikov-Veselov hierarchy (see [31]).

Let $t = \{t_i^1, t_i^2,\ i = 1, 2 \ldots\}$ be two sequences of complex numbers (we assume that only finitely many of them are nonzero). We construct a function ψ on the curve Γ with prescribed exponential essential singularities at points $P_i^{\pm}$ controlled by these t.

LEMMA 4.1

Let $D = D(Z) = \gamma_1 + \cdots + \gamma_{2g}$ be an admissible divisor. Then there exists a unique (up to a constant factor) meromorphic function $\psi_{n,m}(t, P)$ of $P \in \Gamma$, which we call a multiparametric deformation of the Baker-Akhiezer function, such that

(i) *outside of the marked points it has poles only at points γ_s of multiplicity not greater than the multiplicity of γ_s in D;*

(ii) *$\psi_{n,m}(t, P)$ has an at most simple pole at P_3^+;*

(iii) *in the local coordinate k^{-1} mapping a small neighborhood of $P_1^{\pm}$ to a small disk in $\mathbb{C}$ (with the marked point mapping to zero), it has the power series expansion*

$$\psi_{n,m}(t, P) = k^{\mp n} e^{\pm \sum_i t_i^1 k^{-i}} \Big(\sum_{s=0}^{\infty} \xi_s^{\pm}(n, m, t) k^s \Big) \tag{4.1}$$

for some $\xi_s^{\pm}$ (notice that this means that there is an essential singularity and that the expansion starts from k^{-n} at P_1^+ and k^n at P_1^- and goes toward $k^{-\infty}$);

(iv) *in the local coordinate k^{-1} near $P_2^{\pm}$, it has the power series expansion*

$$\psi_{n,m}(t, P) = k^{\mp m} e^{\pm \sum_i t_i^2 k^{-i}} \Big(\sum_{s=0}^{\infty} \chi_s^{\pm}(n, m, t) k^s \Big). \tag{4.2}$$

Proof

This function $\psi_{n,m}$ is given by

$$\psi_{n,m}(t, P) = h_{n,m}(t) \frac{\theta(A(P) + nU + mV + \nu_{nm} W + Z + \sum_i (t_i^1 U_i^1 + t_i^2 U_i^2))}{\theta(A(P) + Z)} \times \exp\Big(n\Omega_1(P) + m\Omega_2(P) + \nu_{nm}\Omega_3(P) + \sum_i (t_i^1 \Omega_i^1(P) + t_i^2 \Omega_i^2(P)) \Big), \tag{4.3}$$

where Ω_1, Ω_2, Ω_3, and the vectors U, V, W are as in Lemma 2.2; Ω_j^μ for $\mu = 1, 2$ is the abelian integral of the differential $d\Omega_j^\mu$, which has poles of the form

$$d\Omega_j^{1(2)} = \pm d\big(k^j + O(1)\big) \tag{4.4}$$

at punctures $P_{1(2)}^{\pm}$, is holomorphic everywhere else, and is uniquely determined by the normalization conditions

$$\oint_{a_k} d\Omega_j^{1(2)} = 0, \quad k = 0, \dots, 2g\,; \tag{4.5}$$

coordinates of the vectors $U_j^{1(2)}$ are defined by b-periods of these differentials; that is,

$$2\pi i U_{k,j}^{1(2)} = \oint_{b_k} d\Omega_j^{1(2)}, \quad k = 1, \dots, g. \tag{4.6}$$

Note that, as before, if $\nu_{nm} = 0$, then $\psi_{n,m}$ is in fact holomorphic at P_3^+, and if $\nu_{nm} = 1$, then $\psi_{n,m}$ does have a pole at P_3^+ but also has a zero at P_3^-. As before, we normalize $\psi_{n,m}$ by conditions (3.25). □

Notation

In what follows, we deal with formal pseudodifference operators, shifting n and m, with coefficients being functions of the variables n and m, and of the t's. For the remainder of this section, when we write functions $f, g, \dots$ as coefficients of pseudodifferential operators, they are meant to be functions of n, m, and t.

Denote by $\mathcal{R}$ the ring of functions of variables n, m, and t. We denote by $\mathcal{O}_1^{\pm}$ the rings of pseudodifference operators in two variables that are Laurent series in $T_1^{\mp}$ (i.e., for elements in $\mathcal{O}_1^+$, only finitely many positive powers of T_1 are allowed) and polynomials in $T_2^{\pm}$; that is,

$$\mathcal{O}_1^{\pm} := \mathcal{R}(T_1^{\mp})[T_1^{\pm}, T_2, T_2^{-1}] = \Big\{D = \sum_{j=M_1}^{M_2} \sum_{i=N}^{\infty} r_{ij} T_1^{\mp i} T_2^j\Big\},$$

where $r_{ij} \in \mathcal{R}$. The intersection

$$\mathcal{O} := \mathcal{O}_1^+ \cap \mathcal{O}_1^- = \mathcal{R}[T_1, T_1^{-1}, T_2, T_2^{-1}]$$

is the ring of difference operators. We further denote by $\mathcal{O}_{1,0}^{\pm}$ the ring of pseudodifference operators in one variable that are Laurent polynomials in $T_1^{\mp}$, thought of as subrings of $\mathcal{O}_1^{\pm}$, respectively; that is,

$$\mathcal{O}_{1,0}^{\pm} := \mathcal{R}\big((T_1^{\mp})\big) = \Big\{D = \sum_{i=N}^{\infty} r_i T_1^{\mp i}\Big\}.$$

Finally, we denote by $\mathcal{O}_H$ the left principal ideal in $\mathcal{O}$ generated by the operator $H = T_1T_2 - u(T_1 - T_2) - 1$, that is, $\mathcal{O}_H := \mathcal{O}H$, and we similarly set $\mathcal{O}_H^{\pm} := \mathcal{O}_1^{\pm}H$.

Moreover, while performing computations in these rings, it is often convenient to compute only a couple of the highest terms. To this end, we use for $k > 0$ notation $O(T_1^{-k}) = T_1^{-k}\mathcal{R}[[T_1^{-1}]]$ for the operators in $\mathcal{O}_1^+$ only having terms with T_1^n for $n \leq -k$, and $O(T_1^k) = T_1^k\mathcal{R}[[T_1]]$ for the operators in $\mathcal{O}_1^-$ only having terms with T_1^n for $n \geq k$.

We now want to show that the multiparametric deformations of the Baker-Akhiezer functions satisfy a hierarchy of difference-differential equations.

PROPOSITION 4.2
The Baker-Akhiezer function $\psi = \psi_{n,m}(t, P)$ satisfies (2.1) with u_{nm} as in (2.2), with Z replaced by $Z + \sum_j (t_j^1 U_j^1 + t_j^2 U_j^2)$ (this can be written as $H\psi = 0$). There exist unique difference operators of the form

$$L_j^{(\mu)} = \Big(f_{0j} + \sum_{i=1}^{j-1} f_{ij}^{(\mu)} T_\mu^i + T_\mu^{-i} f_{ij}^{(\mu)}\Big)(T_\mu - T_\mu^{-1}),\ \mu = 1, 2,\ j = 1, 2, \ldots \tag{4.7}$$

such that the equations

$$\frac{\partial}{\partial t_j^\mu}\,\psi = L_j^{(\mu)}\psi \tag{4.8}$$

hold.

Proof
The proof of the first statement is identical to that in Lemma 2.3. The proof of the statement that there exist operators of the form

$$L_j^{(\mu)} = \sum_{i=-j}^{j} g_{ij}^{(\mu)} T_\mu^i \tag{4.9}$$

such that the equations in (4.8) hold is standard. Indeed, for each formal series (4.1), there exists a unique operator $L_j^{(1)}$ such that

$$\Big(\frac{\partial}{\partial t_j^\mu} - L_j^{(1)}\Big)\psi = k^{\pm n} e^{\pm \sum_i t_i^1 k^i}\Big(\sum_{s=0}^{\infty} \tilde{\xi}_s^{\pm} k^{-s}\Big), \quad \tilde{\xi}_0^+ = 0. \tag{4.10}$$

The coefficients $g_j^{(1)}$ of the operator are difference polynomials in terms of the coefficients ξ_s of the series (4.1). Now note that the left-hand side of (4.10) satisfies all the properties that ψ satisfies, and thus must be proportional to it. However, since

$\tilde{\xi}_0^+ = 0$, the constant of proportionality must be equal to zero, and thus the left-hand side vanishes as desired. The same arguments prove the existence of $L_j^{(2)}$.

It remains to show that the operators $L_j^{(\mu)}$ have the form as in (4.7). This is a matter of showing that the coefficients of $L_j^{(\mu)}$ satisfy certain identities; that is, that the generally constructed $g_{ij}^{(\mu)}$ for $-j \leq i \leq j$ can in fact be expressed in terms of $f_{ij}^{(\mu)}$ for $1 \leq i \leq j-1$. One easily checks that if $L_j^{(\mu)}$ is given by (4.7), then its formal adjoint operator satisfies

$$(L_j^{(\mu)})^* = -(T_\mu - T_\mu^{-1})L_j^{(\mu)}(T_\mu - T_\mu^{-1})^{-1}. \tag{4.11}$$

This equation is in fact equivalent to (4.7), as it determines the coefficients of all the negative powers of T_μ uniquely, given the coefficients of the positive powers. It thus remains to prove this identity.

We denote by ψ^σ the composition $\psi^\sigma(P) = \psi(\sigma(P))$; notice that by (4.1), we know the expansion of both ψ and ψ^σ near $P_1^\pm$. Now consider the differential $\psi^\sigma\left(T_\mu^n(T_\mu - T_\mu^{-1})\,\psi\right)d\Omega$, where, as before, $d\Omega$ is a holomorphic differential having zeros at the poles of ψ and ψ^σ. Then this expression is a meromorphic differential on Γ, which a priori has poles only at $P_\mu^\pm$ and $P_3^\pm$. Due to normalization in (3.25), it is holomorphic at the punctures $P_3^\pm$; the pole of ψ cancels with the zero of ψ^σ and vice versa. Therefore, for $n > 0$, this differential has a pole only at P_μ^+, and hence its residue at this point must vanish:

$$\mathrm{res}_{P_\mu^+}\left(\psi^\sigma(T_\mu^n(T_\mu - T_\mu^{-1})\,\psi)\,d\Omega\right) = 0, \quad \forall n > 0. \tag{4.12}$$

The normalization in (3.25) also implies that

$$\mathrm{res}_{P_\mu^+}\left(\psi^\sigma(T_\mu\,\psi)d\Omega\right) = 1. \tag{4.13}$$

Equations (4.12) and (4.13) recursively define coefficients of the power series expansion of $\psi^\sigma d\Omega$ at P_μ^+ in terms of the coefficients of the power series for ψ. The corresponding expressions can be explicitly written in terms of the so-called wave operator.

We first observe that in the ring $\mathcal{O}_{1,0}^+$ there exists a unique pseudodifference operator

$$\Phi = \sum_{s=0}^{\infty} \varphi_s T_1^{-s} \tag{4.14}$$

such that the expansion (4.1) of ψ at P_1^+ is equal to

$$\psi = \Phi\, k^n e^{\sum_i t_i^1 k^i}. \tag{4.15}$$

Indeed, this identity provides a unique way of determining the coefficients φ_s recursively.

LEMMA 4.3
The following identity holds:

$$\psi^\sigma d\Omega = \left(k^{-n} e^{-\sum_i t_i^1 k^i} (T_1 - T_1^{-1})\, \Phi^{-1} (T_1 - T_1^{-1})^{-1}\right) \frac{dk}{k^2 - 1}. \tag{4.16}$$

Here and below, the right action of pseudodifference operators is defined as the formal adjoint action; that is, we set $fT = T^{-1} f$.

Proof
Recall that by definition, the residue of a pseudodifference operator $D = \sum_s d_s T^s$ is $\operatorname{res}_T D := d_0$. It is easy to check (by verifying that this holds for the basis, that is, by checking this for $D_1 = T_1^a$ and $D_2 = T_1^b$) that for any two pseudodifference operators D_1, D_2, we have

$$\operatorname{res}_k (k^{-n} e^{-\sum_i t_i^1 k^i} D_1)(D_2 k^n e^{\sum_i t_i^1 k^i})\, d \ln k = \operatorname{res}_T (D_2 D_1). \tag{4.17}$$

The last equation implies that

$$\begin{aligned}&\operatorname{res}_k \left(k^{-n} e^{-\sum_i t_i^1 k^i} (T_1 - T_1^{-1})\, \Phi^{-1} (T_1 - T_1^{-1})^{-1}\right)\\ &\quad\times \left(T_1^n (T_1 - T_1^{-1})\, \psi\right) \frac{dk}{k^2 - 1} = \operatorname{res}_k \left(k^{-n} e^{-\sum_i t_i^1 k^i}\, \Phi^{-1} (T_1 - T_1^{-1})^{-1}\right)\\ &\quad\times \left(T_1^n (T_1 - T_1^{-1}) \Phi\, k^n e^{\sum_i t_i^1 k^i}\right) d \ln k = \operatorname{res}_T\, T_1^n = \delta_{n,0};\end{aligned}$$

that is, the formal series defined by the right-hand side of (4.16) satisfies equations (4.12) and (4.13), which are the defining equations (solving term by term; see above) for $\psi^\sigma d\Omega$. □

Now we are ready to complete the proof that the adjoints of L_j^μ satisfy (4.11), thus proving Proposition 4.2. Consider the pseudodifference operator

$$\mathcal{L} := \Phi T_1 \Phi^{-1},$$

for which ψ is an eigenvector; indeed, we have

$$\mathcal{L}\psi = \Phi T_1 k^n e^{\sum_i t_i^1 k^i} = \Phi k^{n+1} e^{\sum_i t_i^1 k^i} = k\psi. \tag{4.18}$$

Considering the expansion of (4.8) in a neighborhood of P_1^+, we see that the positive parts of the pseudodifference operators $L_j^{(1)}$ and $\mathcal{L}^j$ coincide; hence

$$(L_j^{(1)})_+ = \mathcal{L}_+^j \tag{4.19}$$

(where by the positive part of a pseudodifference operator $D = \sum_s d_s T^s$ we mean $D_+ := \sum_{s>0} d_s T^s$).

The differential $d\Omega$ is independent of n. Therefore, from (4.16) it follows that the operator

$$\widetilde{\mathcal{L}} := (T_1 - T_1^{-1})^{-1} \mathcal{L}^* (T_1 - T_1^{-1}) \tag{4.20}$$

has ψ^σ as an eigenfunction:

$$\widetilde{\mathcal{L}} \psi^\sigma = k \psi^\sigma. \tag{4.21}$$

Equation (4.8) considered in the neighborhood of P_1^- implies that the negative parts of $L_j^{(1)}$ and $\widetilde{\mathcal{L}}^j$ coincide; hence

$$(L_j^{(1)})_- = -\widetilde{\mathcal{L}}_-^j. \tag{4.22}$$

The last two equations prove (4.11) and then (4.7) for $\mu = 1$. The case $\mu = 2$ is analogous, and the proposition is thus proved. □

COROLLARY 4.4

The operators H and $L_j^{(\mu)}$ satisfy the equations

$$\frac{\partial}{\partial t_j^\mu} H \equiv [L_j^\mu, H] \bmod \mathcal{O}_H. \tag{4.23}$$

Proof

It is easy to show that the ideal of difference operators D (i.e., of $D \in \mathcal{O}$) such that $D\psi = 0$ is $\mathcal{O}_H$. From (4.8), it follows that

$$\Big(\frac{\partial}{\partial t_{j^\mu}} H - [L_j^\mu, H] \Big) \psi = 0.$$

Hence, the right- and left-hand sides of (4.23) are equal in the factor-ring $\mathcal{O}/\mathcal{O}_H$. □

We show below that the system of nonlinear equations (4.23) can be regarded as a discrete analog of the Novikov-Veselov hierarchy. The basic equation of this hierarchy—the discrete analog of the Novikov-Veselov equation—is given by (4.23) for $j = 1$. The operator $L_1^{(1)}$ is of the form

$$L_1^{(1)} = v(T_1 - T_1^{-1}). \tag{4.24}$$

Equation (4.23) is equivalent to the system of two equations for the two functions $u = u_{n,m}(t),\ v = v_{n,m}(t)$:

$$v(\mathbf{t}_1^{-1}u) = u(\mathbf{t}_2 v) \tag{4.25}$$

$$\partial_t u = [(\mathbf{t}_1\mathbf{t}_2 v)(\mathbf{t}_1 u) - u(\mathbf{t}_2 v)]\, u - [\mathbf{t}_1\mathbf{t}_2 v - v] \tag{4.26}$$

The discrete Novikov-Veselov hierarchy

The discrete analog of the Novikov-Veselov hierarchy is of independent interest. In what follows, we consider only the part of the hierarchy corresponding to times $t_j := t_j^1$, and set all $t_j^2 = 0$.

Let us write out this part of the hierarchy in a closed form. We think of it as a system of evolution equations on the space

$$\mathcal{S} := \left\{H, \mathcal{L}\ \middle|\ H = T_1T_2 - u(T_1 - T_2) - 1,\ \mathcal{L} = \sum_{i=0}^{\infty} v_i T_1^{-i+1}\right\} \tag{4.27}$$

satisfying

$$[H, \mathcal{L}] \equiv 0 \bmod \mathcal{O}_H^+, \tag{4.28}$$

and such that, moreover, u and v_0 are of the form

$$u = C\frac{(\mathbf{t}_1\tau)\,(\mathbf{t}_2\tau)}{(\mathbf{t}_1\mathbf{t}_2\tau)\,\tau}, \qquad v_0 = \frac{(\mathbf{t}_1\tau)\,(\mathbf{t}_1^{-1}\tau)}{\tau^2}, \tag{4.29}$$

where C is a constant and $\tau = \tau(n, m)$ is some function.

The meaning of (4.28) is as follows. A priori, the operator $[H, \mathcal{L}]$ has a unique representation of the form

$$[H, \mathcal{L}] = \Big(\sum_{s=0}^{\infty} h_s T_1^{-s+2}\Big) + DH,$$

with $D \in \mathcal{O}_1^+$. Therefore, the constraint (4.28) is equivalent to equations $h_s = 0$. The first of these equations $h_0 = 0$ is an equation for u and v_0, which is automatically satisfied due to (4.29).

By a direct computation of the series expansion of $[H, \mathcal{L}]$, it is easy to see that equations $h_s = 0$ for $s > 0$ have the form

$$(\mathbf{t}_2 v_s)(\mathbf{t}_1^{-s}u) - (\mathbf{t}_1^{-1}u)v_s = R_s(\tau, v_1, \ldots, v_{s-1}), \tag{4.30}$$

where R_s is some difference polynomial. They recursively define $v_s(n, m)$, if the initial data $v_s|_{m=0}$ are fixed. Therefore, the space $\mathcal{S}$ of operators H, $\mathcal{L}$ with the leading coefficients u, v_0 of the form (4.29) satisfying (4.28) can be identified with the space of one function of two variables, and an infinite number of functions of one variable, that is, $\{\tau(n,m), v_s(n),\ s>0\}$.

Our next goal is to define a hierarchy of commuting flows on $\mathcal{S}$. Any operator in $\mathcal{O}^+_{1,0}$, and, in particular, $\mathcal{L}^j$, has a unique representation of the form

$$\mathcal{L}^j = \sum_{i=-\infty}^{j-1} f_{ij} T_1^i (T_1 - T_1^{-1}). \tag{4.31}$$

Then the formula (4.7) with $\mu = 1$ defines a unique operator $L_j := L_j^{(1)}$ such that (4.19) holds, and also satisfies condition (4.11) with $\mu = 1$ for the adjoint.

THEOREM 4.5
The equations

$$\partial_{t_j} \mathcal{L} = [L_j, \mathcal{L}], \qquad \partial_{t_j} H \equiv [L_j, H] \bmod \mathcal{O}_H \tag{4.32}$$

define commuting flows on the space $\mathcal{S}$.

Proof
Note that the highest power of T_1 in $\mathcal{L}$ is T_1, and $\partial_{t_j} H = (\partial_{t_j} u)(T_1 - T_2)$. Thus in order to show that equations (4.32) are well defined, we need to prove the following:
(a) $[L_j, \mathcal{L}]$ is of degree not greater than 1;
(b) $[L_j, H] \equiv a_j(T_1 - T_2) \bmod \mathcal{O}_H$;
(c) the corresponding equations for v_0 and u are consistent with the ansatz (4.29).

The proof of (a) is standard. We compute

$$L_j = \mathcal{L}^j + F_j + F_j^1 T_1^{-1} + O(T^{-2}), \tag{4.33}$$

where

$$F_j = \mathbf{t}_1^{-1} f_{1,j} - f_{-1,j}, \ \ F_j^1 = \mathbf{t}_1^{-2} f_{2,j} - f_{-2,j}. \tag{4.34}$$

Using $[\mathcal{L}, \mathcal{L}^j] = 0$, we get

$$[L_j, \mathcal{L}] = [F_j + O(T_1^{-1}), \mathcal{L}] = (F_j - \mathbf{t}_1 F_j) v_0 T_1 + O(1),$$

thus proving (a). Note also that by comparing the leading coefficients, we obtain

$$\frac{\partial}{\partial t_j} \ln v_0 = F_j - \mathbf{t}_1 F_j. \tag{4.35}$$

The proof of (b) is much harder. The difference operator HL_j is of order one in T_2. Hence it has a unique representation of the form

$$HL_j = D_1 - a_j T_2 + DH, \tag{4.36}$$

where $D \in \mathcal{O}$ and $D_1 \in \mathcal{O}_{1,0}$.

Our next goal is to show that D_1 is of degree one in T_1; that is, it has the form $D_1 = b_j T_1 + c_j$. From the equation $T_1^{-1} H \equiv 0 \bmod \mathcal{O}_H$, we get

$$T_2 = \mathbf{t}_1^{-1} u + T_1^{-1} - \mathbf{t}_1^{-1} u T_1^{-1} T_2 = \mathbf{t}_1^{-1} u + (1 - \mathbf{t}_1^{-1} u \mathbf{t}^{-2} u) T^{-1} + O(T_1^{-2}). \tag{4.37}$$

Equations $[\mathcal{L}^j, H] = 0$ and (4.33) imply that in $\mathcal{O}_H^+$, the left-hand side of (4.36) is equal to

$$\begin{aligned} HL_j = \big((\mathbf{t}_1\mathbf{t}_2 F_j - \mathbf{t}_1 F_j)u\big)T_1 + &+\big((1 - u\mathbf{t}_1^{-1}u)\mathbf{t}_1\mathbf{t}_2 F_j + (u\mathbf{t}_1^{-1}u)\mathbf{t}_2 F_j \\ &- F_j + (\mathbf{t}_1^{-1}u)\mathbf{t}_1\mathbf{t}_2 F_j^1 - u\mathbf{t}_1 F_j^1\big). + O(T^{-1}). \end{aligned} \tag{4.38}$$

Substituting this expression and the formula for T_2 in (4.36), we get $D_1 = b_j T_1 + c_j + O(T_1^{-1})$, where

$$b_j := (\mathbf{t}_1\mathbf{t}_2 F_j - \mathbf{t}_1 F_j)\, u \tag{4.39}$$

$$c_j : = a_j \mathbf{t}_1^{-1} u + (1 - u\mathbf{t}_1^{-1}u)\mathbf{t}_1\mathbf{t}_2 F_j + (u\mathbf{t}_1^{-1}u)\mathbf{t}_2 F_j - F_j + (\mathbf{t}_1^{-1}u)\mathbf{t}_1\mathbf{t}_2 F_j^1 - u\mathbf{t}_1 F_j^1. \tag{4.40}$$

Now we compute the left- and right-hand sides of (4.36) in $\mathcal{O}_H^-$. Indeed, in $\mathcal{O}_1^-$ we have

$$L_j = -\widetilde{\mathcal{L}}^j - \widetilde{F}_j - \widetilde{F}_j^1 T_1 + O(T_1^2), \tag{4.41}$$

where, as before, $\tilde{\mathcal{L}} = (T_1 - T_1^{-1})^{-1} \mathcal{L}^* (T_1 - T_1^{-1})$. If f_{ij} are coefficients of $\widetilde{L}$ in (4.31), then

$$\widetilde{\mathcal{L}}^j = - \sum_{i=-\infty}^{j-1} T_1^{-i} \cdot f_{ij} \cdot (T_1 - T_1^{-1}). \tag{4.42}$$

Hence

$$\widetilde{F}_j = \mathbf{t}_1 F_j = f_{1,j} - \mathbf{t}_1 f_{-1,j}, \ \widetilde{F}_j^1 = \mathbf{t}_1^2 F_j^1 = f_{2,j} - \mathbf{t}_2 f_{-2,j}. \tag{4.43}$$

In order to proceed, we now need the following statement.

LEMMA 4.6
If (4.28) is satisfied, then the equation

$$[H, \widetilde{\mathcal{L}}] \equiv 0 \bmod \mathcal{O}_H^- \tag{4.44}$$

holds.

Proof
We prove the lemma by inverting the arguments used above in the proof of Lemma 3.2. First, for a pair of operators $\mathcal{L}$ and H satisfying (4.28), we introduce a formal solution $\psi = \psi_{nm}$ of equations

$$\mathcal{L}\psi = k\psi, \qquad H\psi = 0 \tag{4.45}$$

of the form

$$\psi_{nm} = k^n\Big(\sum_{s=0}^{\infty} \xi_s(n,m)k^{-s}\Big). \tag{4.46}$$

Substitution of (4.46) into (4.45) gives a system of difference equations, which recursively define ξ_s. They have the form

$$(T_2\,\xi_{s+1}) - u\,\xi_{s+1} = \xi_s - u\,(T_2\,\xi_s), \qquad v_0\,(T_1\,\xi_{s+1}) - \xi_{s+1} = \tilde{R}_s, \tag{4.47}$$

where $\tilde{R}_s$ are explicit expressions linear in the coefficients v_r of $\mathcal{L}$ and difference polynomial in $\xi_r,\ r < s$. If u, v_0 are of the form (4.29), then the first equation for $s = -1$ is satisfied by

$$\xi_0 = \frac{\mathbf{t}_1^{-1}\tau}{\tau}. \tag{4.48}$$

The compatibility condition of equations (4.47) is equivalent to (4.28). These equations uniquely define ξ_{s+1} for all (n,m) if the initial data $\xi_{s+1}(0,0)$ for (4.47) is fixed. Therefore, the solution ψ is unique up to multiplication by an (n,m)-independent Laurent series in the variable k.

The function ψ defines a unique operator Φ of the form (4.14) such that equation (4.15) holds (with $t_i = 0$). Now we define a formal series

$$\psi^\sigma = k^{-n}\Big(\sum_{s=0}^{\infty} \xi_s^\sigma(n,m)k^{-s}\Big), \quad \xi_0^\sigma = \frac{\mathbf{t}_1\tau}{\tau} \tag{4.49}$$

by the formula

$$\psi^\sigma = \big((T_1 - T_1^{-1})\,\Phi^{-1}(T_1 - T_1^{-1})^{-1}\big)^*\,k^{-n}. \tag{4.50}$$

This formal series is an eigenfunction of the operator $\widetilde{L}$; that is, $\widetilde{\mathcal{L}}\psi^\sigma = k\psi^\sigma$. Therefore, in order to prove (4.44), it is sufficient to prove that $H\psi^\sigma = 0$.

From equations (4.29) and (4.48), it follows that

$$\widetilde{\psi}^\sigma := H\psi^\sigma = k^{-n}\Big(\sum_{s=1}^{\infty}\widetilde{\xi}^\sigma(n,m)k^{-s}\Big). \tag{4.51}$$

Hence, to prove that $\tilde{\psi}^\sigma = 0$, it is enough to show that

$$\Big[\widetilde{\psi}^\sigma(T_1^j\psi)\Big]_R := \operatorname{res}_k\Big(\frac{\tilde{\psi}^\sigma(T_1^j\psi)\,dk}{k^2-1}\Big) = 0, \quad \forall j \geq 2. \tag{4.52}$$

From the definition of ψ^σ, it follows that

$$[\psi^\sigma(\mathbf{t}_1^{2j}\psi)]_R = 0, \qquad [\psi^\sigma(\mathbf{t}_1^{2j+1}\psi)]_R = 1, \quad j \geq 0 \tag{4.53}$$

(compare to (4.12), (4.13)). Using the equation $H\psi = 0$, we get

$$\begin{aligned}\mathbf{t}_2\left[\psi^\sigma t^{2j}\psi\right]_R = (\mathbf{t}_1^{2j-1}u)\,\left[\mathbf{t}_2\psi^\sigma t^{2j-1}\psi)\right]_R \\ -(\mathbf{t}_1^{2j-1}u)\,\mathbf{t}_2\left[\psi^\sigma t^{2j-1}\psi\right]_R + \left[\mathbf{t}_2\psi^\sigma t^{2j-1}\right]_R.\end{aligned} \tag{4.54}$$

Then, by induction, it is easy to show that (4.53) and (4.54) imply that

$$\left[\mathbf{t}_2\psi^\sigma t^{2j+2}\psi\right]_R = 1 - \prod_{i=0}^{2j+1}(\mathbf{t}_1^i u)^{-1}, \qquad \left[\mathbf{t}_2\psi^\sigma t^{2j+1}\psi\right]_R = \prod_{i=0}^{2j}(\mathbf{t}_1^i u)^{-1}, \quad j \geq 0. \tag{4.55}$$

Direct substitution of (4.55) into (4.52) completes the proof of the lemma. □

Now we compute both sides of (4.36):

$$T_2 \equiv \frac{1}{u} + \Big(1 - \frac{1}{u\mathbf{t}_1 u}\Big)T_1^1 + O(T_1^2) \bmod \mathcal{O}_H^-. \tag{4.56}$$

Equations (4.41) and (4.44) imply that $[L_j, H] = H(\widetilde{F}_j + \widetilde{F}_j^1 T_1 + O(T_1^2)) \in \mathcal{O}_H^-$. Therefore, the operator D_1 in (4.36) has no negative powers of T_1, and hence, it is indeed of the form $b_j T_1 + c_j$.

Straightforward computations of the first two coefficients of $[L_j, H]$ give the following formulas:

$$c_j - \frac{a_j}{u} = \widetilde{F}_j - \mathbf{t}_2\widetilde{F}_j, \tag{4.57}$$

$$\Big(1-\frac{1}{u\mathbf{t}_1 u}\Big)a_j - b_j = \frac{1}{\mathbf{t}_1 u}\left(\mathbf{t}_1\mathbf{t}_2\widetilde{F}_j + (u\mathbf{t}_1 u - 1)\mathbf{t}_2\widetilde{F}_j - (u\mathbf{t}_1 u)\,\mathbf{t}_1\widetilde{F}_j - \mathbf{t}_1 u\widetilde{F}_j^1 + u\,\mathbf{t}_2\widetilde{F}_j^1\right). \tag{4.58}$$

From (4.39), (4.43), and (4.57), we get the equations

$$c_j u = (a_j - b_j), \tag{4.59}$$

and then

$$c_j(u\mathbf{t}_1 u - 1) = \mathbf{t}_1\mathbf{t}_2\widetilde{F}_j + u\mathbf{t}_1 u\left(\mathbf{t}_2\widetilde{F}_j - \mathbf{t}_1\widetilde{F}_j\right) - \mathbf{t}_1 u\widetilde{F}_j^1 + u\,\mathbf{t}_2\widetilde{F}_j^1 - \widetilde{F}_j. \tag{4.60}$$

In order to complete the proof of (b), it is enough now to show that the right-hand side of (4.60) is zero. For that we need the following.

LEMMA 4.7
The equations

$$\widetilde{\mathcal{F}} := -k + (k^2-1)\sum_{j=1}^{\infty}\widetilde{F}_j\,k^{-j-1} = (\mathbf{t}_1\psi^\sigma)\psi - \psi^\sigma(\mathbf{t}_1\psi), \tag{4.61}$$

$$\widetilde{\mathcal{F}}^1 := -\frac{(\mathbf{t}_1\tau)^2}{\tau\mathbf{t}_2\tau}\,k^2 + (k^2-1)\sum_{j=1}^{\infty}\widetilde{F}_j^1\,k^{-j-1} = \psi^\sigma(\mathbf{t}_1^2\psi) - (\mathbf{t}_1^2\psi^\sigma)\psi \tag{4.62}$$

hold.

Proof
The expression for the leading coefficients of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^1$ follows from (4.48) and (4.50). In order to prove (4.61), we need to show that

$$\begin{aligned}\widetilde{F}_j &= \operatorname{res}_k\Big(\,[(\mathbf{t}_1\psi^\sigma)\psi - \psi^\sigma(\mathbf{t}_1\psi)]\,\frac{k^j\,dk}{k^2-1}\Big)\\ &= \operatorname{res}_k\Big(\,\big[(T_1\psi^\sigma)(\mathcal{L}^j\psi) - \psi^\sigma(T_1\mathcal{L}^j\psi)\big]\,\frac{dk}{k^2-1}\Big).\end{aligned} \tag{4.63}$$

From (4.50), using the relation (4.17), we see that the right-hand side of (4.64) is equal to

$$\operatorname{res}_T\left((\mathcal{L}^jT_1^{-1} - T_1\mathcal{L}^j)(T_1 - T_1^{-1})^{-1}\right) = f_{1,j} - \mathbf{t}_1 f_{-1,j}, \tag{4.64}$$

which proves (4.61). The proof of (4.62) is identical. □

From (4.61) and the equation $H\psi = 0$, it follows that

$$\mathbf{t}_2\widetilde{\mathcal{F}} = -\mathcal{A}\big\{(\mathbf{t}_2\psi^\sigma(u\mathbf{t}_1\psi - u\mathbf{t}_2\psi + \psi)\big\} = -\mathcal{A}\big\{(\mathbf{t}_2\psi^\sigma(u\mathbf{t}_1\psi + \psi)\big\}. \tag{4.65}$$

Here and below, $\mathcal{A}\{\cdot\}$ stands for the antisymmetrization of the corresponding expression with respect to the interchange of ψ^σ and ψ.

In the same way, we get

$$\begin{aligned}\mathbf{t}_1\mathbf{t}_2\widetilde{\mathcal{F}} &= -\mathcal{A}\big\{(u\mathbf{t}_1\psi^\sigma - u\mathbf{t}_2\psi^s + \psi^\sigma)\mathbf{t}_1^2\mathbf{t}_2\psi\big\}\\ &= -(u\mathbf{t}_1 u)\mathbf{t}_1\mathcal{A}\big\{\psi^\sigma(\mathbf{t}_1\psi - \mathbf{t}_2\psi)\big\} - u\mathbf{t}_2\widetilde{\mathcal{F}}^1\\ &\quad -\mathcal{A}\big\{\psi^\sigma(\mathbf{t}_1(u\mathbf{t}_1\psi - u\mathbf{t}_2\psi + \psi)\big\}.\end{aligned}$$

Further direct use of the equation $H\psi = 0$ and (4.65) finally gives the equation

$$\mathbf{t}_1\mathbf{t}_2\widetilde{\mathcal{F}} + u\mathbf{t}_1 u\left(\mathbf{t}_1\widetilde{\mathcal{F}} - \mathbf{t}_2\widetilde{\mathcal{F}}\right) - \mathbf{t}_1 u\widetilde{\mathcal{F}}^1 + u\,\mathbf{t}_2\widetilde{\mathcal{F}}^1 - \widetilde{\mathcal{F}} = 0. \tag{4.66}$$

The proof of (b) is complete. The comparison of the coefficients at T_1 in the left- and right-hand sides of (4.32) gives

$$\partial_{t_j}\ln u = b_j = \mathbf{t}_2\widetilde{F}_j - \widetilde{F}_j. \tag{4.67}$$

We now proceed to prove (c) and to derive the evolution equation for τ. The left and right actions of pseudodifference operators are formally adjoint; that is, for any two operators the equality $(k^{-n}\mathcal{D}_1)(\mathcal{D}_2k^n) = k^{-n}(\mathcal{D}_1\mathcal{D}_2k^n) + (T_1-1)(k^{-n}(\mathcal{D}_3k^n))$ holds. Here $\mathcal{D}_3$ is a pseudodifference operator whose coefficients are difference polynomials in the coefficients of $\mathcal{D}_1$ and $\mathcal{D}_2$. Therefore, from (4.61) and (4.50), it follows that

$$\widetilde{\mathcal{F}}^0 = -k - (T_1-1)\Big((k^2-1)\sum_{s=2}^{\infty} Q_j k^{-j}\Big), \tag{4.68}$$

where the coefficients of the series Q are difference polynomials in the coefficients of the wave operator Φ. Equation (4.68) implies that

$$\widetilde{F}_j = (1-T_1)Q_j = Q_j - \mathbf{t}_1 Q_j. \tag{4.69}$$

Taking into account the ansatz (4.29), we see that equations (4.35) and (4.69) are equivalent to one equation for the function τ

$$\partial_{t_j}\ln\tau = Q_j. \tag{4.70}$$

Remark 4.8
It is necessary to mention that the Q_j are defined only up to an additive term that is invariant under T_1. This ambiguity reflects the fact that the ansatz (4.29) is invariant under the transformation

$$\tau(n,m) \longmapsto f(m)\tau(n,m),$$

where $f(m)$ is an arbitrary function.

Equation (4.70) completes the proof of the statement that equations (4.32) are well defined. The proof of the statement that the corresponding flows on $\mathcal{S}$ commute with each other is standard. □

5. Bloch (quasi-periodic) wave solutions

We begin by proving the implication (A) ⇒ (C) in the main theorem. As mentioned above, this does not require the knowledge of the explicit theta-functional form of the function ψ. An implication of this kind was proved for the first time in [2].

Throughout this section, $\nu = 0, 1$ and is considered as an element of the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

LEMMA 5.1

Let $V \in \mathbb{C}^d$, and let $\tau_n^\nu(z)$ for $n \in \mathbb{N}, \nu \in \mathbb{Z}_2$ be two sequences of holomorphic functions on $\mathbb{C}^d$ such that each divisor $\mathcal{T}_n^\nu := \{z \in \mathbb{C}^d : \tau_n^\nu(z) = 0\}$ is not invariant as a set under the shift by V; that is, $\mathcal{T}_n^\nu \neq \mathcal{T}_n^\nu + V$. Suppose that the system of equations (considered as a joint system for $\nu = 0$ and $\nu = 1$, intertwining ψ^0 and ψ^1)

$$\psi_{n+1}^\nu(z+V) - u_n^\nu(z)\left(\psi_{n+1}^{\nu+1}(z) - \psi_n^{\nu+1}(z+V)\right) - \psi_n^\nu(z) = 0, \tag{5.1}$$

where for some constant $C \in \mathbb{C}$

$$u_n^\nu(z) = C\,\frac{\tau_{n+1}^{\nu+1}(z)\,\tau_n^{\nu+1}(z+V)}{\tau_{n+1}^\nu(z+V)\,\tau_n^\nu(z)}, \tag{5.2}$$

has solutions ψ_n^ν of the form

$$\psi_n^\nu(z) = \frac{\alpha_n^\nu(z)}{\tau_n^\nu(z)}, \tag{5.3}$$

where α_n^ν is a holomorphic function. Then the equation

$$\begin{aligned}
&\tau_{n+1}^{\nu+1}(z_n^\nu)\,\tau_n^{\nu+1}(z_n^\nu+V)\ \tau_{n-1}^\nu(z_n^\nu-V) + \tau_{n+1}^\nu(z_n^\nu+V)\,\tau_n^{\nu+1}(z_n^\nu-V)\ \tau_{n-1}^{\nu+1}(z_n^\nu)\\
&\quad= \big(\tau_{n+1}^{\nu+1}(z_n^\nu)\,\tau_n^{\nu+1}(z_n^\nu-V)\ \tau_{n-1}^\nu(z_n^\nu+V) + \tau_{n+1}^\nu(z_n^\nu-V)\,\tau_n^{\nu+1}\\
&\qquad\times(z_n^\nu+V)\ \tau_{n-1}^{\nu+1}(z_n^\nu)\big)C^2
\end{aligned}\tag{5.4}$$

is valid for all n, ν, and for any point $z_n^\nu \in \mathcal{T}_n^\nu$.

Proof

Let $I_n^\nu(z)$ be the left-hand side of (5.1). A priori, it may have poles at the divisors $\mathcal{T}_n^\nu$ and $\mathcal{T}_{n+1}^\nu - V$. The vanishing of the residue of I_n^ν at $\mathcal{T}_n^\nu$ implies that

$$\psi_{n+1}^{\nu+1}(z_n^\nu) - \psi_n^{\nu+1}(z_n^\nu+V) = -\alpha_n^\nu(z_n^\nu)\frac{\tau_{n+1}^\nu(z_n^\nu+V)}{\tau_{n+1}^{\nu+1}(z_n^\nu)\tau_n^{\nu+1}(z_n^\nu+V)}\,C^{-1}, \tag{5.5}$$

while the vanishing of the residue of I_{n-1}^{ν} at $\mathcal{T}_{n-1}^{\nu} - V$ implies that

$$\psi_n^{\nu+1}(z_n^\nu - V) - \psi_{n-1}^{\nu+1}(z_n^\nu) = \alpha_n^\nu(z_n^\nu)\frac{\tau_{n-1}^\nu(z_n^\nu - V)}{\tau_n^{\nu+1}(z_n^\nu - V)\tau_{n-1}^{\nu+1}(z_n^\nu)}\, C^{-1}. \tag{5.6}$$

On the other hand, the evaluation of $I_n^{\nu+1}$ at the divisor $\mathcal{T}_n^{\nu} - V$ implies that

$$\psi_{n+1}^{\nu+1}(z_n^\nu) - \psi_n^{\nu+1}(z_n^\nu - V) = -\alpha_n^\nu(z_n^\nu)\frac{\tau_{n+1}^\nu(z_n^\nu - V)}{\tau_{n+1}^{\nu+1}(z_n^\nu)\tau_n^{\nu+1}(z_n^\nu - V)}\, C, \tag{5.7}$$

while the evaluation of $I_{n-1}^{\nu+1}$ at the divisor $\mathcal{T}_n^{\nu}$ implies that

$$\psi_n^{\nu+1}(z_n^\nu + V) - \psi_{n-1}^{\nu+1}(z_n^\nu) = \alpha_n^\nu(z_n^\nu)\frac{\tau_{n-1}^\nu(z_n^\nu + V)}{\tau_n^{\nu+1}(z_n^\nu + V)\tau_{n-1}^{\nu+1}(z_n^\nu)}\, C. \tag{5.8}$$

The left-hand side of the difference of (5.5) and (5.6) is the same as that of the difference of (5.7) and (5.8); equating the right-hand sides of these differences yields (5.4). □

Formulation (A) of our main theorem implies that the assumption of Lemma 5.1 is satisfied for $C = c_3$, $z \in \mathbb{C}^g$ and that

$$\tau_n^\nu(z) = \theta\,(Un + (1-\nu)W + z)\,(c_1^{(l,z)}c_2^n)^{\nu-1/2}, \tag{5.9}$$

where $l \in \mathbb{C}^g$ is a vector such that $(l, V) = 1$. Then from (5.4) for $\nu = 0$, we get on the divisor $\mathcal{T}_0^0$ (that is, for $\theta(Z) = \theta(z+W) = 0$),

$$\begin{aligned}&\tau_1^1(z)\,\tau_0^1(z+V)\,\tau_{-1}^0(z-V) + \tau_1^0(z+V)\,\tau_0^1(z-V)\,\tau_{-1}^1(z)\\ &\quad = c_3^2\big(\tau_1^1(z)\,\tau_0^1(z-V)\,\tau_{-1}^0(z+V) + \tau_1^0(z-V)\,\tau_0^1(z+V)\,\tau_{-1}^0(z)\big),\end{aligned} \tag{5.10}$$

which, after substituting (5.9) and canceling the common factors, yields

$$\begin{aligned}&c_1^2c_2^2\,\theta(Z+U-W)\theta(Z+V-W)\theta(Z-U-V)\\ &\qquad + \theta(Z+U+V)\theta(Z-V-W)\theta(Z-U-W)\\ &\quad = c_2^2c_3^2\,\theta(Z+U-W)\theta(Z-V-W)\theta(Z+V-U)\\ &\qquad + c_1^2c_3^2\,\theta(Z-V+U)\theta(Z+V-W)\theta(Z-U-W),\end{aligned} \tag{5.11}$$

which is identical to equation (2.6) with the minus sign chosen for W (and, correspondingly, the constants c_1 and c_2 appearing in positive power). Similarly, the case of $\nu = 1, n = 0$ of formula (5.4) yields the plus-sign case of (2.6). The implication that (A) $\Rightarrow$ (C) in the main theorem is thus proved.

We now show that (C) can also be obtained as a corollary of a more general *fourth order* relation for Prym theta functions. As mentioned above, it was proved in [8] that equation (2.1) implies the five-term equation (3.26). Note that all pairs of indices have sums of the same parity; that is, equation (3.26) is in fact a pair of equations on two functions ψ defined on two sublattices of the variables (n, m).

The statement that $\psi_{n,m}$ satisfy (3.26) can be proved directly. Indeed, all the functions involved in the equation are in

$$H^0\big(D + (n+1)P_1^+ - (n-1)P_1^- + (m+1)P_2^+ - (m-1)P_2^- + \nu(P_3^+ - P_3^-)\big).$$

By the Riemann-Roch theorem, the dimension of the latter space is 4. Hence, any five elements of this space are linearly dependent, and it remains to find the coefficients of (3.26) by a comparison of singular terms at the points $P_1^\pm$, $P_2^\pm$. For $n+m = 0 \bmod 2$, we get

$$\begin{aligned}
\tilde{a}_{n,m} &= c_1^2 c_3^2 \frac{\theta(Z_{n,m}+V)\theta(Z_{n,m}+U-V+W)}{\theta(Z_{n,m}-V)\theta(Z_{n,m}+U+V+W)}, \\
\tilde{b}_{n,m} &= c_2^2 c_3^2 \frac{\theta(Z_{n,m}+U)\theta(Z_{n,m}-U+V+W)}{\theta(Z_{n,m}-U)\theta(Z_{n,m}+U+V+W)}, \\
\tilde{c}_{nm} &= c_1^2 c_2^2 \frac{\theta(Z_{n,m}+U)\theta(Z_{n,m}+V)\theta(Z_{n,m}-U-V+W)}{\theta(Z_{n,m}-U)\theta(Z_{n,m}-V)\theta(Z_{n,m}+U+V+W)},
\end{aligned} \tag{5.12}$$

where $Z_{n,m} = Z + Un + Vm$. From the normalization of $\psi_{n,m}$, it follows that

$$\tilde{d}_{nm} = 1 - \tilde{a}_{n,m} - \tilde{b}_{n,m} + \tilde{c}_{n,m}. \tag{5.13}$$

Substituting (3.23), (5.12), and (5.13) here proves the following statement.

PROPOSITION 5.2

For any four points A, U, V, W on the image $\Gamma \hookrightarrow \mathcal{P}(\Gamma)$ and for any $Z \in \mathcal{P}(\Gamma)$, the following equation holds:

$$\begin{aligned}
\theta(Z+W) \times [&\theta(A+U+V+Z)\theta(Z-U)\theta(Z-V) \\
&- c_1^2 c_3^2 \theta(A+U-V+Z)\theta(Z-U)\theta(Z+V) \\
&- c_2^2 c_3^2 \theta(A-U+V+Z)\theta(Z+U)\theta(Z-V) \\
&+ c_1^2 c_2^2 \theta(A-U-V+Z)\theta(Z+U)\theta(Z+V)] \\
= \theta(A+Z) \times [&\theta(W+U+V+Z)\theta(Z-U)\theta(Z-V) \\
&- c_1^2 c_3^2 \theta(W+U-V+Z)\theta(Z-U)\theta(Z+V) \\
&- c_2^2 c_3^2 \theta(W-U+V+Z)\theta(Z+U)\theta(Z-V) \\
&+ c_1^2 c_2^2 \theta(W-U-V+Z)\theta(Z+U)\theta(Z+V)].
\end{aligned} \tag{5.14}$$

To the best of our knowledge, equation (5.14) is a new identity for Prym theta functions. For Z such that $\theta(W+Z)=0$, it is equivalent to equation (2.6) with the minus sign chosen. The second equation of the pair (2.6) can be obtained from (3.26) considered for the odd case (i.e., for $n+m=1 \mod 2$). Using theta-functional formulas, it can be shown using (3.26) that equation (5.14) is equivalent to (2.1).

Wave solutions

In Section 2, we proved that if $\theta(Z)$ is the Prym theta function, then equation (2.1) with u as in (2.2) has not just one solution ψ of the form (2.4), but a family of solutions parameterized by points A in the image $\Gamma \longrightarrow \mathcal{P}(\Gamma)$ under the Abel-Prym map. Note, however, that formulation (C) of the main theorem does not involve (A). The first step in proving the *only if* part of (C) (and thus also of (A) and (B), which imply (C)) is to introduce a *spectral parameter* in the problem, that is, to show that equations (2.6) are sufficient for the existence of certain formal solutions of equations (5.1). These solutions are functions of the form

$$\psi_n^\nu(z) = k^n C^{(l,z)}\,\phi_n^\nu(z,k), \tag{5.15}$$

where k^{-1} is a formal parameter (eventually to be identified with the local coordinate on the curve), $\phi_n^\nu(z,k)$ is a regular series in k^{-1}; that is,

$$\phi_n^\nu(z,k) = \sum_{s=0}^{\infty} \xi_{n,s}^\nu(z) k^{-s}, \tag{5.16}$$

and $l \in \mathbb{C}^d$ is such that $(l,V)=1$.

The ultimate goal of this section is to show that such solutions exist with $\xi_{n,s}^\nu$ being holomorphic functions of $z \in \mathbb{C}^g$ defined outside the divisor $\theta(z+Un+(1-\nu)W)=0$.* As we see below, an obstruction for the existence of such solutions is the "bad locus."

$$\Sigma := \Sigma^0 \cup \Sigma^1,$$

where Σ^ν is the V-invariant subvariety of the divisor $\Theta+(\nu-1)W$ that is not U-invariant; that is,

$$\Sigma^\nu := \left\{ Z \in X : \begin{array}{ll} \forall n \in \mathbb{Z} & \theta(Z+nV+(1-\nu)W)=0; \\ \exists n \in \mathbb{Z} & \theta(Z+U+nV+(1-\nu)W) \neq 0 \end{array} \right\}. \tag{5.17}$$

*In [17], [18], [19] the corresponding solutions were called λ-*periodic*, reflecting the normalization leading to their definition. The idea of that normalization goes back to [21].

We prove in Lemma 5.10 that the bad locus is empty, but until then we construct wave solutions with the desired properties only along certain affine subspaces of $\mathbb{C}^g$; then we patch these together.[†]

Notation

Denote by $\pi : \mathbb{C}^g \to X = \mathbb{C}^g/\Lambda$ the universal cover map for X. Let Y be the Zariski closure of the group $\langle \mathbb{Z}V \rangle \subset X$. As an abelian subvariety, it is generated by its irreducible component Y^0, containing zero, and by the point V_0 of finite order in X, such that $V - V_0 \in Y^0$, $NV_0 = \lambda_0 \in \Lambda$. Shifting Y if needed, we may assume, without loss of generality, that zero is not in the bad locus Σ. Since any subset of Y that is invariant under the shift by V is dense in Y, this implies that $Y \cap \Sigma = \emptyset$.

We denote $\mathcal{C} := \pi^{-1}(Y)$. Then $\mathcal{C}$ is a union of its connected component passing through zero (which is a linear subspace $\mathbb{V} \cong \mathbb{C}^d \subset \mathbb{C}^g$) and shifts by a preimage of a vector of finite order; that is, we have $\mathcal{C} = \cup_{r\in\mathbb{Z}}(\mathbb{V} + rV_0)$. Denoting then $\Lambda_0 := \Lambda \cap \mathcal{C}$, we have $Y = \mathcal{C}/\Lambda_0$, and we can also write $\Lambda_0 = \widetilde{\Lambda}_0 + \mathbb{Z}V_0$, where $\widetilde{\Lambda}_0 := \Lambda \cap \mathbb{V}$.

In what follows, we assume that $\tau_n^\nu(z)$ are holomorphic functions of the variable $z \in \mathcal{C}$ that do not vanish identically and have the following factors of automorphy with respect to Λ_0:

$$\tau_n^\nu(z+\lambda) = \tau_n^\nu(z)\, e^{(z,\,\alpha_\lambda)+n\beta_\lambda+w_\lambda^\nu}, \tag{5.18}$$

where α_λ, β_λ^ν are independent of n, and we define for further use

$$b_\lambda^\nu := e^{\beta_\lambda+w_\lambda^\nu-w_\lambda^{\nu+1}}. \tag{5.19}$$

This means that $u_n^\nu(z)$ given by (5.2) is a section of some degree zero line bundle on Y.

PROPOSITION 5.3

Suppose that equation (5.4) for $\tau_n^\nu(z)$ holds. Then equations (5.1) with potentials $u_n^\nu(z)$ given by (5.2) have wave solutions of the form (5.15) such that

(i) *the coefficients $\xi_{n,s}^\nu(z)$ of the formal series $\phi_n^\nu(z,k)$ are meromorphic functions of the variable $z \in \mathcal{C}$ with a simple pole at the divisor $\mathcal{T}_n^\nu$,*

$$\xi_{n,s}^\nu(z) = \frac{\tau_{n,s}^{\nu+1}(z)}{\tau_n^\nu(z)}, \tag{5.20}$$

[†] The locus Σ is an analog of singular locus considered in [28]. We are grateful to Enrico Arbarello for an explanation of its crucial role, which helped us focus on the heart of the problem.

where $\tau_{n,s}^{\nu+1}(z)$ is a holomorphic function (the shift from ν to $\nu+1$ is only for notational ease to simplify further formulas), and

$$\tau_{n,0}^{\nu}(z) = \tau_{n-1}^{\nu}(z); \tag{5.21}$$

(ii) *each of the individual terms in the power series expansion of ϕ have the following automorphy properties (note we are not yet making any claims regarding ϕ as a whole):*

$$b_{\lambda}^{\nu}\xi_{n,s}^{\nu}(z+\lambda) - \xi_{n,s}^{\nu}(z) = \sum_{i=1}^{s} B_{i,n-s+i}^{\lambda}\xi_{n,s-i}^{\nu}(z) \tag{5.22}$$

for any $\lambda \in \Lambda_0$ (notice that the coefficients depend on i and in a sort of diagonal way on n, but do not depend on ν, which is important for future computations).

Proof
Writing down the equation for ψ in terms of the power series expansions in k^{-s} and equating coefficient of k^{-s} to zero (i.e., substituting (5.2), (5.15), (5.16) into (5.1)) yields

$$C\,\xi_{n+1,\,s+1}^{\nu}(z+V) - u_n^{\nu}(z)\left(\xi_{n+1,\,s+1}^{\nu+1}(z) - C\,\xi_{n,\,s}^{\nu+1}(z+V)\right) + \xi_{n,s}^{\nu}(z) = 0. \tag{5.23}$$

For $s=-1$, equation (5.23) is satisfied with $\tau_{n,0}^{\nu}$ given by (5.21), that is, with

$$\xi_{n,\,0}^{\nu}(z) = \frac{\tau_{n-1}^{\nu+1}(z)}{\tau_n^{\nu}(z)}. \tag{5.24}$$

We now prove the lemma by induction in s. Let us assume inductively that for $r \le s-1$ the functions $\xi_{n,r}^{\nu}(z)$ are known *for all n and ν* and satisfy the quasi-periodicity condition (5.22) above (it is customary in the subject to call such solutions Bloch solutions or Bloch functions).

The idea of the proof of the inductive step is as follows. We write down the equation relating $\tau_{n+1,s+1}^{\nu}$ (we are using $n+1$ instead of n solely for the ease of notation; recall that the inductive assumption is for all n) to the τ for smaller values of s (which we know inductively to exist and be holomorphic). From this equation we then get an explicit formula for $\tau_{n+1,s+1}^{\nu}$ on the divisor $\mathcal{T}_n^{\nu}$ (i.e., for $\tau_n^{\nu}(z)=0$). We also get an explicit formula for $\tau_{n+1,s+1}^{\nu}$ for z such that $\tau_n^{\nu}(z+V)=0$, which after translating the argument gives another formula for $\tau_{n+1,s+1}^{\nu}$ on the divisor $\mathcal{T}_n^{\nu}$. Once we verify that the two resulting formulas agree (this is a hard computation using the step of the induction), it follows that $\tau_{n+1,s+1}^{\nu}$ restricted to $\mathcal{T}_n^{\nu}$ is in fact holomorphic, and thus can be extended from this divisor holomorphically to $\mathbb{C}^d$. We now give the details of this argument.

Writing down equation (5.23) in terms of τ's for arbitrary s and clearing denominators yields

$$C\tau^{\nu+1}_{n+1,s+1}(z+V)\tau^{\nu}_{n}(z) - C\tau^{\nu+1}_{n}(z+V)\tau^{\nu}_{n+1,s+1}(z) - C^2\tau^{\nu}_{n,s}(z+V)\tau^{\nu+1}_{n+1}(z) + \tau^{\nu}_{n+1}(z+V)\tau^{\nu+1}_{n,s}(z) = 0. \tag{5.25}$$

These equations can be easily solved on the divisor $\mathcal{T}^{\nu}_{n}$. Indeed, if we take $z = z^{\nu}_{n} \in \mathcal{T}^{\nu}_{n}$ here, the first term vanishes and we get the following formula:

$$C\tau^{\nu}_{n+1,s+1}(z^{\nu}_{n}) = \frac{\tau^{\nu+1}_{n,s}(z^{\nu}_{n})\tau^{\nu}_{n+1}(z^{\nu}_{n}+V) - C^2\tau^{\nu}_{n,s}(z^{\nu}_{n}+V)\tau^{\nu+1}_{n+1}(z^{\nu}_{n})}{\tau^{\nu+1}_{n}(z^{\nu}_{n}+V)}. \tag{5.26}$$

Alternatively, using equation (5.25) for $\nu+1$ instead of ν and setting $z = z^{\nu}_{n} - V$, for $z^{\nu}_{n} \in \mathcal{T}^{\nu}_{n}$ as above, we get

$$C\tau^{\nu}_{n+1,s+1}(z^{\nu}_{n}) = \frac{\tau^{\nu+1}_{n,s}(z^{\nu}_{n})\tau^{\nu}_{n+1}(z^{\nu}_{n}-V) - C^2\tau^{\nu}_{n,s}(z^{\nu}_{n}-V)\tau^{\nu+1}_{n+1}(z^{\nu}_{n})}{\tau^{\nu+1}_{n}(z^{\nu}_{n}-V)}. \tag{5.27}$$

For $\tau^{\nu}_{n+1,s+1}$ to have a chance to exist, these two expressions must agree.

LEMMA 5.4
If the inductive assumption (and the conditions of the proposition, in particular formula (5.4)) is satisfied for s, then the two expressions above for the function $\tau^{\nu}_{n+1,s+1}(z)$ restricted to the divisor $\mathcal{T}^{\nu}_{n}$ are equal.

Proof
Equating the two expressions obtained for $\tau^{\nu}_{n+1,s+1}$ on $\mathcal{T}^{\nu}_{n}$, we see that what we need to prove is the following identity:

$$\begin{aligned}&\tau^{\nu+1}_{n,s}(z^{\nu}_{n})\tau^{\nu}_{n+1}(z^{\nu}_{n}-V)\tau^{\nu+1}_{n}(z^{\nu}_{n}+V) - C^2\tau^{\nu}_{n,s}(z^{\nu}_{n}-V)\tau^{\nu+1}_{n+1}(z^{\nu}_{n})\tau^{\nu+1}_{n}(z^{\nu}_{n}+V)\\ &\quad = \tau^{\nu+1}_{n,s}(z^{\nu}_{n})\tau^{\nu}_{n+1}(z^{\nu}_{n}+V)\tau^{\nu+1}_{n}(z^{\nu}_{n}-V) - C^2\tau^{\nu}_{n,s}(z^{\nu}_{n}+V)\tau^{\nu+1}_{n+1}(z^{\nu}_{n})\tau^{\nu+1}_{n}(z^{\nu}_{n}-V).\end{aligned} \tag{5.28}$$

To prove that this is the case, we use the inductive assumption for $n-1$, $s-1$, and equation (5.4). Indeed, for $n-1$, $s-1$, equation (5.25) reads

$$\begin{aligned}&C\tau^{\nu+1}_{n,s}(z+V)\tau^{\nu}_{n-1}(z) - C\tau^{\nu+1}_{n-1}(z+V)\tau^{\nu}_{n,s}(z)\\ &\quad - C^2\tau^{\nu}_{n-1,s-1}(z+V)\tau^{\nu+1}_{n}(z) + \tau^{\nu}_{n}(z+V)\tau^{\nu+1}_{n-1,s-1}(z) = 0.\end{aligned}$$

By the inductive assumption, we know that this is satisfied. If we now take $z = z_n^\nu - V$, that is, set $\tau_n^\nu(z+V) = 0$ here, we get

$$C^2 \tau^\nu_{n-1,s-1}(z_n^\nu) = \frac{C\tau^{\nu+1}_{n,s}(z_n^\nu)\tau^\nu_{n-1}(z_n^\nu - V) - C\tau^{\nu+1}_{n-1}(z_n^\nu)\tau^\nu_{n,s}(z_n^\nu - V)}{\tau_n^{\nu+1}(z_n^\nu - V)}.$$

Similarly, if we instead take the equation with $\nu+1$ instead of ν, and take $z = z_n^\nu$, we get

$$\tau^\nu_{n-1,s-1}(z_n^\nu) = \frac{C\tau^\nu_{n-1}(z_n^\nu + V)\tau^{\nu+1}_{n,s}(z_n^\nu) - C\tau^\nu_{n,s}(z_n^\nu + V)\tau^{\nu+1}_{n-1}(z_n^\nu)}{\tau_n^{\nu+1}(z_n^\nu + V)}.$$

Since we inductively assumed the existence and uniqueness of $\tau^\nu_{n-1,s-1}$, these two expressions must agree, which is to say that we have the following identity:

$$\begin{aligned}
&\tau^{\nu+1}_{n,s}(z_n^\nu)\tau^\nu_{n-1}(z_n^\nu - V)\tau_n^{\nu+1}(z_n^\nu + V) - \tau^\nu_{n,s}(z_n^\nu - V)\tau^{\nu+1}_{n-1}(z_n^\nu)\tau_n^{\nu+1}(z_n^\nu + V)\\
&\quad = C^2\tau^{\nu+1}_{n,s}(z_n^\nu)\tau^\nu_{n-1}(z_n^\nu + V)\tau_n^{\nu+1}(z_n^\nu - V) - C^2\tau^\nu_{n,s}(z_n^\nu + V)\tau^{\nu+1}_{n-1}(z_n^\nu)\tau_n^{\nu+1}(z_n^\nu - V).
\end{aligned} \tag{5.29}$$

Notice now how similar this known identity is to formula (5.28) that we need to prove. Indeed, the coefficient of $\tau^{\nu+1}_{n,s}(z_n^\nu)$ in (5.28) is equal to

$$\tau^\nu_{n+1}(z_n^\nu - V)\tau_n^{\nu+1}(z_n^\nu + V) - \tau^\nu_{n+1}(z_n^\nu + V)\tau_n^{\nu+1}(z_n^\nu - V).$$

Now using formula (5.4), which we know holds for τ, we see that this coefficient is equal to

$$\big(\tau_n^{\nu+1}(z_n^\nu - V)\tau^\nu_{n-1}(z_n^\nu + V) - \tau_n^{\nu+1}(z_n^\nu + V)\tau^\nu_{n-1}(z_n^\nu - V)\big)\,\frac{\tau^{\nu+1}_{n+1}(z_n^\nu)}{\tau^{\nu+1}_{n-1}(z_n^\nu)}. \tag{5.30}$$

Substituting this expression into (5.28) is equivalent to the identity

$$\begin{aligned}
&\tau^{\nu+1}_{n,s}(z_n^\nu)\frac{\tau^{\nu+1}_{n+1}(z_n^\nu)}{\tau^{\nu+1}_{n-1}(z_n^\nu)}\big(\tau_n^{\nu+1}(z_n^\nu - V)\tau^\nu_{n-1}(z_n^\nu + V) - C^2\tau_n^{\nu+1}(z_n^\nu + V)\tau^\nu_{n-1}(z_n^\nu - V)\big)\\
&\quad = \tau^\nu_{n,s}(z_n^\nu - V)\tau^{\nu+1}_{n+1}(z_n^\nu)\tau_n^{\nu+1}(z_n^\nu + V) - C^2\tau^\nu_{n,s}(z_n^\nu + V)\tau^{\nu+1}_{n+1}(z_n^\nu)\tau_n^{\nu+1}(z_n^\nu - V).
\end{aligned} \tag{5.31}$$

Multiplying this identity by $\tau^{\nu+1}_{n-1}(z_n^\nu)/\tau^{\nu+1}_{n+1}(z_n^\nu)$ yields formula (5.29), which we inductively know to hold. Thus formula (5.28) holds, and the lemma is proved. □

LEMMA 5.5

The function $\tau^{\nu}_{n+1,s+1}(z^{\nu}_n)$ given by (5.26) and (5.27) can be extended to a holomorphic function on the entire divisor $\mathcal{T}^{\nu}_n$.

Proof

The expression (5.26) for $\tau^{\nu}_{n+1,s+1}(z^{\nu}_n)$ is certainly holomorphic when $\tau^{\nu+1}_n(z^{\nu}_n+V)$ is nonzero, that is, is holomorphic outside of $\mathcal{T}^{\nu}_n \cap (\mathcal{T}^{\nu+1}_n - V)$. Similarly, the expression for $\tau^{\nu}_{n+1,s+1}$ given by formula (5.27) is holomorphic away from $\mathcal{T}^{\nu}_n \cap (\mathcal{T}^{\nu+1}_n + V)$.

We have assumed that the closure of the abelian subgroup generated by V is everywhere dense. Thus for any $z^{\nu}_n \in \mathcal{T}^{\nu}_n$, there must exist some $N \in \mathbb{N}$ such that $z^{\nu}_n + (N+1)V \notin \mathcal{T}^{\nu+1}_n$; let N moreover be the minimal such N. From (5.26), it then follows that $\tau^{\nu}_{n+1,s+1}$ can be extended holomorphically to the point $z^{\nu}_n + NV$. However, by Lemma 5.4 we know that the expressions (5.27) and (5.26) agree. Thus expression (5.27) must also be holomorphic at $z^{\nu}_n + NV$; since its denominator there vanishes, it means that the numerator must also vanish; that is, we must have

$$C\tau^{\nu+1}_{n,s}(z^{\nu}_n+NV)\tau^{\nu}_{n+1}(z^{\nu}_n+(N-1)V) - \tau^{\nu}_{n,s}(z^{\nu}_n+(N-1)V)\tau^{\nu+1}_{n+1}(z^{\nu}_n+NV) = 0.$$

But this expression is equal to the numerator of (5.26) at $z^{\nu}_n + (N-1)V$; thus $\tau^{\nu}_{n+1,s+1}$ defined from (5.26) is also holomorphic at $z^{\nu}_n + (N-1)V$ (the numerator vanishes, and the vanishing order of the denominator is one, since we are talking exactly about points on its vanishing divisor). Thus unless $N=0$, we have a contradiction, since N was chosen minimal. For $N=0$, however, $z^{\nu}_n + V \notin \mathcal{T}^{\nu+1}_n$, and thus (5.26) defines $\tau^{\nu}_{n+1,s+1}$ holomorphically at z^{ν}_n. □

Recall now that an analytic function on an analytic divisor in $\mathbb{C}^d$ has a holomorphic extension to all of $\mathbb{C}^d$ (see [27]). Therefore, there exists a holomorphic function $\widetilde{\tau}^{\nu}_{n+1,s+1}(z)$ extending the function given on the divisor $\mathcal{T}^{\nu}_n$ by the right-hand side of (5.26) (by Lemma 5.5, it is holomorphic, and thus the extension is holomorphic). It is then natural to attempt to use the function $\widetilde{\xi}^{\nu}_{n+1,s+1} := \widetilde{\tau}^{\nu+1}_{n+1,s+1}/\tau^{\nu}_{n+1}$ for the proposition, but this cannot be done immediately, as such an extension does not need to be quasi-periodic, nor is it going to be a solution of equation (5.23). We thus need to adjust this extension appropriately.

We start by determining the quasi-periodicity properties; indeed, for $z^{\nu+1}_n \in \mathcal{T}^{\nu+1}_n$, where we know that $\widetilde{\tau}^{\nu+1}_{n+1,s+1}$ is given by (5.26), we have

$$\widetilde{\xi}^{\nu}_{n+1,s+1}(z^{\nu+1}_n) = -C\xi^{\nu}_{n,s}(z^{\nu+1}_n+V) + \frac{\tau^{\nu}_{n,s}(z^{\nu+1}_n)\tau^{\nu+1}_{n+1}(z^{\nu+1}_n+V)}{\tau^{\nu}_n(z^{\nu+1}_n+V)\tau^{\nu}_{n+1}(z^{\nu+1}_n)}, \tag{5.32}$$

from which by using the quasi periodicity of τ_n (5.18) and that of $\tau_{n,s}$ (5.22), it follows that

$$b_\lambda^\nu \widetilde{\xi}_{n+1,s+1}^\nu(z_n^{\nu+1}+\lambda) = -C\Big(\xi_{n,s}^\nu(z_n^{\nu+1}+V) - \sum_{i=1}^s B_{i,n-s+i}^\nu \xi_{n,s-i}^\nu(z_n^{\nu+1}+V)\Big)$$
$$+\frac{(\tau_{n,s}^\nu(z_n^{\nu+1}) + \sum_{i=1}^s B_{i,n-s+i}^\lambda \tau_{n,s-i}^\nu(z_n^{\nu+1}))\tau_{n+1}^{\nu+1}(z_n^{\nu+1}+V)}{\tau_n^\nu(z_n^{\nu+1}+V)\tau_{n+1}^\nu(z_n^{\nu+1})}, \tag{5.33}$$

since the $e^{(2z+V,\alpha_\lambda)+(2n+1)\beta_\lambda}$ factors for the second term of (5.32) coming from (5.18) cancel in the numerator and denominator, and the remaining $e^{2\omega_\lambda^{\nu+1}-2\omega_\lambda^\nu}$ cancels with $b_\lambda^\nu/b_\lambda^{\nu+1}$. We now note that the terms in the right-hand side split in pairs similar to those in (5.32) and we can thus simplify this to get

$$0 = b_\lambda^\nu \widetilde{\xi}_{n+1,s+1}^\nu(z_n^{\nu+1}+\lambda) - \widetilde{\xi}_{n+1,s+1}^\nu(z_n^{\nu+1}) - \sum_{i=1}^s B_{i,n-s+i}^\lambda \xi_{n+1,s+1-i}^\nu(z_n^{\nu+1}). \tag{5.34}$$

This says that the function on the right-hand side here (denote it by $g_{n+1,s+1}^{\lambda,\nu}(z)$) vanishes for $z = z_n^{\nu+1} \in \mathcal{T}_n^{\nu+1}$ and has a pole for $z \in \mathcal{T}_{n+1}^\nu$. Using formula (5.24) for $\xi_{n,0}^\nu$, we can then write

$$g_{n+1,s+1}^{\lambda,\nu}(z) = f_{n+1,s+1}^{\lambda,\nu}(z)\xi_{n+1,0}^\nu(z),$$

where $f_{n+1,s+1}^{\lambda,\nu}(z)$ is now holomorphic and satisfies the twisted homomorphism relations

$$f_{n+1,s+1}^{\lambda+\mu,\nu}(z) = f_{n+1,s+1}^{\lambda,\nu}(z+\mu) + f_{n+1,s+1}^{\mu,\nu}(z). \tag{5.35}$$

We only know the function $\widetilde{\xi}$ to have the desired quasi periodicity on the divisor $\mathcal{T}_n^{\nu+1}$, and would now like to adjust it so that the corrected function would have computable quasi periodicity for all z. To achieve this, we need to add to $\widetilde{\xi}$ a summand involving f.

Indeed, f defines an element of the first cohomology group of Λ_0 with coefficients in the sheaf of holomorphic functions, $f \in H_{gr}^1(\Lambda_0, H^0(\mathbb{C}^d, \mathcal{O}))$. Arguments identical to that in the proof of part (b) in [28, Lemma 12] show that there must then exist a holomorphic function $h_{n+1,s+1}^\nu(z)$ such that

$$f_{n+1,s+1}^{\lambda,\nu}(z) = h_{n+1,s+1}^\nu(z+\lambda) - h_{n+1,s+1}^\nu(z) + E_{n+1,s+1}^{\lambda,\nu}, \tag{5.36}$$

where $E^{\lambda,\,\nu}_{n+1,s+1}$ is a (z-independent!) constant. By using equation (5.35), we observe that E depends on λ linearly, that is, that

$$E^{\lambda+\mu,\nu}_{n+1,s+1} = E^{\lambda,\nu}_{n+1,s+1} + E^{\mu,\nu}_{n+1,s+1}. \tag{5.37}$$

We then define

$$\zeta^{\,\nu}_{n+1,s+1}(z) := \widetilde{\xi}^{\,\nu}_{n+1,s+1}(z) - h^{\nu}_{n+1,s+1}(z)\xi^{\,\nu}_{n+1,0}(z).$$

Using (5.24) and (5.18), we first compute

$$\xi^{\,\nu}_{n+1,0}(z+\lambda) = \frac{\tau^{\nu+1}_{n}(z+\lambda)}{\tau^{\nu}_{n+1}(z+\lambda)} = e^{\omega^{\nu+1}_{\lambda}-\omega^{\nu}_{\lambda}-\beta_{\lambda}}\frac{\tau^{\nu+1}_{n}(z)}{\tau^{\nu}_{n+1}(z)} = \frac{\xi^{\,\nu}_{n+1,0}(z)}{b^{\nu}_{\lambda}} \tag{5.38}$$

and then compute the quasi periodicity

$$\begin{aligned}
&b^{\nu}_{\lambda}\zeta^{\,\nu}_{n+1,s+1}(z+\lambda) - \zeta^{\,\nu}_{n+1,s+1}(z)\\
&\quad= \big(b^{\nu}_{\lambda}\widetilde{\xi}^{\,\nu}_{n+1,s+1}(z+\lambda) - \widetilde{\xi}^{\,\nu}_{n+1,s+1}(z)\big)\\
&\qquad - b^{\nu}_{\lambda}h^{\lambda,\nu}_{n+1,s+1}(z+\lambda)\xi^{\,\nu}_{n+1,0}(z+\lambda) + h^{\lambda,\nu}_{n+1,s+1}(z)\xi^{\,\nu}_{n+1,0}(z)\\
&\quad= \Big(g^{\lambda,\nu}_{n+1,s+1}(z) + \sum_{i=1}^{s} B^{\lambda}_{i,n-s+i}\xi^{\,\nu}_{n+1,s+1-i}(z)\Big)\\
&\qquad + \big(E^{\lambda,\,\nu}_{n+1,s+1} - f^{\lambda,\nu}_{n+1,s+1}(z)\big)\xi^{\,\nu}_{n+1,0}(z)\\
&\quad= E^{\lambda,\,\nu}_{n+1,s+1}\xi^{\,\nu}_{n+1,0}(z) + \sum_{i=1}^{s} B^{\lambda}_{i,n-s+i}\xi^{\,\nu}_{n+1,s+1-i}(z).
\end{aligned} \tag{5.39}$$

We have now constructed a function ζ having the correct quasi-periodicity properties (although the first coefficient depends on ν, so we need to deal with this below) but we still cannot take it to be the function $\xi^{\,\nu}_{n+1,s+1}$ that we are trying to define, as it may not satisfy equation (5.23). We thus define $R^{\,\nu}_{n+1,s+1}$ to be the error obtained by plugging ζ into (5.23):

$$\begin{aligned}
R^{\,\nu}_{n+1,s+1}(z)\xi^{\,\nu}_{n+1,0}(z+V) :&= C\,\zeta^{\,\nu}_{n+1,\,s+1}(z+V) - u^{\nu}_{n}(z)\big(\zeta^{\,\nu+1}_{n+1,\,s+1}(z)\\
&\qquad -C\,\xi^{\,\nu+1}_{n,\,s}(z+V)\big) + \xi^{\,\nu}_{n,s}(z).
\end{aligned} \tag{5.40}$$

Notice that for this to make sense we need to assume that we have been performing all of the above computations simultaneously for ν and $\nu+1$, so that indeed both ζ's above are defined at this point.

From Lemma 5.5, we know that the right-hand side of this formula has no pole at $\mathcal{T}^{\,\nu}_{n}$ and vanishes at $\mathcal{T}^{\,\nu+1}_{n} - V$, and thus we know that $R^{\,\nu}_{n+1,s+1}$ is a holomorphic function of z. We can use (5.22) and (5.39) to compute the transformation properties of R under a shift by a vector $\lambda \in \Lambda_0$. Indeed, using (5.18) to compute $b^{\nu}_{\lambda}u^{\nu}_{n}(z+\lambda) = u^{\nu}_{n}(z)b^{\nu+1}_{\lambda}$, and using (5.38) for the left-hand side, we get, shifting by λ and multiplying by b_λ^ν and subtracting the original function,

$$
\begin{aligned}
&\left(R^\nu_{n+1,s+1}(z+\lambda) - R^\nu_{n+1,s+1}(z)\right)\xi^\nu_{n+1,0}(z+V)\\
&= C E^{\lambda,\nu}_{n+1,s+1}\xi^\nu_{n+1,0}(z+V) + \sum_{i=1}^{s} B^\lambda_{i,n-s+i}\xi^\nu_{n+1,s+1-i}(z+V)\\
&\quad -u^\nu_n(z)\Big(E^{\lambda,\nu+1}_{n+1,s+1}\xi^{\nu+1}_{n+1,0}(z) + \sum_{i=1}^{s} B^\lambda_{i,n-s+i}\xi^{\nu+1}_{n+1,s+1-i}(z)\\
&\quad -C\sum_{i=1}^{s} B^\lambda_{i,n-s+i}\xi^{\nu+1}_{n,s-i}(z+V)\Big) + \sum_{i=1}^{s} B^\lambda_{i,n-s+i}\xi^\nu_{n,s-i}(z). \qquad (5.41)
\end{aligned}
$$

Now note that for each constant $B^\lambda_{i,n-s+i}$ in the above expression, the function it multiplies is exactly the right-hand side of (5.23) for n and some $j \le s$ and thus vanishes identically (this uses in a crucial way the fact that B's do not depend on ν). Using formulas (5.2), (5.24) for u^ν_n and $\xi_{n+1,0}$, we get

$$R^\nu_{n+1,s+1}(z+\lambda) - R^\nu_{n+1,s+1}(z) = C(E^{\lambda,\nu}_{n+1,s+1} - E^{\lambda,\nu+1}_{n+1,s+1}).$$

Moreover, by (5.37) we know that the E's are linear functions of λ, that is,

$$E^{\lambda,\nu}_{n+1,s+1} - E^{\lambda,\nu+1}_{n+1,s+1} = 2\ell^\nu_{n+1,s+1}(\lambda)$$

for some linear function ℓ; note that $\ell^\nu_{n+1,s+1}(z) = -\ell^{\nu+1}_{n+1,s+1}(z)$. It then follows that the difference $R - 2\ell$ is periodic with respect to shifts by Λ_0 and is thus constant; that is, we have then $R^\nu_{n+1,s+1}(z) = 2C\ell^\nu_{n+1,s+1}(z) + 2A^\nu$. We can now introduce one last correction and finally define

$$\xi^\nu_{n+1,s+1}(z) := \zeta^\nu_{n+1,s+1}(z) - \left(\ell^\nu_{n+1,s+1}(z - V/2) + A^\nu + l(z)\right)\xi^\nu_{n+1,0}(z), \qquad (5.42)$$

where $l(z)$ is a linear function such that $l(V) = A^\nu + A^{\nu+1}$. These functions are solutions of (5.23); indeed, the new error term is equal to

$$
\begin{aligned}
R^\nu_{n+1,s+1}(z)\xi^\nu_{n+1,0}(z+V) &-\left(\ell^\nu_{n+1,s+1}(z+V/2) + A^\nu + l(z+V)\right)\xi^\nu_{n+1,0}(z+V)\\
&+u^\nu_n(z)\left(\ell^{\nu+1}_{n+1,s+1}(z-V/2) + A^{\nu+1} + l(z)\right)\xi^{\nu+1}_{n+1,0}(z)\\
&= \xi^\nu_{n+1,0}(z+V)\big(R^\nu_{n+1,s+1}(z) - \ell^\nu_{n+1,s+1}(z+V/2)\\
&\quad -A^\nu - l(z+V) + \ell^{\nu+1}_{n+1,s+1}(z-V/2) + A^{\nu+1} + l(z)\big)\\
&= \xi^\nu_{n+1,0}(z+V)\big(2\ell^\nu_{n+1,s+1}(z) + 2A^\nu - \ell^\nu_{n+1,s+1}(z)\\
&\quad -A^\nu - l(V) - \ell^\nu_{n+1,s+1}(z) + A^{\nu+1}\big) = 0,
\end{aligned}
$$

where we used definitions (5.2), (5.21) and the definitions of ℓ, l, and A.

We now need to check that the functions ξ satisfy the quasi-periodicity conditions (5.22). From (5.38) and (5.39), it follows that

$$b_\lambda^\nu \xi_{n+1,s+1}^\nu(z+\lambda) - \xi_{n+1,s+1}^\nu(z) = \left(E_{n+1,s+1}^{\lambda,\nu} - \ell_{n+1,s+1}^\nu(\lambda) - l(\lambda)\right)\xi_{n+1,0}^\nu(z) + \sum_{i=1}^{s} B_{i,n-s+i}^\nu \xi_{n+1,s+1-i}^\nu(z),$$

which means that the function $\xi_{n+1,s+1}^\nu$ satisfies the quasi-periodicity condition (5.22) if we take

$$B_{n+1,s+1}^\lambda := E_{n+1,s+1}^{\lambda,\nu} - \ell_{n+1,s+1}^\nu(\lambda) - l(\lambda) = \frac{E_{n+1,s+1}^{\lambda,\nu} + E_{n+1,s+1}^{\lambda,\nu+1}}{2} - l(\lambda)$$

(notice that this does not depend on ν, as required in formula (5.22)). Observe that the B we construct depends on the choice of the linear function $l(\lambda)$. We have thus constructed a quasi-periodic solution for $s+1$ and proved the inductive step of the proposition. □

COROLLARY 5.6
For $\xi_{n,s}^\nu$ and $\xi_{n,s}^{\nu+1}$ fixed, the solutions of (5.23), for both ν and $\nu+1$, are unique up to the transformation

$$\xi_{n+1,s+1}^\nu(z) \longmapsto \xi_{n+1,s+1}^\nu(z) + \left(c + l(z)\right)\xi_{n+1,0}^\nu(z), \tag{5.43}$$

where c is a constant, and where l is a linear function on $\mathcal{C}$ such that $l(V)=0$ and both c and l are independent of ν.

Proof
This follows by tracing the ambiguity of the choices involved in the proof of the above lemma. Alternatively, one can prove this directly by investigating the quasi-periodicity properties of the difference of two solutions of (5.23). □

To eliminate the freedom of choosing $\xi_{n+1,s+1}^\nu$, we would now like to fix the quasi-periodicity condition to be the same for all n, and to be as simple as possible. Similarly to the case of a nondegenerate trisecant treated in [19], there may be a problem here in that the functions $\xi_{n,s}^\nu$ may turn out to be periodic (in our case, by periodic we mean $b_{\lambda_j}\xi_{n,s}^\nu(z+\lambda_j) = \xi_{n,s}^\nu(z)$). Similarly to the situation in [19], note that the space of periodic functions with a pole on the divisor $\mathcal{T}_n^\nu$ is the space of sections of some line bundle, and thus finite-dimensional. Since all divisors $\mathcal{T}_n^\nu$ differ by shifts, there is an upper bound on this dimension independent of n and ν.

It then follows that the functions $\xi_{n,s}^\nu$, for n fixed and for s and ν varying, are linearly independent. Indeed, suppose that there were some linear relation among them, with the maximal value of s involved in this relation being equal to S. But then solving equations (5.23) with ν and with $\nu+1$, allows one to express $\xi^{\nu}_{n,S}$ in terms of $\xi^{\nu}_{n-1,S-1}$ and $\xi^{\nu+1}_{n-1,S-1}$, and thus obtain a linear relation among the ξ's with index $n-1$, and with maximal s being equal to $S-1$. By downward induction, we can get to $s=0$ and arrive at a contradiction with the fact that $\xi^{\nu}_{n,0}\neq 0$ and is not proportional to $\xi^{\nu+1}_{n,0}$. Note, moreover, that if for some s the function $\xi^{\nu}_{n,s}$ is not periodic, this would mean that some B is nonzero, and thus $\xi^{\nu}_{n,s+i}$ could not be periodic for any $i>0$, as the term in (5.22) with this nonzero B would be linearly independent with all the other terms on the right-hand side there.

LEMMA 5.7
Let $\lambda_0,\lambda_1,\ldots,\lambda_d$ be a set of $\mathbb{C}$-linear independent vectors in Λ_0. Suppose that equation (5.23) has periodic solutions for $i<r$ (and any n and ν), that is, that there are some $\Xi^{\nu}_{n,i}(z)$ such that

$$b^{\nu}_{\lambda_j}\,\Xi^{\nu}_{n,i}(z+\lambda_j)-\Xi^{\nu}_{n,i}(z)=0$$

for all $i<r$, all n and ν, and such that $\Xi^{\nu}_{n,0}=\xi^{\nu}_{n,0}$ is given by (5.24). Suppose also that there are quasi-periodic solutions $\Xi^{\nu}_{n,r}$ with

$$b^{\nu}_{\lambda_j}\,\Xi^{\nu}_{n,r}(z+\lambda_j)-\Xi^{\nu}_{n,r}(z)=A_j\xi^{\nu}_{n,0}(z),\quad \forall j=0,\ldots,d \tag{5.44}$$

for all n, where A_j are some constants such that there does not exist a linear form l on $\mathcal{C}$ with $l(\lambda_j)=A_j$, and $l(V)=0$ (i.e., such that the scalar product of the vector $\vec{A}=(A_1,\ldots,A_d)$ and V is nonzero). Then for all $s\geq r$, and all n and ν equations, (5.23) has quasi-periodic solutions satisfying (5.22) with $B^{\lambda_j}_{i,n}=A_j\delta_{i,r}$; that is, there exist functions $\xi^{\nu}_{n,s}(z)$ for all $s\geq r$ and all n and ν such that

$$b^{\nu}_{\lambda_j}\xi^{\nu}_{n,s}(z+\lambda_j)-\xi^{\nu}_{n,s}(z)=A_j\xi^{\nu}_{n,s-r}(z). \tag{5.45}$$

(Note that we do not necessarily have $\xi^{\nu}_{n,i}(z)=\Xi^{\nu}_{n,i}(z)$ for $i\leq r$, but they satisfy the same quasi periodicity and solve the same equation (5.23).) Moreover, such $\xi^{\nu}_{n,s}(z)$ are unique up to adding $c_{n,s}\xi^{\nu}_{n,0}(z)$, with $c_{n,s}$ being a constant dependent only on the remainder of n modulo r.

Proof
We prove the lemma by induction in s, starting with $s=0$, with the inductive assumption being that functions $\Xi^{\nu}_{n,i}$ satisfying (5.23) and the quasi-periodicity condition (5.45) have been constructed for all n and ν and for all $i\leq r+s$, that they are periodic for $i<r$, and that, moreover, $\xi^{\nu}_{n,i}(z):=\Xi^{\nu}_{n,i}(z)$ for $i\leq s$ (so that the inductive assumption for $s=0$ is the assumption of the lemma).

From (5.43) we know that there must exist solutions $\widetilde{\xi}^{\nu}_{n,s+r+1}(z)$ of (5.23) for all n and ν, with quasi periodicity

$$b^{\nu}_{\lambda_j}\widetilde{\xi}^{\nu}_{n,s+r+1}(z+\lambda_j) = A_j\,\Xi^{\nu}_{n,s+1}(z) + B^{\lambda_j}_{n,s+r+1}\xi^{\nu}_{n,0}(z), \tag{5.46}$$

where B are some new constants. The idea now is that we adjust all the $\Xi^{\nu}_{n,s+i}$ for $0 < i \le r$ to another set of solutions of (5.23) with the same quasi periodicity, so that $\Xi_{n,s+r+1}$ satisfying the quasi-periodicity condition (5.45) would exist.

Indeed, suppose we take $\xi^{\nu}_{n+1,s+1}(z) := \Xi^{\nu}_{n+1,s+1}(z) + c_{n+1,s+1}\xi^{\nu}_{n+1,0}(z)$ for some constant $c_{n+1,s+1}$, independent of ν (if we added $l(z)\xi^{\nu}_{n+1,0}(z)$, the quasi periodicity of $\xi^{\nu}_{n+1,s+1}(z)$ would no longer be the same as that of $\Xi^{\nu}_{n+1,s+1}(z)$). If we make such a change, we also need to add something (call it $f^{\nu}(z)$), to $\Xi^{\nu}_{n+2,s+2}(z)$, so that (5.23) is still satisfied. Since the Ξ's themselves satisfied (5.23), the corrections we introduce must also satisfy it; that is, we must then have

$$Cf^{\nu}(z+V) - u^{\nu}_{n+1}(z)\left(f^{\nu+1}(z) - Cc_{n+1,s+1}\xi^{\nu+1}_{n+1,0}(z+V)\right) + c_{n+1,s+1}\xi^{\nu}_{n+1,0}(z) = 0,$$

and the same for $\nu+1$. However, this is exactly the equation (5.23) that is satisfied by $c_{n+1}\,\Xi^{\nu}_{n+2,1}(z)$, and thus it follows that $f^{\nu}(z) = c_{n+1,s+1}\,\Xi^{\nu}_{n+2,1}(z)$ would work. Similarly, we need to add $c_{n+1,s+1}\,\Xi^{\nu}_{n+i+1,i}(z)$ to each $\Xi^{\nu}_{n+i+1,s+i+1}(z)$, so that all of the equations (5.23) are satisfied. Finally, in this way we see that the necessary adjustment of $\widetilde{\xi}^{\nu}_{n+r+1,s+r+1}$ is

$$\Xi^{\nu}_{n+r+1,s+r+1}(z) := \widetilde{\xi}^{\nu}_{n+r+1,s+r+1}(z) + c_{n+1,s+1}\,\Xi^{\nu}_{n+r+1,r}(z) + l_{n+r+1}(z)\xi^{\nu}_{n+r+1,0}(z),$$

where we now need to allow the presence of a linear term to make the quasi periodicity be (5.45) as desired. From (5.46) and (5.44) we can compute the quasi periodicity to be

$$\begin{aligned}
b^{\nu}_{\lambda_j}\,\Xi^{\nu}_{n+r+1,s+r+1}(z+\lambda_j) - \Xi^{\nu}_{n+r+1,s+r+1}(z) &= A_j\,\Xi^{\nu}_{n+r+1,s+1}(z)\\
+\big(B^{\lambda_j}_{n+r+1,s+r+1} + c_{n+1,s+1}A_j + l_{n+r+1}(\lambda_j)\big)\xi^{\nu}_{n+r+1,0}(z) &= A_j\xi^{\nu}_{n+r+1,s+1}(z)\\
+\big(B^{\lambda_j}_{n+r+1,s+r+1} + (c_{n+1,s+1} - c_{n+r+1,s+1})A_j + l_{n+r+1}(\lambda_j)\big)\xi^{\nu}_{n+r+1,0}(z).&
\end{aligned}$$

For this to be the desired property (5.45), we must have

$$B^{\lambda_j}_{n+r+1,s+r+1} + (c_{n+1,s+1} - c_{n+r+1,s+1})A_j + l_{n+r+1}(\lambda_j) = 0, \quad \forall j = 0,\dots,d.$$

For fixed n, this is a system of linear equations for the difference of the constants $c_{n+1,s+1} - c_{n+r+1,s+1}$ and the coefficients of the linear form l. Recall that l can be chosen arbitrarily such that $l(V) = 0$; that is, if $\lambda_0 \neq 0$, then the coefficients of l span the d-dimensional space, in which by assumption $\vec{A}$ does not lie. Thus the rank of the matrix of coefficients is $d+1$, and this system of $d+1$ linear equations has a unique solution. If $\lambda_0 = 0$, then the dimension of the space of linear forms l is d, but the periodicity condition for λ_0 is trivially satisfied. The inductive assumption is thus proved; note that as a result we are able to fix the differences $c_{n+1,s+1} - c_{n+r+1,s+1}$, and thus the constants only depend on the remainder of n modulo r. $\square$

From local to global considerations

Up until this point we have only been working on $\mathcal{C}$, under the assumption that for all n the functions $\tau_n^\nu(z)$ do not vanish identically. For τ_n^ν given by (5.9), this is equivalent to the assumption that $Un \notin \Sigma$ for all n. We now observe that if a vector $Z \in \mathbb{C}^g$ is such that $Z + Un \notin \Sigma$ for all n, then by the same arguments we can construct wave solutions along the shifted affine subspaces $Z + \mathcal{C} \subset \mathbb{C}^d$. Since all the constructions are explicitly analytic, if we perturb Z (while still staying away from $\Sigma - Un$), the solutions constructed along $Z + \mathcal{C}$ will change holomorphically with Z. Of course, such solutions can only be constructed locally, while globally there may be a choice involved, and we may thus have a monodromy for this choice as we go around $\Sigma - Un$. Thus, we cannot a priori expect $\xi_{n,s}(Z+z)$ (for $z \in \mathcal{C}$, $Z \in \mathbb{C}^g$) to be a global holomorphic function of Z.

Note that for fixed n the functions $\xi^\nu_{n+1,s+1}(Z+z)$ exist if $Z + nU \notin \Sigma$, and $\xi^\nu_{n-i,s-i}(Z+z)$ exist for $0 \le i \le s$. We pass now from a local to a global setting. In this setting, the recursive equation (5.23) takes the form

$$C\,\xi^\nu_{s+1}(Z+U+V) - u^\nu(Z)\,(\xi^{\nu+1}_{s+1}(Z+U) - C\,\xi^{\nu+1}_s(Z+V)) + \xi^\nu_s(Z) = 0 \quad (5.47)$$

with

$$u^\nu(Z) = C\frac{\tau^{\nu+1}(Z+U)\,\tau^{\nu+1}(Z+V)}{\tau^\nu(Z+U+V)\,\tau^\nu(Z)}, \quad (5.48)$$

where

$$\tau^\nu(Z) = \theta\big(Z + (1-\nu)W\big)\left(c_1^{(l_1,\,Z)} c_2^{(l_2,Z)}\right)^{\nu-(1/2)} \quad (5.49)$$

and l_1, l_2 are vectors such that $l_1(V) = l_2(U) = 1,\ l_1(U) = l_2(V) = 0$. In these notation the arguments in the proof of Proposition 5.3 yield the following.

PROPOSITION 5.8

If equations (2.6) are satisfied, then

(i) *for $Z \notin \cup_{i=0}^N(\Sigma - iU)$, there exist functions $\tau_s^\nu(Z+z)$, $0 \le s \le N$, which are local holomorphic functions of Z and global holomorphic functions of $z \in \mathcal{C}$, such that equations (5.47) hold for $\xi_s^\nu(Z) = \tau_s^{\nu+1}(Z)/\tau^\nu(Z)$, with $\tau_0^\nu(Z) = \tau^\nu(Z-U)$ (this is (5.24);*

(ii) *the functions ξ_s satisfy the monodromy relations*

$$b_\lambda^\nu \xi_s^\nu(Z+z+\lambda) - \xi_s^\nu(Z+z) = \sum_{i=1}^{s} B_i^\lambda(Z)\, \xi_{s-i}^\nu(Z+z), \qquad \lambda \in \Lambda_0; \tag{5.50}$$

(iii) *if ξ_{s-1} is fixed, then ξ_s is unique up to the transformation*

$$\xi_s(z+Z) \longmapsto \xi_s(Z+z) + \big(c_s(Z) + l_s(Z,z)\big)\xi_0, \tag{5.51}$$

where $l_s(Z,z)$ is a linear form in z such that $l_s(Z,V)=0$.

LEMMA 5.9
Let r be the minimal integer such that $\xi_1^\nu, \dots, \xi_{r-1}^\nu$ are periodic functions of z with respect to Λ_0, and such that there is no periodic solution ξ_r^ν of (5.47). Then the inductive assumptions of Lemma (5.7) are satisfied; that is, the quasi-periodicity coefficients $B_i^\lambda(Z)$ in (5.50) do not depend on Z or $i-s$.

Proof
By assumption, $\xi_{r-1}^\nu(z)$ is periodic; that is, we have

$$0 = b_\lambda^\nu \xi_{r-1}^\nu(Z+z+\lambda) - \xi_{r-1}^\nu(Z+z) = \sum_{i=1}^{r-1} B_i^\lambda(Z)\, \xi_{r-1-i}^\nu(Z+z).$$

However, as noted above, the functions τ_s^ν, for $0 \le s \le r-1$, are linearly independent (recall that if not, by applying (5.47) we could produce a linear dependence having only one term, which is impossible), which means that all coefficients B_i^λ are zero for all $i \le r-1$. Thus the monodromy of the next function is given by

$$b_\lambda^\nu \xi_r^\nu(Z+z+\lambda) - \xi_r^\nu(Z+z) = B_r^\lambda(Z)\xi_0^\nu(Z+z),$$

where we of course know ξ_0^ν explicitly, and B_r^λ is a local function of Z defined locally for

$$Z \in X \setminus \bigcup_{i=0}^{r-1}(\Sigma - iV).$$

From Lemma 5.8, we know that the only ambiguity in the choice of the solutions $\xi_{n,r}^\nu(z)$ is given by (5.51). Recall that adding a linear function multiple changes the equation to be satisfied, while adding a constant multiple does not change the quasi-periodicity properties, so that finally $B_r^\lambda(Z)$ is independent of the ambiguity, and is well defined as a holomorphic function of $Z \in X'$. Since the locus $\Sigma \subset X$ is of codimension at least 2, by Hartogs's theorem the function $B_r^\lambda(Z)$ can be extended holomorphically to all of X. Since X is compact, this means that $B_r^\lambda(Z)$ is a constant, which we can denote A_λ for the inductive assumption of Lemma 5.7. If we had $\vec{A} \cdot V = 0$, then by a transformation (5.43) with a suitable linear term we could get a new solution with $A_{\lambda_i} = 0$ for $i = 0, \ldots, d$; that is, the function $\xi_{n,r}^\nu(z)$ could be made periodic, contradicting the way we chose r. □

LEMMA 5.10
In the setup of our construction, the "bad locus" Σ is actually empty; that is, if equation (2.6) (part (C), the weakest assumption of our main theorem) is satisfied, then $\Sigma = \emptyset$.

Proof
The proof of this lemma is analogous to the proof of the similar statement for the fully discrete trisecant characterization of Jacobians treated in [19], once we first prove that $\Sigma^0 = \Sigma^1$.

The only ambiguity in the definition of $\tau_1^\nu(Z)$ is in the choice of the coefficient c_1 in (5.51). Suppose there exists a point $A \in \Sigma^0 \setminus \Sigma^1$, such that $\theta(A + NV) \neq 0 = \theta(A + NV + W)$ for some N. Then locally near the point $A + NV$, choose some holomorphic branch of the function $\xi_1^1(Z) = \tau_1^0(Z)/\tau^1(Z)$. Doing this fixes the value of $c_1(Z)$ for all Z near $A + NV$. However, since the ambiguity in the choice of $\xi_1^0(Z) = \tau_1^1(Z)/\tau^0(Z)$ is given by the same function $c_1(Z)$ (which did not depend on ν!), it means that for Z in a neighborhood of $A + NV$, but away from Σ^0, we also have a fixed choice of $\xi_1^0(Z)$, and thus also of the holomorphic function $\tau_1^1(Z)$. Since Σ^0 has codimension at least 2 in X, the function $\tau_1^1(Z)$ can thus be extended to all points in a neighborhood of A, which is a contradiction. Thus we must have $\Sigma^0 \subset \Sigma^1$, and of course, by symmetry, they are in fact equal.

We now prove in the same manner that $\Sigma = \Sigma + rU$; above we used the fact that in (5.51) c_1 is independent of ν, and now we use $c_1(Z) = c_1(Z + rU)$. Indeed, suppose we have $A \in (\Sigma - rU) \setminus \Sigma$. This means that in a neighborhood of A, we can choose a locally holomorphic function $\tau_1^\nu(Z)$; that is, choose a local holomorphic branch of $c_1(Z)$. However, since $c_1(Z) = c_1(Z + rU)$, this also fixes the choice of c_1 in a neighborhood of the point $A + rU$, and thus in a neighborhood of A, outside of $\Sigma - rU$, we have a holomorphic function $\tau_1^\nu(Z + rU)$, which now can be extended across $\Sigma - rU$, and which we know to be of codimension at least two. This constructs a solution $\tau_1^\nu(A + rU)$, which contradicts the assumption $A \in (\Sigma - rU)$.

Since by definition Σ has no subset invariant under a shift by U, this implies that either Σ is empty, or $r > 1$. Suppose now that Σ is nonempty, so $r > 1$. Recall that we have $\tau_0^\nu(Z) = \tau^\nu(Z - U) = c\theta(Z + (1 - \nu)W - U)$ for some constant c, and thus since $\Sigma^0 = \Sigma^1$, we have $\tau_0^\nu|_{\Sigma+U} = 0$. Thus for any $Z \in \Sigma + U$ and for $s = 1$, the last two terms in (5.47) vanish, yielding

$$C\xi_1^\nu(Z+U+V)-u^\nu(Z)\xi_1^{\nu+1}(Z+U)=0.$$

However, this is exactly equation (5.47) for $s=0$, which is solved by ξ_0^ν, and thus all *periodic with respect to* Λ_0 solutions are constant multiples of ξ_0^ν. By using (5.51), we can subtract this constant and get a solution such that $\xi_1^\nu(Z+U)=0$ for any $Z\in\Sigma+U$; that is, we have $\xi_1^\nu|_{\Sigma+2U}=0$.

Now we can repeat this process. Indeed, for $s=2$ and $Z\in\Sigma+2U$, the last two terms in (5.47) include $\xi_1^\nu(Z+V)$ and $\xi_1^\nu(Z)$, and thus vanish, so that as a result we see that ξ_2^ν on $\Sigma+3U$ is a constant multiple of ξ_0^ν. By using (5.51) again, we can make this multiple to be zero again. Repeating this a number of times, we eventually get $\xi_{r-1}^\nu|_{\Sigma+rU}=0$.

Since $\Sigma=\Sigma+rU$, we also have $\tau^\nu|_{\Sigma+rU}=0$, and thus for $Z\in\Sigma+rU$ and $s=r-1$, the last two terms in (5.47) vanish to the second order (both factors of each summand vanish). Thus ξ_r^ν can be defined in a neighborhood of $\Sigma+rU=\Sigma$ as a holomorphic function vanishing on $\Sigma+rU$. However, this implies in particular that $b_\lambda^\nu\xi_r^\nu(Z+\lambda_j)-\xi_r^\nu(Z)=0$ for $Z\in\Sigma$ and any $\lambda_j\in\Lambda_0$, which contradicts the assumption that ξ_r^ν could not be periodic. The lemma is thus proved. □

As shown above, if Σ is empty, then the functions τ_s^ν can be defined as global holomorphic functions of $Z\in\mathbb{C}^g$. Then, as a corollary of the previous lemmas, we get the following statement.

LEMMA 5.11
Suppose that (2.6) for $\theta(Z)$ holds. Then there exists a pair of formal solutions

$$\phi^\nu=\sum_{s=0}^{\infty}\xi_s^\nu(Z)\,k^{-s} \tag{5.52}$$

of the equation

$$kC\phi^\nu(Z+U+V,k)-u^\nu(Z)\big(k\phi^{\nu+1}(Z+U,k)-C\phi^{\nu+1}(Z+V,k)\big)-\phi^\nu(Z,k)=0, \tag{5.53}$$

with $C=c_3$ and

$$u^\nu(Z)=\frac{\tau^{\nu+1}(Z+U)\,\tau^{\nu+1}(Z+V)}{\tau^\nu(Z+U+V)\,\tau^\nu(Z)}, \tag{5.54}$$

where τ^ν is given by (5.49), such that

(i) the coefficients ξ_s^ν of the formal series ϕ^ν are of the form $\xi_s^\nu(Z)=\tau_s^{\nu+1}(Z)/\tau^\nu(Z)$, where $\tau_s^\nu(Z)$ are holomorphic functions;

(ii) $\phi^\nu(Z,k)$ *is quasi-periodic with respect to the lattice* Λ, *and for the basis vectors* λ_j *in* $\mathcal{C}$ *its monodromy relations have the form*

$$\phi^\nu(Z+\lambda_j) = (1 + A_{\lambda_j} k^{-1})\,\phi^\nu(Z,k), \quad j = 1, \ldots, g, \tag{5.55}$$

where A_{λ_j} *are constants such that there is no linear form on* $\mathcal{C}$ *vanishing at* V *(i.e.,* $l(V) = 0$*), and such that* $l(\lambda_j) = A_{\lambda_j}$*;*

(iii) ϕ^ν *is unique up to multiplication by a constant in* Z *factor.*

6. The spectral curve

In this section, we finish a proof of the fact that condition (C) of the main theorem characterizes Prym varieties. Indeed, in the previous section we showed that if (C) holds, some quasi-periodic wave solutions can be constructed. In this section, we also show that these wave solutions are eigenfunctions of commuting difference operators and identify X with the Prym variety of the spectral curve of these operators. Much of the argument is analogous to that in [17] and [19].

The formal series $\phi^\nu(Z,k)$ constructed in the previous section defines a wave function

$$\psi = \psi_{nm}(k) := k^n \phi^{\nu_{nm}}(nU + mV + Z, k).$$

This wave function determines a unique pseudodifference operator $\mathcal{L}$ such that $\mathcal{L}\psi = k\psi$ (the coefficients of this $\mathcal{L}$ can be computed inductively term by term); we note that the ambiguity in the definition of $\phi^\nu(Z)$ (it is only defined up to a factor that is T_1-invariant) does not affect the coefficients of the wave operator L. Therefore, its coefficients are of the form

$$\mathcal{L} = \sum_{s=-1}^{\infty} w_s^{\nu_{nm}}(Z + nU + mV)\, T_1^{-s}, \tag{6.1}$$

where $w_s^\nu(Z)$ are well-defined meromorphic sections of line bundles on X with automorphy properties given by (5.19).

As before, we define functions $\widetilde{F}_j$ by formula (4.63); that is, we set

$$\widetilde{F}_j := \operatorname{res}_T \left((\mathcal{L}^j T_1^{-1} - T_1 \mathcal{L}^j)(T_1 - T_1^{-1})^{-1} \right).$$

The definition of ψ implies that these functions are of the form

$$\widetilde{F}_j = \widetilde{F}_j^{\nu_{nm}}(Un + Vm + Z), \tag{6.2}$$

where $\widetilde{F}_j^{\nu}(Z)$ are meromorphic functions on X.

LEMMA 6.1
There exist vectors $V_m = \{V_{m,k}\} \in \mathbb{C}^g$ and constants $v_m \in \mathbb{C}$ such that

$$\widetilde{F}_j^{\nu}(Z) = v_j + \frac{\partial}{\partial V_j}\left(\ln \tau^{\nu}(Z) - \ln \tau^{\nu+1}(Z+U)\right). \tag{6.3}$$

Proof
Consider the formal series ψ^{σ} given by (4.50). It has the form

$$\psi^{\sigma}_{n,m} = k^{-n}\phi^{\sigma,\, \nu_{n,m}}(Un + Vm + Z, k), \tag{6.4}$$

where the coefficients of the formal series

$$\phi^{\sigma,\,\nu}(Z,k) = \sum_{s=0}^{\infty} \xi_s^{\sigma,\,\nu}(Z)\, k^{-s} \tag{6.5}$$

are difference polynomials in the coefficients of ϕ^{ν} and $\phi^{\nu+1}$. Therefore, we know a priori that $\xi^{\sigma,\nu(Z)}$ are meromorphic functions, which may have poles for $Z \in \mathcal{T}^{\nu+nU}$ or $Z \in \mathcal{T}^{\nu+1} + nU$. We claim that in fact these coefficients are of the form

$$\xi_s^{\sigma,\,\nu} = \frac{\tau_s^{\sigma,\,\nu}(Z)}{\tau^{\nu}(Z)}, \tag{6.6}$$

where $\tau_s^{\sigma,\,\nu}(Z)$ are some holomorphic functions; that is, that they have only simple poles at $\mathcal{T}^{\nu}$.

Indeed, we showed in Section 3 that ψ^{σ} solves the equation $H\psi^{\sigma} = 0$. In Section 4, we deduced from statement (C) of the main theorem the fact that ψ^{ν} can have only a simple pole at $\mathcal{T}^{\nu}$. By replacing ψ^{ν} by $\psi^{\sigma,\,\nu}$ and replacing U by $-U$, we get from statement (C) functional equations for $\tau_s^{\sigma,\,\nu}$, and in the same way deduce also that $\psi^{\sigma,\nu}$ can have only a simple pole at $\mathcal{T}^{\nu}$.

Equation (4.61) then implies that $\widetilde{\mathcal{F}}_j^{\nu}$ are the coefficients of the formal series

$$\begin{aligned} -k + (k^2 - 1)\sum_{j=1}^{\infty} \widetilde{F}_j^{\nu}(Z) \\ = k^{-1}\phi^{\sigma,\,\nu+1}(Z+U,k)\,\phi^{\nu}(Z,k) - k\phi^{\sigma,\,\nu}(Z,k)\,\phi^{\nu+1}(Z+U,k). \end{aligned} \tag{6.7}$$

It thus follows that $\widetilde{F}_j^{\nu}(Z)$ have simple poles only at the divisors $\mathcal{T}^{\nu}$ and $\mathcal{T}^{\nu+1} - U$; these are the only possible poles of the right-hand side. Moreover, equation (4.69) says (recall that $\mathbf{t_1}$ is shifting the variable n, i.e., adding U) that there exist meromorphic functions Q_j^{ν} such that

$$\widetilde{F}_j^{\nu}(Z) = Q_j^{\nu}(Z) - Q_j^{\nu+1}(Z+U). \tag{6.8}$$

We know a priori that Q_j^ν may have poles only at $\mathcal{T}^\nu$ and $\mathcal{T}^{\nu+1} - U$. However, if there were a pole at $\mathcal{T}^{\nu+1} - U$, it would then mean that $Q_j^{\nu+1}(Z+U)$ would have a pole at $\mathcal{T}^\nu - 2U$, and since by our initial assumptions U was not a point of order 2, this is impossible. Thus Q_j^ν has simple poles only on $\mathcal{T}^\nu$, as desired for expression (6.3) for $\widetilde{F}_j^\nu$ to be valid. The functions $\widetilde{F}_j^\nu$ are abelian functions. Therefore, the residue of Q_j^ν is a well-defined section of the theta bundle restricted on $\mathcal{T}^\nu$, that is,

$$(Q_j^\nu \tau^\nu)|_{\mathcal{T}^\nu} \in H^0(\tau^\nu|_{\mathcal{T}^\nu}).$$

It is known that the later space is spanned by the directional derivatives of the theta function. Thus we see that there must exist some vector $V_j^\nu \in \mathbb{C}^g$ such that $Q_j^\nu - (\partial \ln \tau^\nu(Z)/\partial V_j^\nu)$ is a holomorphic function. The periodicity of $\widetilde{F}_j^\nu$ with respect to the lattice implies that $V_j^\nu = V_j^{\nu+1}$, and thus (6.3) holds. □

Consider now the linear space spanned by the functions $\{\widetilde{F}_j^\nu(Z),\ j = 1, \ldots\}$. From (6.3), we see that there are only $(g+1)$ parameters involved in determining $\widetilde{F}_j^\nu$, and thus this space is at most $(g+1)$-dimensional. Therefore, for all but $\widetilde{g} := \dim\{\widetilde{F}_j^\nu(Z)\} - 1 \leq g$ positive integers j, there exist constants $c_{i,j}$ such that

$$\widetilde{F}_j^\nu(Z) = c_{0,j} + \sum_{i=1}^{j-1} c_{i,j} \widetilde{F}_i^\nu(Z). \tag{6.9}$$

Let I denote the subset of integers j for which there are no such constants. We call this subset the *gap sequence*; the set of $\{\widetilde{F}_j^\nu\}$ with j in this subset forms a basis of the space spanned by all $\widetilde{F}_j^\nu$.

LEMMA 6.2
Let $\mathcal{L}$ be the pseudodifference operator corresponding to the quasi-periodic (Bloch) wave function ψ constructed above. Then, for the difference operators

$$\widehat{L}_j := L_j + \sum_{i=1}^{j-1} c_{i,j} L_{n-i} = 0, \quad \forall j \notin I, \tag{6.10}$$

the following equations are satisfied with some constants $a_{s,j}$:

$$\widehat{L}_j\, \psi = a_j(k)\, \psi, \qquad a_j(k) = k^j + \sum_{s=1}^{\infty} a_{s,j} k^{j-s}. \tag{6.11}$$

Proof
From the proof of Theorem 3.5, we get

$$[L_j, H] \equiv (\mathbf{t}_2 \widetilde{F}_j - \widetilde{F}_j)(T_1 - T_2) \bmod \mathcal{O}_H.$$

Therefore, operators $\widehat{L}_j$ and H commute in $\mathcal{O}/\mathcal{O}_H$. Hence, if ψ is a Bloch wave solution of (5.1) (i.e., $H\psi = 0$), then $\widehat{L}_j\psi$ is also a Bloch solution of the same equation. Since (5.1) has a unique solution up to multiplication by a constant (i.e., the kernel of H is 1-dimensional), we must have $\widehat{L}_j\psi = a_j(Z,k)\psi$, where a_j is T_1-invariant; that is, $a_j(Z,k) = a_j(Z+U,k)$.

Note that the constant factor ambiguity in the definition of ψ does not affect a_j, and thus a_j are well-defined *global* meromorphic functions on $\mathbb{C}^g \setminus \Sigma$. Since the closure of $\mathbb{Z}U$ is dense in X, the T_1 invariance of a_j implies that a_j is a holomorphic function of $Z \in X$, and thus it is constant in Z (note that we in fact have $a_{s,n} = -c_{s,n}$ for $s \leq n$). $\square$

If we now set $m = 0$, the operator $\widehat{L}_j$ can be regarded as a Z-parametric family of ordinary difference operators $\widehat{L}_j^Z$.

COROLLARY 6.3
The operators $\widehat{L}_j^Z$ commute with each other:

$$[\widehat{L}_i^Z, \widehat{L}_j^Z] = 0. \tag{6.12}$$

A theory of commuting difference operators containing a pair of operators of coprime orders was developed in [25], [15]. It is analogous to the theory of rank 1 commuting differential operators [4], [5], [13], [14], [25] (this theory was recently generalized to the case of commuting difference operators of arbitrary rank in [20]).

LEMMA 6.4
Let $\mathcal{A}^Z$ be the commutative ring of ordinary difference operators spanned by the operators $\widehat{L}_j^Z$. Then there exists an irreducible algebraic curve Γ of arithmetic genus $\hat{g}$ with involution $\sigma : \Gamma \longrightarrow \Gamma$ such that for a generic Z, the ring $\mathcal{A}^Z$ is isomorphic to the ring of meromorphic functions on Γ with the only poles at two smooth points $P_1^{\pm}$, which are odd with respect to the involution σ. The correspondence $Z \to \mathcal{A}^Z$ defines a holomorphic map of X to the space of odd torsion-free rank 1 sheaves $\mathcal{F}$ on Γ

$$j : X \longrightarrow \overline{\mathrm{Prym}}(\Gamma) = \mathrm{Ker}(1+\sigma) \subset \overline{\mathrm{Pic}}(\Gamma). \tag{6.13}$$

Proof
As shown in [25], [15], there is a natural correspondence

$$\mathcal{A} \longleftrightarrow \{\Gamma, P_{\pm}, \mathcal{F}\} \tag{6.14}$$

between commutative rings $\mathcal{A}$ of ordinary linear difference operators containing a pair of monic operators of coprime orders, and sets of algebro-geometric data

$\{\Gamma, P_\pm, [k^{-1}]_\pm, \mathcal{F}\}$, where Γ is an algebraic curve with fixed first jets $[k^{-1}]_\pm$ of local coordinates $k_\pm^{-1}$ in the neighborhoods of smooth points $P_1^\pm \in \Gamma$, and $\mathcal{F}$ is a torsion-free rank 1 sheaf on Γ such that

$$h^0(\Gamma, \mathcal{F}) = h^1\big(\Gamma, \mathcal{F}(nP_+ - nP_-)\big) = 0. \tag{6.15}$$

The correspondence becomes one-to-one if the rings $\mathcal{A}$ are considered modulo the conjugation $\mathcal{A}' = g\mathcal{A}g^{-1}$.

The construction of the correspondence (6.14) depends on a choice of initial point $n_0 = 0$. The spectral curve and the sheaf $\mathcal{F}$ are defined by the evaluations of the coefficients of generators of $\mathcal{A}$ at a finite number of points of the form $n_0 + n$. In fact, the spectral curve is independent on the choice of x_0, but the sheaf does depend on it; that is, $\mathcal{F}$ depends on the choice of n_0.

Using the shift of the initial point, it is easy to show that the correspondence (6.14) extends to the commutative rings of operators whose coefficients are *meromorphic* functions of x. The rings of operators having poles at $n = 0$ correspond to sheaves for which the condition (6.15) for $n = 0$ is violated.

A commutative ring $\mathcal{A}$ of linear ordinary difference operators is called maximal if it is not contained in any larger commutative ring. The algebraic curve Γ corresponding to a maximal ring is called the spectral curve of $\mathcal{A}$. The ring $\mathcal{A}$ is isomorphic to the ring $A(\Gamma, P_1^\pm)$ of meromorphic functions on Γ with the only pole at P_1^+, and vanishing at P_1^-. The isomorphism is given by the equation

$$L_a\psi = a\psi, \quad L_a \in \mathcal{A}, \ a \in A(\Gamma, P_1^\pm), \tag{6.16}$$

where ψ is a common eigenfunction of the commuting operators.

Let Γ^Z be the spectral curve corresponding to the maximal ring $\hat{\mathcal{A}}^Z$ containing $\mathcal{A}^Z$. The eigenvalues $a_j(k)$ of the operators $\hat{L}_j^Z$ defined in (6.11) coincide with the Laurent expansions at P_1^+ of the meromorphic functions $a_j \in A(\Gamma^Z, P_\pm)$, and thus are Z-independent. Hence, the spectral curve Γ^Z in fact does not depend on Z.

The functions ψ^σ are eigenfunctions of $\widehat{L}_j$:

$$\widehat{L}_j\psi^\sigma = -a_j(k)\psi^\sigma. \tag{6.17}$$

Hence, the correspondence $\psi \leftrightarrow \psi^\sigma$ gives rise to an involution σ of the spectral curve. The eigenvalues a_j are odd with respect to the involution, and thus the lemma is proved. □

The next step is to consider deformations of $\mathcal{A}^Z$ defined by the discrete Novikov-Veselov hierarchy introduced in Section 3. Through this hierarchy, we identified the space spanned by functions $\widetilde{F}_j$ with the tangent space to the orbit of the hierarchy. Lemma 5.1 identifies the orbit of the hierarchy with $Z + Y$, where Y is the closure of the group spanned by vectors V_j. The orbit of the NV hierarchy is the odd part of the orbit of two Kadomtsev-Petviashvili flows corresponding to points $P_1^{\pm}$. It follows from [28] that the orbit of the discrete NV hierarchy is isomorphic to $\mathcal{P}(\Gamma)$. For a generic Z, the ring A^Z is a maximal odd ring. Therefore, we get the following.

LEMMA 6.5
For $Z \in \mathbb{C}^g$ generic, the orbit of $\mathcal{A}^Z$ under the NV flows defines an isomorphism:

$$i_Z : \mathcal{P}(\Gamma) \longrightarrow Z + Y \subset X. \tag{6.18}$$

COROLLARY 6.6
The Prym variety $\mathcal{P}(\Gamma)$ of the spectral curve Γ is compact.

The compactness of the Prym variety is not as restrictive as the compactness of the Jacobian (see [7]). Nevertheless, it implies an explicit description of the singular points of the spectral curve. The following result is due to Robert Friedman (see [17, Appendix]).

COROLLARY 6.7
The spectral curve Γ is smooth outside of fixed points q_k of the involution σ. The branches of Γ at q_k are linear and are not permuted by σ.

The arguments identical to that used at the end of [17] prove that in fact the singular points q_k are at most double points. For a curve with at most double singular points, all rank 1 torsion-free sheaves $\mathcal{F}$ are line bundles. Therefore, the map j in (6.13) is inverse to i_Z in (6.18), and the main theorem is thus proved.

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