# Integrable linear equations and the Riemann–Schottky problem

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**Summary.** We prove that an indecomposable principally polarized abelian variety $X$ is the Jacobain of a curve if and only if there exist vectors $U \neq 0$, $V$ such that the roots $x_i(y)$ of the theta-functional equation $\theta(Ux + Vy + Z) = 0$ satisfy the equations of motion of the *formal infinite-dimensional Calogero–Moser system*.

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## 1 Introduction

The Riemann–Schottky problem on the characterization of the Jacobians of curves among abelian varieties is more than 120 years old. Quite a few geometrical characterizations of Jacobians have been found. None of them provides an explicit system of equations for the image of the Jacobian locus in the projective space under the level-two theta imbedding.

The first effective solution of the Riemann–Schottky problem was obtained by T. Shiota [1], who proved the famous Novikov conjecture:

*An indecomposable principally polarized abelian variety* $(X, \theta)$ *is the Jacobian of a curve of a genus* $g$ *if and only if there exist* $g$*-dimensional vectors* $U \neq 0$, $V$, $W$ *such that the function*

$$u(x, y, t) = -2\partial_x^2 \ln \theta(Ux + Vy + Wt + Z) \tag{1.1}$$

*is a solution of the Kadomtsev–Petviashvili* (*KP*) *equation*

$$3u_{yy} = (4u_t + 6uu_x - u_{xxx})_x. \tag{1.2}$$

Here $\theta(Z) = \theta(Z|B)$ is the Riemann theta-function,

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i(z,m)+\pi i(Bm,m)}, \quad (z, m) = m_1 z_1 + \cdots + m_g z_g, \tag{1.3}$$

where $B$ is the corresponding symmetric matrix with positive definite imaginary part.

It is easy to show [2] that the KP equation with $u$ of the form (1.1) is in fact equivalent to the following system of algebraic equations for the fourth-order derivatives of the level-two theta constants:

$$\partial_U^4 \Theta[\varepsilon, 0] - \partial_U \partial_W \Theta[\varepsilon, 0] + \partial_V^2 \Theta[\varepsilon, 0] + c\Theta[\varepsilon, 0] = 0, \quad c = \text{const.} \tag{1.4}$$

Here $\Theta[\varepsilon, 0] = \Theta[\varepsilon, 0](0)$, where $\Theta[\varepsilon, 0](z) = \theta[\varepsilon, 0](2z|2B)$ are level-two theta-functions with half-integer characteristics $\varepsilon \in \frac{1}{2}\mathbb{Z}_2^g$.

The KP equation admits the so-called zero-curvature representation [3, 4], which is the compatibility condition for the following over-determined system of linear equations:

$$(\partial_y - \partial_x^2 + u)\psi = 0, \tag{1.5}$$

$$\left(\partial_t - \partial_x^3 + \frac{3}{2}\partial_x + w\right)\psi = 0. \tag{1.6}$$

The main goal of the present paper is to show that the KP equation contains excessive information and that the Jacobians can be characterized in terms of *only the first* of its auxiliary linear equations.

**Theorem 1.1.** *An indecomposable principally polarized abelian variety* $(X, \theta)$ *is the Jacobian of a curve of genus* $g$ *if and only if there exist* $g$*-dimensional vectors* $U \neq 0$, $V$, $A$ *such that equation* (1.5) *is satisfied for*

$$u = -2\partial_x^2 \ln \theta(Ux + Vy + Z) \tag{1.7}$$

*and*

$$\psi = \frac{\theta(A + Ux + Vy + Z)}{\theta(Ux + Vy + Z)} e^{px+Ey}, \tag{1.8}$$

*where* $p$, $E$ *are constants.*

The "if" part of this statement follows from the exact theta-functional expression for the Baker–Akhiezer function [5, 6].

The addition formula for the Riemann theta-function directly implies that equation (1.5) with $u$ and $\psi$ of the form (1.7) and (1.8) is equivalent to the system of equations

$$(\partial_V - \partial_U^2 - 2p\partial_U + (E - p^2))\Theta[\varepsilon, 0](A/2) = 0, \quad \varepsilon \in \frac{1}{2}\mathbb{Z}_2^g. \tag{1.9}$$

Recently Theorem 1.1 was proved by E. Arbarello and G. Marini and the author [7] under the additional assumption that the closure $\langle A \rangle$ of the subgroup of $X$ generated by $A$ is irreducible. The geometric interpretation of Theorem 1.1 is equivalent to the characterization of Jacobians via flexes of Kummer varieties (see details in [7]), which is a particular case of the so-called *trisecant conjecture*, first formulated in [8].

Theorem 1.1 is not the strongest form of our main result. What we really prove is that the Jacobian locus in the space of principally polarized abelian varieties is characterized by a system of equations which formally can be seen as the equations of motion of the *infinite-dimensional* Calogero–Moser system.

Let $\tau(x, y)$ be an entire function of the complex variable $x$ smoothly depending on a parameter $y$. Consider the equation

$$\mathrm{res}_x(\partial_y^2 \ln \tau + 2(\partial_x^2 \ln \tau)^2) = 0, \tag{1.10}$$

which means that the meromorphic function given by the left-hand side of (1.10) has no *residues* in the $x$ variable. If $x_i(y)$ is a simple zero of $\tau$, i.e., $\tau(x_i(y), y) = 0$, $\partial_x \tau(x_i(y), y) \neq 0$, then (1.10) implies

$$\ddot{x}_i = 2w_i, \tag{1.11}$$

where "dots" stands for the $y$-derivatives and $w_i$ is the third coefficient of the Laurent expansion of $u(x, y) = -2\partial_x^2 \tau(x, y)$ at $x_i$, i.e.,

$$u(x, y) = \frac{2}{(x - x_i(y))^2} + v_i(y) + w_i(y)(x - x_i(y)) + \cdots . \tag{1.12}$$

Formally, if we represent $\tau$ as an infinite product,

$$\tau(x, y) = c(y) \prod_i (x - x_i(y)), \tag{1.13}$$

then equation (1.10) can be written as the infinite system of equations

$$\ddot{x}_i = -4 \sum_{j \neq i} \frac{1}{(x_i - x_j)^3}. \tag{1.14}$$

Equations (1.14) are purely formal because, even if $\tau$ has simple zeros at $y = 0$, in the general case there is no nontrivial interval in $y$ where the zeros stay simple. For the moment, the only reason for representing (1.11) in the form (1.14) is to show that in the case when $\tau$ is a rational, trigonometric or elliptic polynomial the system (1.11) coincides with the equations of motion for the rational, trigonometrical or elliptic Calogero–Moser systems, respectively.

Equations (1.11) for the zeros of the function $\tau = \theta(Ux + Vy + Z)$ were derived in [7] as a direct corollary of the assumptions of Theorem 1.1. Simple expansion of $\theta$ at the points of its divisor $z \in \Theta : \theta(z) = 0$ gives the equation

$$[(\partial_2\theta)^2 - (\partial_1^2\theta)^2]\partial_1^2\theta + 2[\partial_1^2\theta\partial_1^3\theta - \partial_2\theta\partial_1\partial_2\theta]\partial_1\theta + [\partial_2^2\theta - \partial_1^4\theta](\partial_1\theta)^2 \\ = 0 \pmod{\theta} \tag{1.15}$$

*which is valid on* $\Theta$. Here and below $\Theta$ is the divisor on $X$ defined by the equation $\theta(Z) = 0$ and $\partial_1$ and $\partial_2$ are constant vector fields on $\mathbb{C}^g$ corresponding to the vectors $U$ and $V$.

It would be very interesting to understand if any reasonable general theory of equation (1.10) exists. The following form of our main result shows that in any case such a theory has to be interesting and nontrivial.

Let $\Theta_1$ be defined by the equations $\Theta_1 = \{Z : \theta(Z) = \partial_1\theta(Z) = 0\}$. The $\partial_1$-invariant subset $\Sigma$ of $\Theta_1$ will be called the *singular locus*.

**Theorem 1.2.** *An indecomposable principally polarized abelian variety* $(X, \theta)$ *is the Jacobian of a curve of genus g if and only if there exist g-dimensional vectors* $U \neq 0, V$, *such that for each* $Z \in \mathbb{C}^g \setminus \Sigma$ *equation* (1.10) *for the function* $\tau(x, y) = \theta(Ux + Vy + Z)$ *is satisfied, i.e., equation* (1.15) *is valid on* $\Theta$.

The main idea of Shiota's proof of the Novikov conjecture is to show that if $u$ is as in (1.1) and satisfies the KP equation, then it can be extended to a $\tau$-function of the KP *hierarchy*, as a *global* holomorphic function of the infinite number of variables $t = \{t_i\}$, $t_1 = x$, $t_2 = y$, $t_3 = t$. Local existence of $\tau$ directly follows from the KP equation. The global existence of the $\tau$-function is crucial. The rest is a corollary of the KP theory and the theory of commuting ordinary differential operators developed by Burchnall–Chaundy [9, 10] and the author [5, 6].

The core of the problem is that there is a homological obstruction for the global existence of $\tau$. It is controlled by the cohomology group $H^1(\mathbb{C}^g \setminus \Sigma, \mathcal{V})$, where $\mathcal{V}$ is the sheaf of $\partial_1$-*invariant* meromorphic functions on $\mathbb{C}^g \setminus \Sigma$ with poles along $\Theta$ (see details in [11]). The hardest part of Shiota's work (clarified in [11]) is the proof that the locus $\Sigma$ is empty. That ensures the vanishing of $H^1(\mathbb{C}^g, \mathcal{V})$. Analogous obstructions have occurred in all the other attempts to apply the theory of soliton equations to various characterization problems in the theory of abelian varieties. None of them has been completely successful. Only partial results were obtained. (Note that Theorem 1.1 in one of its equivalent forms was proved earlier in [12] under the additional assumption that $\Theta_1$ does not contain a $\partial_1$-invariant line.)

Strictly speaking, the KP equation and the KP hierarchy are not used in the present paper. But our main construction of *the formal wave solutions* of (1.5) is reminiscent of the construction of the $\tau$-function. All its difficulties can be traced back to those in Shiota's work. The wave solution of (1.5) is a solution of the form

$$\psi(x, y, k) = e^{kx+(k^2+b)y}\left(1 + \sum_{s=1}^{\infty} \xi_s(x, y)k^{-s}\right). \tag{1.16}$$

At the beginning of the next section, we show that the assumptions of Theorem 1.2 are necessary and sufficient conditions for the *local* existence of the wave solutions such that

$$\xi_s = \frac{\tau_s(Ux + Vy + Z, y)}{\theta(Ux + Vy + Z)}, \quad Z \notin \Sigma, \tag{1.17}$$

where $\tau_s(Z, y)$, as a function of $Z$, is holomorphic in some open domain in $\mathbb{C}^g$. The functions $\xi_s$ are defined recursively by the equation $2\partial_1\xi_{s+1} = \partial_y\xi_s - \partial_1^2\xi_s + u\xi_s$. Therefore, the global existence of $\xi_s$ is controlled by the same cohomology group $H^1(\mathbb{C} \setminus \Sigma, \mathcal{V})$ as above. At the local level the main problem is to find a translational invariant normalization of $\xi_s$ which defines wave solutions uniquely up to a $\partial_1$-invariant factor.

In the case of periodic potentials $u(x + T, y) = u(x)$ the normalization problem for the wave functions was solved by D. Phong and the author in [13]. It was shown that the condition that $\xi_s$ *is periodic* completely determines the $y$-dependence of the integration constants and the corresponding wave solutions are related by an $x$-independent factor. In general, the potential $u = -2\partial_x^2\theta(Ux+Vy+Z)$ is only quasi-periodic in $x$. In that case the solution of the normalization problem is technically more involved but mainly goes along the same lines as in the periodic case. The corresponding wave solutions are called $\lambda$-periodic.

In the last section, we showed that for each $Z \notin \Sigma$ a local $\lambda$-periodic wave solution is the common eigenfunction of a commutative ring $\mathcal{A}^Z$ of ordinary differential operators. The coefficients of these operators are independent of ambiguities in the construction of $\psi$. For generic $Z$ the ring $A^Z$ is maximal and the corresponding spectral curve $\Gamma$ is $Z$-independent. The correspondence $j : Z \longmapsto \mathcal{A}^Z$ allows us to take the next crucial step and prove the global existence of the wave function. Namely, on $X \setminus \Sigma$ the wave function can be globally defined as the preimage $j^*\psi_{BA}$ under $j$ of the Baker–Akhiezer function on $\Gamma$ and then can be extended to $X$ by the usual Hartog-type arguments. The global existence of the wave function implies that $X$ contains an orbit of the KP hierarchy, as an abelian subvariety. The orbit is isomorphic to the generalized Jacobian $J(\Gamma) = \mathrm{Pic}^0(\Gamma)$ of the spectral curve [1]. Therefore, the generalized Jacobian is compact. The compactness of $\mathrm{Pic}^0(\Gamma)$ implies that the spectral curve is smooth and the correspondence $j$ extends by linearity and defines an isomorphism $j : X \to J(\Gamma)$.

## 2 λ-periodic wave solutions

As was mentioned above, the formal Calogero–Moser equations (1.11) were derived in [7] as a necessary condition for the existence of a meromorphic solution to equation (1.5).

Let $\tau(x, y)$ be a holomorphic function of the variable $x$ in some open domain $D \in \mathbb{C}$ smoothly depending on a parameter $y$. Suppose that for each $y$ the zeros of $\tau$ are simple,

$$\tau(x_i(y), y) = 0, \qquad \tau_x(x_i(y), y) \neq 0. \tag{2.1}$$

**Lemma 2.1 ([7]).** *If equation* (1.5) *with the potential* $u = -2\partial_x^2 \ln \tau(x, y)$ *has a meromorphic in* $D$ *solution* $\psi_0(x, y)$*, then equations* (1.11) *hold.*

*Proof.* Consider the Laurent expansions of $\psi_0$ and $u$ in the neighborhood of one of the zeros $x_i$ of $\tau$:

$$u = \frac{2}{(x-x_i)^2} + v_i + w_i(x-x_i) + \cdots, \tag{2.2}$$

$$\psi_0 = \frac{\alpha_i}{x-x_i} + \beta_i + \gamma_i(x-x_i) + \delta_i(x-x_i)^2 + \cdots, \tag{2.3}$$

(All coefficients in these expansions are smooth functions of the variable $y$.) Substitution of (2.2), (2.3) in (1.5) gives a system of equations. The first three of them are

$$\alpha_i \dot{x}_i + 2\beta_i = 0, \tag{2.4}$$

$$\dot{\alpha}_i + \alpha_i v_i + 2\gamma_i = 0, \tag{2.5}$$

$$\dot{\beta}_i + v_i\beta_i - \gamma_i\dot{x}_i + \alpha_i w_i = 0. \tag{2.6}$$

Taking the $y$-derivative of the first equation and the using other two, we get (1.11).

Let us show that equations (1.11) are sufficient for the existence of meromorphic wave solutions. □

**Lemma 2.2.** *Suppose that equations* (1.11) *for the zeros of* $\tau(x, y)$ *hold. Then there exist meromorphic wave solutions of equation* (1.5) *that have simple poles at* $x_i$ *and are holomorphic everywhere else.*

*Proof.* Substitution of (1.16) into (1.5) gives a recurrent system of equations

$$2\xi'_{s+1} = \partial_y\xi_s + u\xi_s - \xi''_s. \tag{2.7}$$

We are going to prove by induction that this system has meromorphic solutions with simple poles at all the zeros $x_i$ of $\tau$.

Let us expand $\xi_s$ at $x_i$:

$$\xi_s = \frac{r_s}{x-x_i} + r_{s0} + r_{s1}(x-x_i), \tag{2.8}$$

where for brevity we omit the index $i$ in the notation for the coefficients of this expansion. Suppose that $\xi_s$ are defined and equation (2.7) has a meromorphic solution. Then the right-hand side of (2.7) has zero residue at $x = x_i$, i.e.,

$$\operatorname{res}_{x_i}(\partial_y\xi_s + u\xi_s - \xi''_s) = \dot{r}_s + v_i r_s + 2r_{s1} = 0. \tag{2.9}$$

We need to show that the residue of the next equation also vanishes. From (2.7) it follows that the coefficients of the Laurent expansion for $\xi_{s+1}$ are equal to

$$r_{s+1} = -\dot{x}_i r_s - 2r_{s0}, \tag{2.10}$$

$$2r_{s+1,1} = \dot{r}_{s0} - r_{s1} + w_i r_s + v_i r_{s0}. \tag{2.11}$$

These equations imply

$$\dot{r}_{s+1} + v_i r_{s+1} + 2r_{s+1,1} = -r_s(\ddot{x}_i - 2w_i) - \dot{x}_i(\dot{r}_s - v_i r_s s + 2r_{s1}) = 0, \tag{2.12}$$

and the lemma is proved. □

Our next goal is to fix a *translation-invariant* normalization of $\xi_s$ which defines wave functions uniquely up to an $x$-independent factor. It is instructive to consider first the case of the periodic potentials $u(x+1,y)=u(x,y)$ (see details in [13]).

Equations (2.7) are solved recursively by the formulae

$$\xi_{s+1}(x,y)=c_{s+1}(y)+\xi^0_{s+1}(x,y), \tag{2.13}$$

$$\xi^0_{s+1}(x,y)=\frac{1}{2}\int_{x_0}^{x}(\partial_y\xi_s-\xi_s''+u\xi_s)dx, \tag{2.14}$$

where $c_s(y)$ are *arbitrary* functions of the variable $y$. Let us show that the periodicity condition $\xi_s(x+1,y)=\xi_s(x,y)$ defines the functions $c_s(y)$ uniquely up to an additive constant. Assume that $\xi_{s-1}$ is known and satisfies the condition that the corresponding function $\xi^0_s$ is periodic. The choice of the function $c_s(y)$ does not affect the periodicity property of $\xi_s$, but it does affect the periodicity in $x$ of the function $\xi^0_{s+1}(x,y)$. In order to make $\xi^0_{s+1}(x,y)$ periodic, the function $c_s(y)$ should satisfy the linear differential equation

$$\partial_y c_s(y)+B(y)c_s(y)+\int_{x_0}^{x_0+1}(\partial_y\xi^0_s(x,y)+u(x,y)\xi^0_s(x,y))dx, \tag{2.15}$$

where $B(y)=\int_{x_0}^{x_0+1}udx$. This defines $c_s$ uniquely up to a constant.

In the general case, when $u$ is quasi-periodic, the normalization of the wave functions is defined along the same lines.

Let $Y_U=\langle Ux\rangle$ be the closure of the group $Ux$ in $X$. Shifting $Y_U$ if needed, we may assume, without loss of generality, that $Y_U$ is not in the singular locus, $Y_U\notin\Sigma$. Then for a sufficiently small $y$, we have $Y_U+Vy\notin\Sigma$ as well. Consider the restriction of the theta-function onto the affine subspace $\mathbb{C}^d+Vy$, where $\mathbb{C}^d=\pi^{-1}(Y_U)$, and $\pi:\mathbb{C}^g\to X=\mathbb{C}^g/\Lambda$ is the universal cover of $X$:

$$\tau(z,y)=\theta(z+Vy),\quad z\in\mathbb{C}^d. \tag{2.16}$$

The function $u(z,y)=-2\partial_1^2\ln\tau$ is periodic with respect to the lattice $\Lambda_U=\Lambda\cap\mathbb{C}^d$ and, for fixed $y$, has a double pole along the divisor $\Theta^U(y)=(\Theta-Vy)\cap\mathbb{C}^d$.

**Lemma 2.3.** *Let equation* (1.10) *for* $\tau(Ux+z,y)$ *hold and let* $\lambda$ *be a vector of the sublattice* $\Lambda_U=\Lambda\cap\mathbb{C}^d\subset\mathbb{C}^g$. *Then*

(i) *equation* (1.5) *with the potential* $u(Ux+z,y)$ *has a wave solution of the form* $\psi=e^{kx+k^2y}\phi(Ux+z,y,k)$ *such that the coefficients* $\xi_s(z,y)$ *of the formal series*

$$\phi(z,y,k)=e^{by}\left(1+\sum_{s=1}^{\infty}\xi_s(z,y)k^{-s}\right) \tag{2.17}$$

*are* $\lambda$*-periodic meromorphic functions of the variable* $z\in\mathbb{C}^d$ *with a simple pole along the divisor* $\Theta^U(y)$,

$$\xi_s(z+\lambda,y)=\xi_s(z,y)=\frac{\tau_s(z,y)}{\tau(z,y)}; \tag{2.18}$$

(ii) $\phi(z, y, k)$ *is unique up to a factor* $\rho(z, k)$ *that is* $\partial_1$*-invariant and holomorphic in* $z$*,*

$$\phi_1(z, y, k) = \phi(z, y, k)\rho(z, k), \quad \partial_1\rho = 0. \tag{2.19}$$

*Proof.* The functions $\xi_s(z)$ are defined recursively by the equations

$$2\partial_1\xi_{s+1} = \partial_y\xi_s + (u + b)\xi_s - \partial_1^2\xi_s. \tag{2.20}$$

A particular solution of the first equation $2\partial_1\xi_1 = u + b$ is given by the formula

$$2\xi_1^0 = -2\partial_1 \ln\tau + (l, z)b, \tag{2.21}$$

where $(l, z)$ is a linear form on $\mathbb{C}^d$ given by the scalar product of $z$ with a vector $l \in \mathbb{C}^d$ such that $(l, U) = 1$, and $(l, \lambda) \neq 0$. The periodicity condition for $\xi_1^0$ defines the constant $b$,

$$(l, \lambda)b = (2\partial_1 \ln\tau(z + \lambda, y) - 2\partial_1 \ln\tau(z, y)), \tag{2.22}$$

which depends only on a choice of the lattice vector $\lambda$. A change of the potential by an additive constant does not affect the results of the previous lemma. Therefore, equations (1.11) are sufficient for the local solvability of (2.20) in any domain, where $\tau(z + Ux, y)$ has simple zeros, i.e., outside of the set $\Theta_1^U(y) = (\Theta_1 - Vy) \cap \mathbb{C}^d$. Recall that $\Theta_1 = \Theta \cap \partial_1\Theta$. This set does not contain a $\partial_1$-invariant line because any such line is dense in $Y_U$. Therefore, the sheaf $\mathcal{V}_0$ of $\partial_1$-invariant meromorphic functions on $\mathbb{C}^d \setminus \Theta_1^U(y)$ with poles along the divisor $\Theta^U(y)$ coincides with the sheaf of holomorphic $\partial_1$-invariant functions. That implies the vanishing of $H^1(C^d \setminus \Theta_1^U(y), \mathcal{V}_0)$ and the existence of global meromorphic solutions $\xi_s^0$ of (2.20) which have a simple pole along the divisor $\Theta^U(y)$ (see details in [1, 11]). If $\xi_s^0$ are fixed, then the general global meromorphic solutions are given by the formula $\xi_s = \xi_s^0 + c_s$, where the constant of integration $c_s(z, y)$ is a holomorphic $\partial_1$-invariant function of the variable $z$.

Let us assume, as in the example above, that a $\lambda$-periodic solution $\xi_{s-1}$ is known and that it satisfies the condition that there exists a periodic solution $\xi_s^0$ of the next equation. Let $\xi_{s+1}^*$ be a solution of (2.20) for fixed $\xi_s^0$. Then it is easy to see that the function

$$\xi_{s+1}^0(z, y) = \xi_{s+1}^*(z, y) + c_s(z, y)\xi_1^0(z, y) + \frac{(l, z)}{2}\partial_y c_s(z, y) \tag{2.23}$$

is a solution of (2.20) for $\xi_s = \xi_s^0 + c_s$. A choice of a $\lambda$-periodic $\partial_1$-invariant function $c_s(z, y)$ does not affect the periodicity property of $\xi_s$, but it does affect the periodicity of the function $\xi_{s+1}^0$. In order to make $\xi_{s+1}^0$ periodic, the function $c_s(z, y)$ should satisfy the linear differential equation

$$(l, \lambda)\partial_y c_s(z, y) = 2\xi_{s+1}^*(z + \lambda, y) - 2\xi_{s+1}^*(z, y). \tag{2.24}$$

This equation, together with an initial condition $c_s(z) = c_s(z, 0)$ uniquely defines $c_s(x, y)$. The induction step is then completed. We have shown that the ratio of two periodic formal series $\phi_1$ and $\phi$ is $y$-independent. Therefore, equation (2.19), where $\rho(z, k)$ is defined by the evaluation of the two sides at $y = 0$, holds. The lemma is thus proved. □

**Corollary 2.1.** *Let $\lambda_1, \dots, \lambda_d$ be a set of linear independent vectors of the lattice $\Lambda_U$ and let $z_0$ be a point of $\mathbb{C}^d$. Then, under the assumptions of the previous lemma, there is a unique wave solution of equation* (1.5) *such that the corresponding formal series $\phi(z, y, k; z_0)$ is quasi-periodic with respect to $\Lambda_U$, i.e., for $\lambda \in \Lambda_U$*

$$\phi(z+\lambda, y, k; z_0) = \phi(z, y, k; z_0)\mu_\lambda(k) \tag{2.25}$$

*and satisfies the normalization conditions*

$$\mu_{\lambda_i}(k) = 1, \qquad \phi(z_0, 0, k; z_0) = 1. \tag{2.26}$$

The proof is identical to that of [1, Lemma 12, part (b)]. Let us briefly present its main steps. As shown above, there exist wave solutions corresponding to $\phi$ which are $\lambda_1$-periodic. Moreover, from statement (ii) above it follows that for any $\lambda' \in \Lambda_U$,

$$\phi(z+\lambda, y, k) = \phi(z, y, k)\rho_\lambda(z, k), \tag{2.27}$$

where the coefficients of $\rho_\lambda$ are $\partial_1$-invariant holomorphic functions. Then the same arguments as in [1] show that there exists a $\partial_1$-invariant series $f(z, k)$ with holomorphic in $z$ coefficients and formal series $\mu^0_\lambda(k)$ with constant coefficients such that the equation

$$f(z+\lambda, k)\rho_\lambda(z, k) = f(z, k)\mu_\lambda(k) \tag{2.28}$$

holds. The ambiguity in the choice of $f$ and $\mu$ corresponds to the multiplication by the exponent of a linear form in $z$ vanishing on $U$, i.e.,

$$f'(z, k) = f(z, k)e^{(b(k), z)}, \qquad \mu'_\lambda(k) = \mu_\lambda(k)e^{(b(k),\lambda)}, \quad (b(k), U) = 0, \tag{2.29}$$

where $b(k) = \sum_s b_s k^{-s}$ is a formal series with vector-coefficients that are orthogonal to $U$. The vector $U$ is in general position with respect to the lattice. Therefore, the ambiguity can be uniquely fixed by imposing $(d-1)$ normalizing conditions $\mu_{\lambda_i}(k) = 1, i > 1$. (Recall that $\mu_{\lambda_1}(k) = 1$ by construction.)

The formal series $f\phi$ is quasi-periodic and its multiplicators satisfy (2.26). Then, by these properties it is defined uniquely up to a factor which is constant in $z$ and $y$. Therefore, for the unique definition of $\phi_0$, it is enough to fix its evaluation at $z_0$ and $y = 0$. The corollary is proved.

## 3 The spectral curve

In this section, we show that $\lambda$-periodic wave solutions of equation (1.5), with $u$ as in (1.7), are common eigenfunctions of rings of commuting operators and identify $X$ with the Jacobian of the spectral curve of these rings.

Note that a simple shift $z \to z + Z$, where $Z \notin \Sigma$, gives $\lambda$-periodic wave solutions with meromorphic coefficients along the affine subspaces $Z + \mathbb{C}^d$. These $\lambda$-periodic wave solutions are related to each other by a $\partial_1$-invariant factor. Therefore, choosing, in the neighborhood of any $Z \notin \Sigma$, a hyperplane orthogonal to the vector $U$ and fixing initial data on this hyperplane at $y = 0$, we define the corresponding series $\phi(z + Z, y, k)$ as a *local* meromorphic function of $Z$ and *global* meromorphic function of $z$.

**Lemma 3.1.** *Let the assumptions of Theorem* 1.2 *hold. Then there is a unique pseudodifferential operator*

$$\mathcal{L}(Z, \partial_x) = \partial_x + \sum_{s=1}^{\infty} w_s(Z)\partial_x^{-s} \tag{3.1}$$

*such that*

$$\mathcal{L}(Ux + Vy + Z, \partial_x)\psi = k\psi, \tag{3.2}$$

*where* $\psi = e^{kx+k^2y}\phi(Ux+Z, y, k)$ *is a* $\lambda$*-periodic solution of* (1.5). *The coefficients* $w_s(Z)$ *of* $\mathcal{L}$ *are meromorphic functions on the abelian variety* $X$ *with poles along the divisor* $\Theta$.

*Proof.* The construction of $\mathcal{L}$ is standard for the KP theory. First we define $\mathcal{L}$ as a pseudodifferential operator with coefficients $w_s(Z, y)$, which are functions of $Z$ and $y$.

Let $\psi$ be a $\lambda$-periodic wave solution. The substitution of (2.17) in (3.2) gives a system of equations that recursively define $w_s(Z, y)$ as differential polynomials in $\xi_s(Z, y)$. The coefficients of $\psi$ are local meromorphic functions of $Z$, but the coefficients of $\mathcal{L}$ are well-defined *global meromorphic functions* on $\mathbb{C}^g \setminus \Sigma$, because different $\lambda$-periodic wave solutions are related to each other by a $\partial_1$-invariant factor, which does not affect $\mathcal{L}$. The singular locus is of codimension $\geq 2$. Then Hartog's holomorphic extension theorem implies that $w_s(Z, y)$ can be extended to a global meromorphic function on $\mathbb{C}^g$.

The translational invariance of $u$ implies the translational invariance of the $\lambda$-periodic wave solutions. Indeed, for any constant $s$ the series $\phi(Vs + Z, y - s, k)$ and $\phi(Z, y, k)$ correspond to $\lambda$-periodic solutions of the same equation. Therefore, they coincide up to a $\partial_1$-invariant factor. This factor does not affect $\mathcal{L}$. Hence $w_s(Z, y) = w_s(Vy + Z)$.

The $\lambda$-periodic wave functions corresponding to $Z$ and $Z + \lambda'$ for any $\lambda' \in \Lambda$ are also related to each other by a $\partial_1$-invariant factor:

$$\partial_1(\phi_1(Z + \lambda', y, k)\phi^{-1}(Z, y, k)) = 0. \tag{3.3}$$

Hence $w_s$ are periodic with respect to $\Lambda$ and therefore are meromorphic functions on the abelian variety $X$. The lemma is proved. □

Consider now the differential parts of the pseudodifferential operators $\mathcal{L}^m$. Let $\mathcal{L}^m_+$ be the differential operator such that $\mathcal{L}^m_- = \mathcal{L}^m - \mathcal{L}^m_+ = F_m\partial^{-1} + O(\partial^{-2})$. The leading coefficient $F_m$ of $\mathcal{L}^m_-$ is the residue of $\mathcal{L}^m$:

$$F_m = \operatorname{res}_\partial \mathcal{L}^m. \tag{3.4}$$

From the construction of $\mathcal{L}$ it follows that $[\partial_y - \partial_x^2 + u, \mathcal{L}^n] = 0$. Hence

$$[\partial_y - \partial_x^2 + u, \mathcal{L}^m_+] = -[\partial_y - \partial_x^2 + u, \mathcal{L}^m_-] = 2\partial_x F_m. \tag{3.5}$$

The functions $F_m$ are differential polynomials in the coefficients $w_s$ of $\mathcal{L}$. Hence $F_m(Z)$ are meromorphic functions on $X$. The next statement is crucial for the proof of the existence of commuting differential operators associated with $u$.

**Lemma 3.2.** *The abelian functions $F_m$ have at most a second-order pole along the divisor $\Theta$.*

*Proof.* We need a few more standard constructions from the KP theory. If $\psi$ is as in Lemma 3.1, then there exists a unique pseudodifferential operator $\Phi$ such that

$$\psi = \Phi e^{kx+k^2y}, \qquad \Phi = 1 + \sum_{s=1}^{\infty} \varphi_s(Ux + Z, y)\partial_x^{-s}. \tag{3.6}$$

The coefficients of $\Phi$ are universal differential polynomials on $\xi_s$. Therefore, $\varphi_s(z + Z, y)$ is a global meromorphic function of $z \in \mathbb{C}^d$ and a local meromorphic function of $Z \notin \Sigma$. Note that $\mathcal{L} = \Phi(\partial_x)\Phi^{-1}$.

Consider the dual wave function defined by the left action of the operator $\Phi^{-1}$: $\psi^+ = (e^{-kx-k^2y})\Phi^{-1}$. Recall that the left action of a pseudodifferential operator is the formal adjoint action under which the left action of $\partial_x$ on a function $f$ is $(f\partial_x) = -\partial_x f$. If $\psi$ is a formal wave solution of (3.5), then $\psi^+$ is a solution of the adjoint equation

$$(-\partial_y - \partial_x^2 + u)\psi^+ = 0. \tag{3.7}$$

The same arguments, as before, prove that if equations (1.11) for poles of $u$ hold then $\xi_s^+$ have simple poles at the poles of $u$. Therefore, if $\psi$ as in Lemma 2.3, then the dual wave solution is of the form $\psi^+ = e^{-kx-k^2y}\phi^+(Ux + Z, y, k)$, where the coefficients $\xi_s^+(z + Z, y)$ of the formal series

$$\phi^+(z + Z, y, k) = e^{-by}\left(1 + \sum_{s=1}^{\infty} \xi_s^+(z + Z, y)k^{-s}\right) \tag{3.8}$$

are $\lambda$-periodic meromorphic functions of the variable $z \in \mathbb{C}^d$ with a simple pole along the divisor $\Theta^U(y)$.

The ambiguity in the definition of $\psi$ does not affect the product

$$\psi^+\psi = (e^{-kx-k^2y}\Phi^{-1})(\Phi e^{kx+k^2y}). \tag{3.9}$$

Therefore, although each factor is only a local meromorphic function on $\mathbb{C}^g \setminus \Sigma$, the coefficients $J_s$ of the product

$$\psi^+\psi = \phi^+(Z, y, k)\phi(Z, y, k) = 1 + \sum_{s=2}^{\infty} J_s(Z, y)k^{-s}. \tag{3.10}$$

are *global meromorphic functions* of $Z$. Moreover, the translational invariance of $u$ implies that they have the form $J_s(Z, y) = J_s(Z + Vy)$. Each of the factors in the left-hand side of (3.10) has a simple pole along $\Theta - Vy$. Hence $J_s(Z)$ is a meromorphic function on $X$ with a second-order pole along $\Theta$.

From the definition of $\mathcal{L}$, it follows that

$$\operatorname{res}_k(\psi^+(\mathcal{L}^n\psi)) = \operatorname{res}_k(\psi^+k^n\psi) = J_{n+1}. \tag{3.11}$$

On the other hand, using the identity

$$\operatorname{res}_k (e^{-kx}\mathcal{D}_1)(\mathcal{D}_2 e^{kx}) = \operatorname{res}_\partial (\mathcal{D}_2\mathcal{D}_1), \tag{3.12}$$

which holds for any two pseudodifferential operators [14], we get

$$\operatorname{res}_k (\psi^+\mathcal{L}^n\psi) = \operatorname{res}_k (e^{-kx}\Phi^{-1})(\mathcal{L}^n\Phi e^{kx}) = \operatorname{res}_\partial \mathcal{L}^n = F_n. \tag{3.13}$$

Therefore, $F_n = J_{n+1}$ and the lemma is proved. □

Let $\hat{\mathbf{F}}$ be a linear space generated by $\{F_m, m = 0, 1, \dots\}$, where we set $F_0 = 1$. It is a subspace of the $2^g$-dimensional space of the abelian functions that have at most second-order pole along $\Theta$. Therefore, for all but $\hat{g} = \dim \hat{\mathbf{F}}$ positive integers $n$, there exist constants $c_{i,n}$ such that

$$F_n(Z) + \sum_{i=0}^{n-1} c_{i,n} F_i(Z) = 0. \tag{3.14}$$

Let $I$ denote the subset of integers $n$ for which there are no such constants. We call this subset the gap sequence.

**Lemma 3.3.** *Let $\mathcal{L}$ be the pseudodifferential operator corresponding to a $\lambda$-periodic wave function $\psi$ constructed above. Then for the differential operators*

$$L_n = \mathcal{L}_+^n + \sum_{i=0}^{n-1} c_{i,n}\mathcal{L}_+^{n-i} = 0, \quad n \notin I, \tag{3.15}$$

*the equations*

$$L_n\psi = a_n(k)\psi, \quad a_n(k) = k^n + \sum_{s=1}^{\infty} a_{s,n} k^{n-s}, \tag{3.16}$$

*where $a_{s,n}$ are constants, hold.*

*Proof.* First, note that from (3.5), it follows that

$$[\partial_y - \partial_x^2 + u, L_n] = 0. \tag{3.17}$$

Hence if $\psi$ is a $\lambda$-periodic wave solution of (1.5) corresponding to $Z \notin \Sigma$, then $L_n\psi$ is also a formal solution of the same equation. This implies the equation $L_n\psi = a_n(Z,k)\psi$, where $a$ is $\partial_1$-invariant. The ambiguity in the definition of $\psi$ does not affect $a_n$. Therefore, the coefficients of $a_n$ are well-defined *global* meromorphic functions on $\mathbb{C}^g \setminus \Sigma$. The $\partial_1$-invariance of $a_n$ implies that $a_n$, as a function of $Z$, is holomorphic outside of the locus. Hence it has an extension to a holomorphic function on $\mathbb{C}^g$. Equations (3.3) imply that $a_n$ is periodic with respect to the lattice $\Lambda$. Hence $a_n$ is $Z$-independent. Note that $a_{s,n} = c_{s,n}$, $s \le n$. The lemma is proved. □

The operator $L_m$ can be regarded as a $Z \notin \Sigma$-parametric family of ordinary differential operators $L_m^Z$ whose coefficients have the form

$$L_m^Z = \partial_x^n + \sum_{i=1}^{m} u_{i,m}(Ux + Z)\partial_x^{m-i}, \quad m \notin I. \tag{3.18}$$

**Corollary 3.1.** *The operators $L_m^Z$ commute with each other,*

$$[L_n^Z, L_m^Z] = 0, \quad Z \notin \Sigma. \tag{3.19}$$

From (3.16) it follows that $[L_n^Z, L_m^Z]\psi = 0$. The commutator is an ordinary differential operator. Hence the last equation implies (3.19).

**Lemma 3.4.** *Let $\mathcal{A}^Z$, $Z \notin \Sigma$, be a commutative ring of ordinary differential operators spanned by the operators $L_n^Z$. Then there is an irreducible algebraic curve $\Gamma$ of arithmetic genus $\hat{g} = \dim \hat{\mathbf{F}}$ such that $\mathcal{A}^Z$ is isomorphic to the ring $A(\Gamma, P_0)$ of meromorphic functions on $\Gamma$ with the only pole at a smooth point $P_0$. The correspondence $Z \to \mathcal{A}^Z$ defines a holomorphic imbedding of $X \setminus \Sigma$ into the space of torsion-free rank-1 sheaves $\mathcal{F}$ on $\Gamma$,*

$$j : X\backslash\Sigma \longmapsto \overline{\mathrm{Pic}}(\Gamma). \tag{3.20}$$

*Proof.* It is a fundamental fact of the theory of commuting linear ordinary differential operators [5, 6, 9, 10, 15] that there is a natural correspondence

$$\mathcal{A} \longleftrightarrow \{\Gamma, P_0, [k^{-1}]_1, \mathcal{F}\} \tag{3.21}$$

between *regular* at $x = 0$ commutative rings $\mathcal{A}$ of ordinary linear differential operators containing a pair of monic operators of coprime orders, and sets of algebraic-geometrical data $\{\Gamma, P_0, [k^{-1}]_1, \mathcal{F}\}$, where $\Gamma$ is an algebraic curve with a fixed first jet $[k^{-1}]_1$ of a local coordinate $k^{-1}$ in the neighborhood of a smooth point $P_0 \in \Gamma$ and $\mathcal{F}$ is a torsion-free rank-1 sheaf on $\Gamma$ such that

$$H^0(\Gamma, \mathcal{F}) = H^1(\Gamma, \mathcal{F}) = 0. \tag{3.22}$$

The correspondence becomes one-to-one if the rings $\mathcal{A}$ are considered modulo conjugation, $\mathcal{A}' = g(x)\mathcal{A}g^{-1}(x)$.

Note that in [5, 6, 9, 10] the main attention was paid to the generic case of commutative rings corresponding to smooth algebraic curves. The invariant formulation of the correspondence given above is due to Mumford [15].

The algebraic curve $\Gamma$ is called the spectral curve of $\mathcal{A}$. The ring $\mathcal{A}$ is isomorphic to the ring $A(\Gamma, P_0)$ of meromorphic functions on $\Gamma$ with the only pole at the puncture $P_0$. The isomorphism is defined by the equation

$$L_a\psi_0 = a\psi_0, \quad L_a \in \mathcal{A}, \quad a \in A(\Gamma, P_0). \tag{3.23}$$

Here $\psi_0$ is a common eigenfunction of the commuting operators. At $x = 0$, it is a section of the sheaf $\mathcal{F} \otimes \mathcal{O}(-P_0)$.

*Important Remark.* The construction of the correspondence (3.21) depends on a choice of initial point $x_0 = 0$. The spectral curve and the sheaf $\mathcal{F}$ are defined by the evaluations of the coefficients of generators of $\mathcal{A}$ and a finite number of their derivatives at the initial point. In fact, the spectral curve is independent of the choice of $x_0$, but the sheaf does depend on it, i.e., $\mathcal{F} = \mathcal{F}_{x_0}$.

Using the shift of the initial point it is easy to show that the correspondence (3.21) extends to commutative rings of operators whose coefficients are *meromorphic* functions of $x$ at $x = 0$. The rings of operators having poles at $x = 0$ correspond to sheaves for which the condition (3.22) is violated.

Let $\Gamma^Z$ be the spectral curve corresponding to $\mathcal{A}^Z$. Note that, due to the remark above, it is well defined for all $Z \notin \Sigma$. The eigenvalues $a_n(k)$ of the operators $L_n^Z$ defined in (3.16) coincide with the Laurent expansions at $P_0$ of the meromorphic functions $a_n \in A(\Gamma^Z, P_0)$. They are $Z$-independent. Hence the spectral curve is $Z$-independent as well, $\Gamma = \Gamma^Z$. The first statement of the lemma is thus proved. □

The construction of the correspondence (3.21) implies that if the coefficients of the operators $\mathcal{A}$ holomorphically depend on parameters then the algebraic-geometrical spectral data are also holomorphic functions of the parameters. Hence $j$ is holomorphic away from $\Theta$. Then using the shift of the initial point and the fact, that $\mathcal{F}_{x_0}$ holomorphically depends on $x_0$, we get that $j$ holomorphically extends over $\Theta \setminus \Sigma$, as well. The lemma is proved.

Recall that a commutative ring $\mathcal{A}$ of linear ordinary differential operators is called maximal if it is not contained in any bigger commutative ring. Let us show that for a generic $Z$ the ring $\mathcal{A}^Z$ is maximal. Suppose that it is not. Then there exists $\alpha \in I$, where $I$ is the gap sequence defined above, such that for each $Z \notin \Sigma$ there exists an operator $L_\alpha^Z$ of order $\alpha$ which commutes with $L_n^Z, n \notin I$. Therefore, it commutes with $\mathcal{L}$. A differential operator commuting with $\mathcal{L}$ up to order $O(1)$ can be represented in the form $L_\alpha = \sum_{m<\alpha} c_{i,\alpha}(Z)\mathcal{L}_+^i$, where $c_{i,\alpha}(Z)$ are $\partial_1$-invariant functions of $Z$. It commutes with $\mathcal{L}$ if and only if

$$F_\alpha(Z) + \sum_{i=0}^{n-1} c_{i,\alpha}(Z)F_i(Z) = 0, \quad \partial_1 c_{i,\alpha} = 0. \tag{3.24}$$

Note the difference between (3.14) and (3.24). In the first equation the coefficients $c_{i,n}$ are constants. The $\lambda$-periodic wave solution of equation (1.5) is a common eigenfunction of all commuting operators, i.e., $L_\alpha\psi = a_\alpha(Z,k)\psi$, where $a_\alpha = k^\alpha + \sum_{s=1}^\infty a_{s,\alpha}(Z)k^{\alpha-s}$ is $\partial_1$-invariant. The same arguments as those used in the proof of equation (3.16) show that the eigenvalue $a_\alpha$ is $Z$-independent. We have $a_{s,\alpha} = c_{s,\alpha}$, $s \le \alpha$. Therefore, the coefficients in (3.24) are $Z$-independent. This contradicts the assumption that $\alpha \notin I$.

Our next goal is to finally prove the global existence of the wave function.

**Lemma 3.5.** *Let the assumptions of Theorem* 1.2 *hold. Then there exists a common eigenfunction of the corresponding commuting operators $L_n^Z$ of the form $\psi = e^{kx}\phi(Ux + Z, k)$ such that the coefficients of the formal series*

$$\phi(Z,k) = 1 + \sum_{s=1}^{\infty} \xi_s(Z)k^{-s} \tag{3.25}$$

*are global meromorphic functions with a simple pole along* $\Theta$.

*Proof.* It is instructive to consider first the case when the spectral curve $\Gamma$ of the rings $\mathcal{A}^Z$ is smooth. Then as shown in [5, 6], the corresponding common eigenfunction of the commuting differential operators (the Baker–Akhiezer function), normalized by the condition $\psi_0|_{x=0} = 1$, is of the form [5, 6]

$$\hat{\psi}_0 = \frac{\hat{\theta}(\hat{A}(P) + \hat{U}x + \hat{Z})\hat{\theta}(\hat{Z})}{\hat{\theta}(\hat{U}x + \hat{Z})\hat{\theta}(\hat{A}(P) + \hat{Z})} e^{x\Omega(P)}. \tag{3.26}$$

Here $\hat{\theta}(\hat{Z})$ is the Riemann theta-function constructed with the help of the matrix of $b$-periods of normalized holomorphic differentials on $\Gamma$; $\hat{A} : \Gamma \to J(\Gamma)$ is the Abel map; $\Omega$ is the abelian integral corresponding to $d\Omega$; $d\Omega$ is the meromorphic differential of the second kind and has the only pole at the puncture $P_0$, where its singularity is of the form $dk$; and $2\pi i\hat{U}$ is the vector of its $b$-periods.

*Remark.* Let us emphasize, that the formula (3.26) is not the result of solution of some differential equations. It is a direct corollary of analytic properties of the Baker–Akhiezer function $\hat{\psi}_0(x, P)$ on the spectral curve:

(i) $\hat{\psi}_0$ *is a meromorphic function of* $P \in \Gamma \setminus P_0$*; its pole divisor is of degree* $\tilde{g}$ *and is* $x$*-independent. It is nonspecial if the operators are regular at the normalization point* $x = 0$.
(ii) *In the neighborhood of* $P_0$ *the function* $\hat{\psi}_0$ *has the form* (1.16) (*with* $y = 0$).

From the Riemann–Roch theorem, it follows that, if $\hat{\psi}_0$ exists, then it is unique. It is easy to check that the function $\hat{\psi}_0$ given by (3.26) is single-valued on $\Gamma$ and has all the desired properties.

The last factors in the numerator and the denominator of (3.26) are $x$-independent. Therefore, the function

$$\hat{\psi}_{BA} = \frac{\hat{\theta}(\hat{A}(P) + \hat{U}x + \hat{Z})}{\hat{\theta}(Ux + \hat{Z})} e^{x\Omega(P)} \tag{3.27}$$

is also a common eigenfunction of the commuting operators.

In the neighborhood of $P_0$ the function $\hat{\psi}_{BA}$ has the form

$$\hat{\psi}_{BA} = e^{kx}\left(1 + \sum_{s=1}^{\infty} \frac{\tau_s(\hat{Z} + \hat{U}x)}{\hat{\theta}(\hat{U}x + \hat{Z})} k^{-s}\right), \quad k = \Omega, \tag{3.28}$$

where $\tau_s(\hat{Z})$ are global holomorphic functions.

According to Lemma 3.4, we have a holomorphic imbedding $\hat{Z} = j(Z)$ of $X \setminus \Sigma$ into $J(\Gamma)$. Consider the formal series $\psi = j^*\hat{\psi}_{BA}$. It is globally well defined away from $\Sigma$. If $Z \notin \Theta$, then $j(Z) \notin \hat{\Theta}$ (which is the divisor on which the condition (3.22) is violated). Hence the coefficients of $\psi$ are regular away from $\Theta$. The singular locus is at least of codimension 2. Hence, once again using Hartog-type arguments, we can extend $\psi$ on $X$.

If the spectral curve is singular, we can proceed along the same lines using the generalization of (3.27) given by the theory of the Sato $\tau$-function [16]. Namely, a set of algebraic-geometrical data (3.21) defines a point of the Sato Grassmannian, and therefore the corresponding $\tau$-function: $\tau(t; \mathcal{F})$. It is a holomorphic function of the variables $t = (t_1, t_2, \dots)$, and is a section of a holomorphic line bundle on $\overline{\mathrm{Pic}}(\Gamma)$.

The variable $x$ is identified with the first time of the KP-hierarchy, $x = t_1$. Therefore, the formula for the Baker–Akhiezer function corresponding to a point of the Grassmannian [16] implies that the function $\hat{\psi}_{BA}$ given by the formula

$$\hat{\psi}_{BA} = \frac{\tau(x-k, -\frac{1}{2}k^2, -\frac{1}{3}k^3, \dots; \mathcal{F})}{\tau(x, 0, 0, \dots; \mathcal{F})} e^{kx} \tag{3.29}$$

is a common eigenfunction of the commuting operators defined by $\mathcal{F}$. The rest of the arguments proving the lemma are the same as in the smooth case. □

**Lemma 3.6.** *The linear space* $\hat{\mathbf{F}}$ *generated by the abelian functions* $\{F_0 = 1, F_m = \mathrm{res}_\partial \mathcal{L}^m\}$, *is a subspace of the space* $\mathbf{H}$ *generated by* $F_0$ *and by the abelian functions* $H_i = \partial_1 \partial_{z_i} \ln \theta(Z)$.

*Proof.* Recall that the functions $F_n$ are abelian functions with at most second-order poles on $\Theta$. Hence a priori $\hat{g} = \dim \hat{\mathbf{F}} \leq 2^g$. In order to prove the statement of the lemma, it is enough to show that $F_n = \partial_1 Q_n$, where $Q_n$ is a meromorphic function with a pole along $\Theta$. Indeed, if $Q_n$ exists, then, for any vector $\lambda$ in the period lattice, we have $Q_n(Z+\lambda) = Q_n(Z) + c_{n,\lambda}$. There is no abelian function with a simple pole on $\Theta$. Hence there exists a constant $q_n$ and two $g$-dimensional vectors $l_n, l_n'$, such that $Q_n = q_n + (l_n, Z) + (l_n', h(Z))$, where $h(Z)$ is a vector with the coordinates $h_i = \partial_{z_i} \ln \theta$. Therefore, $F_n = (l_n, U) + (l_n', H(Z))$.

Let $\psi(x, Z, k)$ be the formal Baker–Akhiezer function defined in the previous lemma. Then the coefficients $\varphi_s(Z)$ of the corresponding wave operator $\Phi$ (3.6) are global meromorphic functions with poles along $\Theta$.

The left and right actions of pseudodifferential operators are formally adjoint, i.e., for any two operators the equality $(e^{-kx}\mathcal{D}_1)(\mathcal{D}_2 e^{kx}) = e^{-kx}(\mathcal{D}_1\mathcal{D}_2 e^{kx}) + \partial_x(e^{-kx}(\mathcal{D}_3 e^{kx}))$ holds. Here $\mathcal{D}_3$ is a pseudodifferential operator whose coefficients are differential polynomials in the coefficients of $\mathcal{D}_1$ and $\mathcal{D}_2$. Therefore, from (3.9)–(3.13) it follows that

$$\psi^+\psi = 1 + \sum_{s=2}^{\infty} F_{s-1} k^{-s} = 1 + \partial_x \left( \sum_{s=2}^{\infty} Q_s k^{-s} \right). \tag{3.30}$$

The coefficients of the series $Q$ are differential polynomials in the coefficients $\varphi_s$ of the wave operator. Therefore, they are global meromorphic functions of $Z$ with poles along $\Theta$. The lemma is proved. □

In order to complete the proof of our main result, we need one more standard fact of the KP theory: flows of the KP hierarchy define deformations of the commutative rings $\mathcal{A}$ of ordinary linear differential operators. The spectral curve is invariant under these flows. For a given spectral curve $\Gamma$ the orbits of the KP hierarchy are isomorphic to the generalized Jacobian $J(\Gamma) = \mathrm{Pic}^0(\Gamma)$, which is the set of equivalence classes of zero degree divisors on the spectral curve (see details in [1, 5, 6, 16]).

The KP hierarchy in the Sato form is a system of commuting differential equation for a pseudodifferential operator $\mathcal{L}$,

$$\partial_{t_n}\mathcal{L} = [\mathcal{L}^n_+, \mathcal{L}]. \tag{3.31}$$

If the operator $\mathcal{L}$ is as above, i.e., if it is defined by $\lambda$-periodic wave solutions of equation (1.5), then equations (3.31) are equivalent to the equations

$$\partial_{t_n} u = \partial_x F_n. \tag{3.32}$$

The first two times of the hierarchy are identified with the variables $t_1 = x$, $t_2 = y$.

Equations (3.32) identify the space $\hat{\mathbf{F}}_1$ generated by the functions $\partial_1 F_n$ with the tangent space of the KP orbit at $\mathcal{A}^Z$. Then from Lemma 3.6, it follows that this tangent space is a subspace of the tangent space of the abelian variety $X$. Hence for any $Z \notin \Sigma$, the orbit of the KP flows of the ring $\mathcal{A}^Z$ is in $X$, i.e., it defines a holomorphic imbedding:

$$i_Z : J(\Gamma) \longmapsto X. \tag{3.33}$$

From (3.33), it follows that $J(\Gamma)$ is *compact*.

The generalized Jacobian of an algebraic curve is compact if and only if the curve is *smooth* [17]. On a smooth algebraic curve a torsion-free rank-1 sheaf is a line bundle, i.e., $\overline{\mathrm{Pic}}(\Gamma) = J(\Gamma)$. Then (3.20) implies that $i_Z$ is an isomorphism. Note that for the Jacobians of smooth algebraic curves the bad locus $\Sigma$ is empty [1], i.e., the imbedding $j$ in (3.20) is defined everywhere on $X$ and is inverse to $i_Z$. Theorem 1.2 is proved.

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