Elliptic solutions to difference non-linear equations and nested Bethe ansatz equations

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Abstract

We outline an approach to a theory of various generalizations of the elliptic Calogero-Moser (CM) and Ruijsenaars-Shneider (RS) systems based on a special inverse problem for linear operators with elliptic coefficients. Hamiltonian theory of such systems is developed with the help of the universal symplectic structure proposed by D.H. Phong and the author. Canonically conjugated action-angle variables for spin generalizations of the elliptic CM and RS systems are found.

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1 Introduction

The elliptic nested Bethe ansatz equations are a system of algebraic equations

$$\prod_{j\neq i}\frac{\sigma(x_i^n - x_j^{n+1})\sigma(x_i^n - \eta - x_j^n)\sigma(x_i^n - x_j^{n-1} + \eta)}{\sigma(x_i^n - x_j^{n-1})\sigma(x_i^n + \eta - x_j^n)\sigma(x_i^n - x_j^{n+1} - \eta)} = -1 \tag{1.1}$$

for N unknown functions $x_i = x_i^n$, $i = 1, \ldots, N$, of a discrete variable n ([1]). (Here and below $\sigma(x) = \sigma(x|\omega_1, \omega_2)$, $\zeta(x) = \zeta(x|\omega, \omega')$, and $\wp(x) = \wp(x|\omega, \omega')$ are the Weierstrass σ-, ζ-, and $\wp$-functions corresponding to the elliptic curve with periods 2ω, $2\omega'$.) This system is an example of the whole family of integrable systems which have attracted renewed interest for years. The most recent burst of interest is due to the unexpected connections of these systems to Seiberg-Witten solution of $N = 2$ supersymmetric gauge theories [2]. It turns out that the low energy effective theory for $SU(N)$ model with matter in the adjoint representation (identified first in [3] with $SU(N)$ Hitchin system) is isomorphic to the elliptic CM system. Using this connection quantum order parameters were found in [5].

The elliptic Calogero-Moser (CM) system [4], [6] is a system of N identical particles on a line interacting with each other via the potential $V(x) = \wp(x)$. Its equations of motion have the form

$$\ddot{x}_i = 4\sum_{j\neq i}\wp'(x_i - x_j). \tag{1.2}$$

The CM system is a completely integrable Hamiltonian system, i.e. it has N independent integrals H_k in involution ([7], [8]). The second integral H_2 is the Hamiltonian of (1.2).

In [9] a remarkable connection of the CM system with a theory of elliptic solutions to the KdV equation was revealed. It was shown that the elliptic solutions of the KdV equations have the form $u(x, t) = 2\sum_{i=1}\wp(x - x_i(t))$ and the poles $x_i(t)$ of the solutions satisfy the constraint $\sum_{j\neq i}\wp'(x_i - x_j) = 0$, which is the *locus* of the stationary points of the CM system. Moreover, it turns out that the dependence of the poles with respect to t coincides with the Hamiltonian flow corresponding to the third integral H_3 of the system. In [10], [11] it was found that this connection becomes an isomorphism in the case of the elliptic solutions to the Kadomtsev-Petviashvilii equation. Since then, the theory of the CM system and its various generalizations is inseparable from the theory of the elliptic solutions to the soliton equations.

In [12] system (1.1) was revealed as the pole system corresponding to the elliptic solutions of completely discretized version of the KP equation on lattice. It was noticed that equations (1.1) have the form of the Bethe ansatz equations for the spin-$\frac{1}{2}$ Heisenberg chain with impurities. Its connection with the nested ansatz equations for the A_k-lattice models was established in [1].

As shown in [16] an intermediate discretization of the KP equation which is the $2D$ Toda lattice equations leads to the Ruijesenaars-Schneider system [13]:

$$\ddot{x}_i = \sum_{s\neq i}\dot{x}_i\dot{x}_s(V(x_i - x_s) - V(x_s - x_i)), \ V(x) = \zeta(x) - \zeta(x + \eta), \tag{1.3}$$

which is a relativistic version of (1.2).

The main goal of this paper is to present a general approach to the theory of the CM type many body systems which is based on a special inverse problem for *linear* operators with the elliptic coefficients. This approach originated in [14] and developed in [1], [15], [16] clarifies the connection of these systems with the soliton equations for which the corresponding linear operator is the Lax operator. We formulate the inverse problem in the next section and show that its solution is equivalent to a finite-dimensional integrable system. We discuss an algebraic-geometric interpretation of the corresponding systems.

The advantage of our approach is that it generates the finite-dimensional system simultaneously with its Lax representation. Until recently, among its disadvantages was the missing connection to the Hamiltonian theory. For example, in [16] spin generalization of the RS system was proposed and explicitly solved in terms of the Riemann theta-functions of auxiliary spectral curves. At the same time, all direct attempts to show that this system is Hamiltonian have failed, so far.

One of the existing general approaches to the Hamiltonian theory of the CM type systems is based on their geometric interpretation as reductions of geodesic flows on symmetric spaces [8]. Equivalently, these models can be obtained from free dynamics on a larger phase space possessing a rich symmetry by means of the Hamiltonian reduction [17]. A generalization to infinite-dimensional phase spaces (cotangent bundles to current algebras and groups) was suggested in [18], [19]. The infinite-dimensional gauge symmetry allows one to make the reduction to a finite number of degrees of freedom.

A further generalization of this approach consists in considering dynamical systems on cotangent bundles to moduli spaces of stable holomorphic vector bundles on Riemann surfaces. Such systems were introduced by Hitchin in the paper [20], where their integrability was proved. An attempt to identify the known many body integrable systems in terms of the abstract formalism developed by Hitchin was made in [21]. To do this, it is necessary to consider vector bundles on algebraic curves with singular points. It turns out that the class of integrable systems corresponding to the Riemann sphere with marked points includes spin generalizations of the CM model as well as integrable Gaudin magnets [22] (see also [23]).

Unfortunately, a geometric interpretation of spin generalization of the elliptic RS system has not been yet found. Recently, such realization and consequently the Hamiltonian theory of the rational degeneration of that system were found in [24].

In section 3 we develop Hamiltonian theory of the CM type systems in the framework of the new approach to the Hamiltonian theory of soliton equations proposed in [25] and [26]. It can be applied evenly to any equation having the Lax representation. The symplectic structure is constructed in terms of the Lax operator, only. We discuss three basic examples: spin generalizations of the CM and the RS systems, and the nested Bethe ansatz equations. We would like to emphasize that the universal form of the symplectic structure provides a universal and direct way to the action-angle type variables.

We would like to refer to [5] for the analysis of connections of the elliptic CM system to Seiberg-Witten theory of $N = 2$ supersymmetric gauge theories.

2 Generating problem

Let $\mathcal{L}$ be a linear differential or difference operator in two variables x, t with coefficients which are scalar or matrix elliptic functions of the variable x (i.e. meromorphic double-periodic functions with the periods $2\omega_\alpha$, $\alpha = 1, 2$). We do not assume any special dependence of the coefficients with respect to the second variable. Then it is natural to introduce a notion of *double-Bloch* solutions of the equation

$$\mathcal{L}\Psi = 0. \tag{2.1}$$

We call a *meromorphic* vector-function $f(x)$ that satisfies the following monodromy properties:

$$f(x + 2\omega_\alpha) = B_\alpha f(x), \quad \alpha = 1, 2, \tag{2.2}$$

a *double-Bloch function*. The complex numbers B_α are called *Bloch multipliers*. (In other words, f is a meromorphic section of a vector bundle over the elliptic curve.)

In the most general form a problem that we are going to address is to *classify* and to *construct* all the operators $\mathcal{L}$ such that equation (2.1) has *sufficiently enough* double-Bloch solutions.

It turns out that existence of the double-Bloch solutions is so restrictive that only in exceptional cases such solutions do exist. A simple and general explanation of that is due to the Riemann-Roch theorem. Let D be a set of points x_i, $i = 1, \ldots, m$, on the elliptic curve Γ_0 with multiplicities d_i and let $V = V(D; B_1, B_2)$ be a linear space of the double-Bloch functions with the Bloch multipliers B_α that have poles at x_i of order less or equal to d_i and holomorphic outside D. Then the dimension of D is equal to:

$$\dim D = \deg D = \sum_i d_i.$$

Now let x_i depend on the variable t. Then for $f \in D(t)$ the function $\mathcal{L}f$ is a double-Bloch function with the same Bloch multipliers but in general with higher orders of poles because taking derivatives and multiplication by the elliptic coefficients increase orders. Therefore, the operator $\mathcal{L}$ defines a linear operator

$$\mathcal{L}|_D : V(D(t); B_1, B_2) \longmapsto V(D'(t); B_1, B_2), \ N' = \deg D' > N = \deg D,$$

and (2.1) is *always* equivalent to an *over-determined* linear system of N' equations for N unknown variables which are the coefficients $c_i = c_i(t)$ of an expansion of $\Psi \in V(t)$ with respect to a basis of functions $f_i(t) \in V(t)$. With some exaggeration one may say that in the soliton theory the representation of a system in the form of the compatibility condition of an over-determined system of the linear problems is considered as equivalent to integrability.

In all of the examples which we are going to discuss $N' = 2N$ and the over-determined system of equations has the form

$$LC = kC, \quad \partial_t C = MC, \tag{2.3}$$

where L and M are $N \times N$ matrix functions depending on a point z of the elliptic curve as on a parameter. A compatibility condition of (2.3) has the standard Lax form $\partial_t L = [M, L]$, and is equivalent to a finite-dimensional integrable system.

The basis in the space of the double-Bloch functions can be written in terms of the fundamental function $\Phi(x, z)$ defined by the formula

$$\Phi(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}. \tag{2.4}$$

Note, that $\Phi(x, z)$ is a solution of the Lame equation:

$$\left(\frac{d^2}{dx^2} - 2\wp(x) \right) \Phi(x, z) = \wp(z)\Phi(x, z). \tag{2.5}$$

From the monodromy properties it follows that Φ considered as a function of z is double-periodic:

$$\Phi(x, z + 2\omega_\alpha) = \Phi(x, z),$$

though it is not elliptic in the classical sense due to an essential singularity at $z = 0$ for $x \neq 0$.

As a function of x the function $\Phi(x, z)$ is double-Bloch function, i.e.

$$\Phi(x + 2\omega_\alpha, z) = T_\alpha(z)\Phi(x, z), \ \ T_\alpha(z) = \exp\left(2\omega_\alpha \zeta(z) - 2\zeta(\omega_\alpha)z \right).$$

In the fundamental domain of the lattice defined by $2\omega_\alpha$ the function $\Phi(x, z)$ has a unique pole at the point $x = 0$:

$$\Phi(x, z) = x^{-1} + O(x). \tag{2.6}$$

The gauge transformation

$$f(x) \longmapsto \tilde{f}(x) = f(x)e^{ax},$$

where a is an arbitrary constant does not change poles of any function and transform a double Bloch-function into a double-Bloch function. If B_α are Bloch multipliers for f than the Bloch multipliers for $\tilde{f}$ are equal to

$$\tilde{B}_1 = B_1 e^{2a\omega_1}, \ \ \tilde{B}_2 = B_2 e^{2a\omega_2}. \tag{2.7}$$

The two pairs of Bloch multipliers that are connected with each other through the relation (2.7) for some a are called equivalent. Note that for all equivalent pairs of Bloch multipliers the product $B_1^{\omega_2} B_2^{-\omega_1}$ is a constant depending on the equivalence class, only.

From (2.6) it follows that a double-Bloch function $f(x)$ with simple poles x_i in the fundamental domain and with Bloch multipliers B_α (such that at least one of them is not equal to 1) may be represented in the form:

$$f(x) = \sum_{i=1}^{N} c_i \Phi(x - x_i, z)e^{kx}, \tag{2.8}$$

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where c_i is a residue of f at x_i and z, k are parameters related by

$$B_\alpha = T_\alpha(z)e^{2\omega_\alpha k}. \tag{2.9}$$

(Any pair of Bloch multipliers may be represented in the form (2.9) with an appropriate choice of the parameters z and k.)

To prove (2.8) it is enough to note that as a function of x the difference of the left and right hand sides is holomorphic in the fundamental domain. It is a double-Bloch function with the same Bloch multipliers as the function f. But a non-trivial double-Bloch function with at least one of the Bloch multipliers that is not equal to 1, has at least one pole in the fundamental domain.

Now we are in a position to present a few examples of the generating problem.

Example 1. The elliptic CM system ([14]).

Let us consider the equation

$$\mathcal{L}\Psi = (\partial_x - \partial_x^2 + u(x,t))\Psi = 0, \tag{2.10}$$

where $u(x,t)$ is an elliptic function. Then as shown in [14] equation (2.10) has N linear independent double-Bloch solutions with equivalent Bloch multipliers and N simple poles at points $x_i(t)$ if and only if $u(x,t)$ has the form

$$u(x,t) = 2\sum_{i=1}^{N} \wp(x - x_i(t)) \tag{2.11}$$

and $x_i(t)$ satisfy the equations of motion of the elliptic CM system (1.2).

The assumption that there exist N linear independent double-Bloch solutions with equivalent Bloch multipliers implies that they can be written in the form

$$\Psi = \sum_{i=1}^{N} c_i(t,k,z)\Phi(x - x_i(t), z)e^{kx+k^2 t}, \tag{2.12}$$

with the same z but different values of the parameter k.

Let us substitute (2.12) into (2.10). Then (2.10) is satisfied if and if we get a function holomorphic in the fundamental domain. First of all, we conclude that u has poles at x_i, only. The vanishing of the triple poles $(x - x_i)^{-3}$ implies that $u(x,t)$ has the form (2.11). The vanishing of the double poles $(x - x_i)^{-2}$ gives the equalities that can be written as a matrix equation for the vector $C = (c_i)$:

$$(L(t,z) - kI)C = 0, \tag{2.13}$$

where I is the unit matrix and the Lax matrix $L(t,z)$ is defined as follows [1] :

$$L_{ij}(t,z) = -\frac{1}{2}\delta_{ij}\dot{x}_i - (1 - \delta_{ij})\Phi(x_i - x_j, z). \tag{2.14}$$

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[1] in order to simplify the consequent formulae in Section 3 and present all the examples in the same framework we choose here and in the next example a normalization of L which differs from that used in [14], [15] by the factor $-1/2$.

Finally, the vanishing of the simple poles gives the equations

$$(\partial_t - M(t, z))C = 0,\tag{2.15}$$

where

$$M_{ij} = \left(\wp(z) - 2\sum_{j \neq i}\wp(x_i - x_j)\right)\delta_{ij} - 2(1 - \delta_{ij})\Phi'(x_i - x_j, z).\tag{2.16}$$

The existence of N linear independent solutions for (2.10) with equivalent Bloch multipliers implies that (2.13) and (2.15) have N independent solutions corresponding to different values of k. Hence, as a compatibility condition we get the Lax equation $\dot{L} = [M, L]$ which is equivalent to (1.2). Note that the last system does not depend on z. Therefore, if (2.13) and (2.15) are compatible for some z then they are compatible for all z. As a result we conclude that if (2.10) has N linear independent double-Bloch solutions with equivalent Bloch multipliers then it has infinitely many of them. All the double-Bloch solutions are parameterized by points of an algebraic curve Γ defined by the characteristic equation

$$R(k, z) \equiv \det(kI - L(z)) = k^N + \sum_{i=1}^{N} r_i(z)k^{N-i} = 0.\tag{2.17}$$

Equation (2.17) can be seen as a dispersion relation between two Bloch multipliers and defines Γ as N-sheet cover of Γ_0.

From (2.4) and (2.14) it follows that

$$L = G\widetilde{L}G^{-1}, \quad G_{ij} = e^{\zeta(z)x_i}\delta_{ij}.\tag{2.18}$$

At $z = 0$ we have the form

$$\widetilde{L} = z^{-1}(F - I) + O(1), \quad F_{ij} = 1.\tag{2.19}$$

The matrix F has zero eigenvalue with multiplicity $N - 1$ and a simple eigenvalue N. Therefore, in a neighborhood of $z = 0$ the characteristic polynomial (2.17) has the form

$$R(k, z) = \prod_{i=1}^{N}(k + \nu_i z^{-1} + h_i + O(z)), \quad \nu_1 = 1 - N, \; \nu_i = 1, \; i > 1.\tag{2.20}$$

We call the sheet of the covering Γ at $z = 0$ corresponding to the branch $k = z^{-1}(N-1)+O(1)$ by upper sheet and mark the point P_1 on this sheet among the preimages of $z = 0$. From (2.20) it follows that in general position when the curve Γ is smooth, its genus equals N.

The coefficient $r_i(z)$ in (2.17) is an elliptic function with a pole at $z = 0$ of order i. The coefficients $I_{i,s}$ of the expansion

$$r_i(z) = I_{i,0} + \sum_{s=0}^{i-2} I_{i,s}\partial_z^s\wp(z)\tag{2.21}$$

are integrals of motion. From (2.20) it is easy to show that there are only N independent among them. Recently, a remarkable explicit representation for the characteristic equation (2.17) was found ([5]:

$$R(k, z) = f(k - \zeta(z), z), \quad f(k, z) = \frac{1}{\sigma(z)} \, \sigma\left(z + \frac{\partial}{\partial k}\right) H(k), \tag{2.22}$$

where H(k) is a polynomial. Note that (2.22) may be written as:

$$f(k, z) = \frac{1}{\sigma(z)} \sum_{n=1}^{N} \frac{1}{n!} \partial_z^n \sigma(z) \left(\frac{\partial}{\partial k}\right)^n H(k).$$

The coefficients of the polynomial $H(k)$ are free parameters of the spectral curve of the CM system.

When the Lax representation and the corresponding algebraic curve Γ are constructed, the next step is to consider analytical properties of the eigenvectors of the Lax operator L on Γ.

Let L be a matrix of the form (2.14). Then the components of the eigenvector $C(P) = (c_i(P))$, $P = (k, z) \in \Gamma$, normalized by the condition

$$\sum_{i=1}^{N} c_i(P)\Phi(-x_i, z) = 1, \tag{2.23}$$

are meromorphic functions on Γ outside the preimages P_i of $z = 0$. They have N poles $\gamma_1, \ldots, \gamma_N$ (in a general position γ_s are distinct). At the points P_j the coordinates $c_i(P)$ have the expansion:

$$c_i = z(c_i^1 + O(z))e^{\zeta(z)x_i}; \quad c_i = (c_i^j + O(z))e^{\zeta(z)x_i}, \quad i > 1. \tag{2.24}$$

If we denote the diagonal elements of L by $-p_i/2$, then the above constructed correspondence

$$p_i, \ x_i \longmapsto \{\Gamma, \mathcal{D} = \{\gamma_s\}\} \tag{2.25}$$

is an isomorphism (on the open set).

Now let $x_i(t)$ be a solution of (1.2) then the divisor $\mathcal{D}$ corresponding to $p_i = \dot{x}_i(t)$, $x_i(t)$ depends on t, $\mathcal{D} = \mathcal{D}(t)$. It turns out that under the Abel transform this dependence becomes linear on the Jacobian $J(\Gamma)$ of the spectral curve. The final result is as follows.

Theorem 2.1 *The coordinates of the particles $x_i(t)$ are roots of the equation*

$$\theta(Ux + Vt + Z_0) = 0, \tag{2.26}$$

where $\theta(\xi) = \theta(\xi|B)$ is the Riemann theta-function corresponding to matrix of b-periods of holomorphic differentials on Γ; the vectors U and V are the vectors of b-periods of normalized meromorphic differentials on Γ, with poles of order 2 and 3 at the point P_1; the vector Z_0 is the Abel transform of the divisor $\mathcal{D}_0 = \mathcal{D}(0)$.

This result can be reformulated in the following more geometric way. Let $J(\Gamma)$ be the Jacobain (N-dimensional complex torus) of a smooth genus N algebraic curve Γ. Abel transform defines imbeding of Γ into $J(\Gamma)$. A point $P \in \Gamma$ defines a vector U in $J(\Gamma)$ that is the tangent vector to the image of Γ at the point. Let us consider a class of curves having the following property: there exists a point on the curve such that the complex linear subspace generated by the corresponding vector U *is compact*, i.e. it is an elliptic curve Γ_0. This means that there exist two complex numbers $2\omega_\alpha$, $\mathrm{Im}\, \omega_2/\omega_1 > 0$, such that $2\omega_\alpha U$ belongs to the lattice of periods of the holomorphic differentials on Γ. From pure algebraic-geometrical point of view the problem of the description of such curves is transcendental. It turns out that this problem has an explicit solution, and algebraic equations that define such curves are the characteristic equation for the Lax operator corresponding to the CM system. Moreover, it turns out that in general position Γ_0 intersects theta-divisor at N points x_i and if we move Γ_0 in the direction that is defined by the vector V of the second jet of Γ at P then the intersections of Γ_0 with the theta-divisor move according to the CM dynamics.

Example 2. Spin generalization of the elliptic CM system ([15]).

Let $\mathcal{L}$ be an operator of the same form (2.10) as in the previous case, but now $u = u^\beta_\alpha(x,t)$ is an elliptic ($l \times l$) *matrix* function of the variable x. We slightly reformulate the results of [15] to a form that would be used later.

Equation (2.10) has $N \geq l$ linear independent double-Bloch solutions with N simple poles at points $x_i(t)$ and such that ($l \times N$) matrix formed by its residues at the poles has rank l if and only if:
(i) the potential u has the form

$$u = \sum_{i=1}^{N} a_i(t) b_i^+(t) \wp(x - x_i(t)), \tag{2.27}$$

where $a_i = (a_{i,\alpha})$ are l-dimensional vectors and $b_i^+ = (b_i^\alpha)$ are l-dimensional co-vectors;
(ii) $x_i(t)$ satisfy the equations

$$\ddot{x}_i = \sum_{j \neq i} (b_i^+ a_j)(b_j^+ a_i) \wp'(x_i - x_j); \tag{2.28}$$

(iii) the vectors a_i and co-vectors b_i^+ satisfy the constraints

$$b_i^+ a_i = \sum_{\alpha=1}^{l} b_i^\alpha(t) a_{i\alpha}(t) = 2 \tag{2.29}$$

and the equations

$$\dot{a}_i = -\sum_{j \neq i} a_j (b_j^+ a_i) \wp(x_i - x_j) - \lambda_i a_i, \quad \dot{b}_i = \sum_{j \neq i} b_j (b_i^+ a_j) \wp(x_i - x_j) + \lambda_i b_i^+ \tag{2.30}$$

where $\lambda_i = \lambda_i(t)$ are scalar functions.

In order to get these results we represent a double-Bloch vector function Ψ in the form:

$$\Psi = \sum_{i=1}^{N} s_i(t, k, z) \Phi(x - x_i(t), z) e^{kx + k^2 t}, \tag{2.31}$$

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where s_i are l-dimensional vectors and substitute it into (2.10). The vanishing of the triple pole $(x - x_i)^{-3}$ implies that u has the form (2.27), the vectors s_i are proportional to a_i, i.e. $s_i = c_i a_i$, where c_i are scalars, and that the constraints (2.29) are fulfilled.

The vanishing of the coefficients in front of $(x - x_i)^{-2}$ implies that the vector C with the coordinates c_i satisfies (2.13), where the Lax matrix has the form

$$L_{ij} = -\frac{1}{2}\delta_{ij}\dot{x}_i - \frac{1}{2}(1 - \delta_{ij})(b_i^+ a_j)\Phi(x_i - x_j, z). \tag{2.32}$$

The vanishing of the coefficients in front of $(x - x_i)^{-1}$ implies (2.30) for the vectors a_i and the equation (2.15), where the matrix M has the form

$$M_{ij} = (\lambda_i + \wp(z))\delta_{ij} - (1 - \delta_{ij})(b_i^+ a_j)\Phi'(x_i - x_j). \tag{2.33}$$

The Lax equation for these matrices implies (2.28) and equations (2.30) for b_i^+ (here we use the assumption that a_i span the whole l-dimensional space).

System (2.30) after the gauge transformation

$$a_i \to a_i q_i, \ b_i^+ \to b_i q_i^{-1}, \quad q_i = \exp\left(\int^t \lambda_i(t)dt\right), \tag{2.34}$$

which does not effect (2.28) and (2.29), becomes

$$\dot{a}_i = -\sum_{j\neq i} a_j(b_j^+ a_i)\wp(x_i - x_j), \quad \dot{b}_i = \sum_{j\neq i} b_j(b_i^+ a_j)\wp(x_i - x_j). \tag{2.35}$$

Equations (2.29), (2.35) are invariant under the transformations

$$a_i \to \lambda_i^{-1}a_i, \ b_i^+ \to \lambda_i b_i^+, \ a_i \to W^{-1}a_i, \ b_i^+ \to b_i^+ W, \tag{2.36}$$

where λ_i are constants and W is a constant $(l \times l)$ matrix. A factorization with respect to these transformations leaves us with a reduced phase space $\mathcal{M}$ of the dimension $\dim \mathcal{M} = 2Nl - l(l-1)$. Let us introduce a canonical system of coordinates on $\mathcal{M}$. First of all, for any set of a_i, b_i^+ we define the matrix

$$S_\alpha^\beta = \sum_{i=1}^N a_{i\alpha}b_i^\beta \tag{2.37}$$

and diagonalize it with the help of the matrix $W_{0\alpha}^j$ which leaves the co-vector $(1, \ldots, 1)$ invariant, i.e.

$$S_\alpha^\beta W_{0\beta}^j = 2\kappa_j W_{0\alpha}^j, \quad \sum_\alpha W_{0\alpha}^j = 1, \ j = 1, \ldots, l. \tag{2.38}$$

Then we define

$$A_i = W_0^{-1}a_i, \ B_i^+ = b_i^+ W_0. \tag{2.39}$$

The vectors A_i and co-vectors B_i satisfy the conditions

$$\sum_{i=1}^N A_{i\alpha}B_i^\beta = 2\kappa_\alpha \delta_\alpha^\beta, \tag{2.40}$$

which destroy the second half of the gauge transformations (2.36).

At $z = 0$ we have

$$L = G\tilde{L}G^{-1}, \ \tilde{L} = z^{-1}(F - I) + O(1), \ \ F_{ij} = \frac{1}{2}(b_i^+ a_j) = \frac{1}{2}(B_i^+ A_j), \tag{2.41}$$

where G is the same as in (2.18). The matrix F has rank l. Its null subspace is a subspace of vectors such that

$$\sum_{i=1}^{N} A_{i,\alpha} c_i = 0. \tag{2.42}$$

Relations (2.40) imply that the eigenvector of F corresponding to non-zero eigenvalue $2\kappa_j$ is identified with B_i^j.

From (2.41) it follows that

$$R(k, z) = \prod_{i=1}^{N}(k + \nu_i z^{-1} + h_i + O(z)), \ \nu_i = 1 - \kappa_i, \ i \leq l, \ \nu_i = 1, \ i > l. \tag{2.43}$$

As shown in [15] expansion (2.43) implies that the spectral curve defined by (2.17) has (in general position) genus $g = Nl - l(l+1)/2 + 1$. At the same time (2.43) implies that a number of independent integrals given by (2.17) is equal to $\frac{1}{2}\dim \mathcal{M}$.

The angle-type variables of our reduced system are the divisor of poles of the solution of (2.13) with the following normalization:

$$\sum_{\alpha=1}^{l} \sum_{i=1}^{N} A_{i\alpha} c_i \Phi(-x_i, z) = 1. \tag{2.44}$$

At the points P_j the components of C has the form

$$c_i = z(c_i^j + O(z))e^{\zeta(z)x_i}, \ i \leq l; \ c_i = (c_i^j + O(z))e^{\zeta(z)x_i}, \ j > l, \tag{2.45}$$

where

$$\sum_{i=1}^{N} A_{i,\alpha} c_i^j = \delta_\alpha^j, \ j \leq l, \ \ \sum_{i=1}^{N} A_{i,\alpha} c_i^j = 0, \ j > l. \tag{2.46}$$

The last formulae identify the normalization (2.44) with a canonical normalization used in the soliton theory. The coordinates

$$\Psi_\alpha = \sum_{i=1}^{N} A_{i\alpha} c_i \Phi(x - x_i, z)^{kx}$$

of the corresponding Baker-Akhiezer function Ψ are meromorphic outside the punctures $P_j, \ j \leq l$. At P_j they have the form:

$$\Psi_\alpha = \left(\delta_\alpha^j + O(z)\right) e^{z^{-1}\kappa_j x}$$

(see details in [15]).

Outside the punctures P_j the canonically normalized vector C has $g + l - 1$ poles $\gamma_1, \ldots, \gamma_{g-l+1}$. This number is equal to $\frac{1}{2} \dim \mathcal{M}$. Note that these poles are independent on the transformations (2.36). Therefore, we have constructed an algebraic-geometric correspondence

$$\mathcal{M} = \{x_i, \ p_i, \ A_i, \ B_i\} \longmapsto \{\Gamma, \mathcal{D} = (\gamma_s)\} \,, \tag{2.47}$$

which is an isomorphism on the open set. The reconstruction formulae for solutions of the elliptic CM system (2.28), (2.29), (2.35) can be found in [15]. It turns out that the coordinates of x_i are defined by the same equation (2.26). The only difference is a set of the corresponding algebraic curves that are defined by the characteristic equation for the Lax matrix L of the form (2.32). In pure algebraic-geometric form they can be described in a way similar to that in the previous example. Namely, they are curves such that there exists a set of l points on it with the following property: a linear space spanned by the tangent vectors to the curve at these points in the Jacobian, contains a vector U which spans in $J(\Gamma)$ an elliptic curve Γ_0.

Example 3. Spin generalization of the elliptic RS system ([16])

Let us consider now the differential-difference equation

$$\mathcal{L}\Psi = \partial_t \Psi(x, t) - \Psi(x + \eta, t) - v(x, t)\Psi(x, t) = 0, \tag{2.48}$$

where η is a complex number and $v(x, t)$ is an elliptic $(l \times l)$ matrix function. It has $N \geq l$ linear independent double-Bloch solutions with N simple poles at points $x_i(t)$ and such that $(l \times N)$ matrix formed by its residues at the poles has rank l if and only if:
(i) the potential u has the form

$$v = \sum_{i=1}^N a_i(t) b_i^+(t) V(x - x_i(t)), \ V = \zeta(x - x_i + \eta) - \zeta(x - x_i), \tag{2.49}$$

where $a_i = (a_{i,\alpha})$ are l-dimensional vectors and $b_i^+ = (b_i^\alpha)$ are l-dimensional co-vectors;
(ii) $x_i(t)$ satisfy the equations

$$\ddot{x}_i = \sum_{j \neq i} (b_i^+ a_j)(b_j^+ a_i) \left(V(x_i - x_j) - V(x_j - x_i) \right); \tag{2.50}$$

(iii) the vectors $a_i = (a_\alpha^i)$ and co-vectors $b_i = (b_i^\alpha)$ satisfy the constraints

$$b_i^+ a_i = \sum_{\alpha=1}^l b_i^\alpha(t) a_{i\alpha}(t) = \dot{x}_i \tag{2.51}$$

and the system of equations

$$\dot{a}_i = \sum_{j \neq i} a_j (b_j^+ a_i) V(x_i - x_j) - \lambda_i a_i, \ \ \dot{b}_i = -\sum_{j \neq i} b_j (b_i^+ a_j) V(x_j - x_i) + \lambda_i b_i^+, \tag{2.52}$$

where $\lambda_i = \lambda_i(t)$ are scalar functions.

The gauge transformation (2.36) allows us to eliminate λ_i in (2.52). The corresponding system was introduced in [16] and is spin generalization of the elliptic RS model, which coincides with (2.50), (2.51) for $l = 1$.

Construction of integrals of system (2.50)-(2.52) and the angle type coordinates is parallel to the previous case but requires some technical modification because $\mathcal{L}$ is a difference operator in x. First of all, we choose a different basis in a space of double-Bloch functions. The proper choice is defined by the formula

$$\Phi(x, z) = \frac{\sigma(z + x + \eta)}{\sigma(z + \eta)\sigma(x)} \left[\frac{\sigma(z - \eta)}{\sigma(z + \eta)} \right]^{x/2\eta}. \tag{2.53}$$

It satisfies the difference analog of the Lame equation (2.5):

$$\Phi(x + \eta, z) + c(x)\Phi(x - \eta, z) = E(z)\Phi(x, z),$$

where

$$c(x) = \frac{\sigma(x - \eta)\sigma(x + 2\eta)}{\sigma(x + \eta)\sigma(x)}, \quad E(z) = \frac{\sigma(2\eta)}{\sigma(\eta)} \frac{\sigma(z)}{(\sigma(z - \eta)\sigma(z + \eta))^{1/2}}.$$

The Riemann surface $\hat{\Gamma}_0$ of the function $E(z)$ is a two-fold covering of the initial elliptic curve Γ_0 with periods $2\omega_\alpha$, $\alpha = 1, 2$. Its genus is equal to 2.

As a function of x the function $\Phi(x, z)$ is double-Bloch function. In the fundamental domain of the lattice defined by $2\omega_\alpha$, the function $\Phi(x, z)$ has a unique pole at the point $x = 0$:

$$\Phi(x, z) = x^{-1} + A + O(x), \quad A = \zeta(z + \eta) + \frac{1}{2\eta} \ln \frac{\sigma(z - \eta)}{\sigma(z + \eta)}. \tag{2.54}$$

Therefore, we may represent a double-Bloch solution Ψ of (2.48) in the form:

$$\Psi = \sum_{i=1}^{N} s_i(t, k, z)\Phi(x - x_i(t), z)k^{x/\eta}, \tag{2.55}$$

substitute this ansatz into the equation and proceed as before. We get (2.50)-(2.52). The corresponding Lax operators have the form

$$L_{ij}(t, z) = (b_i^+ a_j)\Phi(x_i - x_j - \eta, z), \tag{2.56}$$

$$M_{ij}(t, z) = (\lambda_i - (\zeta(\eta) - A)\dot{x}_i)\delta_{ij} + (1 - \delta_{ij})(b_i^+ a_j)\Phi(x_i - x_j, z). \tag{2.57}$$

Explicit formulae in terms of the Riemann theta functions are the same as for spin generalization of the CM system. The only difference is due to the different family of the spectral curves. In the case $l = 1$ they may be defined as a class of curves having the following property: there exists a pair of points on the curve such that the complex linear subspace spanned by the corresponding vector U is an elliptic curve Γ_0. If we move Γ_0 in the direction that is defined by the vector V^+ (V^-) tangent to $\Gamma \in J(\Gamma)$ at the point P^+ (P^-), then the intersections x_i of Γ_0 with the theta-divisor move according to the RS dynamics. The spectral curves for $l > 1$ are characterized by the existence of two sets of points $P_i^\pm$, $i = 1, \ldots, l$ such that in the linear subspace spanned by the vectors corresponding to each pair there exists a vector U with the same property as above.

Example 4. The nested Bethe ansatz equations ([1])

Let us consider two-dimensional difference equation

$$\Psi(x, m+1) = \Psi(x+\eta) + v(x, m)\Psi(x, m) \tag{2.58}$$

with an elliptic in x coefficient of the form

$$v(x, m) = \prod_{i=1}^{N} \frac{\sigma(x - x_i^m)\sigma(x - x_i^{m+1} + \eta)}{\sigma(x - x_i^{m+1})\sigma(x - x_i^m + \eta)} \ . \tag{2.59}$$

It has N linear independent double-Bloch solutions with equivalent Bloch multipliers if and only if the functions x_i^m of the discrete variable m satisfy the nested Bethe ansatz equations. The Lax representation for this system has the form

$$L(m+1)M(m) = M(m)L(m+1), \tag{2.60}$$

where

$$L_{ij}(m) = \lambda_i(m)\Phi(x_i^m - x_j^m - \eta, z), \ M_{ij}(m) = \mu_i(m)\Phi(x_i^{m+1} - x_j^m, z) \tag{2.61}$$

and

$$\lambda_i(m) = \frac{\prod_{s=1}^{M} \sigma(x_i^m - x_s^m - \eta)\sigma(x_i^m - x_s^{m+1})}{\prod_{s=1,\neq i}^{M} \sigma(x_i^m - x_s^m) \prod_{s=1}^{N} \sigma(x_i^m - x_s^{m+1} - \eta)} \ , \tag{2.62}$$

$$\mu_i(m) = \frac{\prod_{s=1}^{M} \sigma(x_i^{m+1} - x_s^{m+1} + \eta)\sigma(x_i^{m+1} - x_s^m)}{\prod_{s=1,\neq i}^{M} \sigma(x_i^{m+1} - x_s^{m+1}) \prod_{s=1}^{N} \sigma(x_i^{m+1} - x_s^m + \eta)} \ . \tag{2.63}$$

A class of the spectral curves is the same as for the RS system. The solution x_i^m of (1.1) corresponding to the spectral curve and the divisor on it is defined by the equation

$$\theta(Ux + Vm + Z) = 0. \tag{2.64}$$

Here V is the vector from the puncture P_+ to the third point Q on Γ. When this point tends to P_+ the vector V becomes the tangent vector to the curve and we come to the RS system as a continuous limit of (1.1).

3 Hamiltonian theory of the CM type systems.

As we have seen various generating problems lead to various integrable finite-dimensional systems which can be explicitly solved via the spectral transform of a phase space $\mathcal{M}$ to algebraic-geometric data. On this way we do not use Hamiltonian description of the system. Moreover, a'priori it's not clear, why all the systems which can be constructed with the help of the generating scheme are Hamiltonian. In this section we clarify this problem using the approach to the Hamiltonian theory of soliton equations proposed in [25] and developed in [26].

First of all, let us outline a framework that was presented in the previous section. The direct spectral transform identifies a space of solutions with a bundle over a space of the

corresponding spectral curves. The fiber over a curve Γ is a symmetric power $S^{g+l-1}\Gamma$ (i.e. an unordered set of $(g+l-1)$ points $\gamma_s \in \Gamma$). Dimension of the space of the spectral curves equals to $g+l-1 = \frac{1}{2}\mathcal{M}$. The spectral curves are realized as N-sheet covering of the elliptic curve Γ_0.

Let us consider the case of spin generalization of the CM system. Entries of $L(z)$ are explicitly defined as functions on $\mathcal{M}$. Therefore, $L(z)$ can be seen as an operator-valued function and its external differential δL as an operator-valued one-form on $\mathcal{M}$. Canonically normalized eigenfunction $C(z, k)$ of $L(z)$ is the vector-valued function on $\mathcal{M}$. Hence, its differential is a vector-valued one-form. Let us define a two-form on $\mathcal{M}$ by the formula

$$\omega = \sum_{i=1}^{N} \mathrm{res}_{P_i} < C^*(z, k)(\delta L(z) + \delta k) \wedge \delta C(z) > dz, \tag{3.1}$$

where $C^*(z, k)$ is the eigen-covector (row vector) of $L(z)$, i.e. the solution of the equation $C^* L = k C^*$, normalized by the condition $< C^*(z, k) C(z, k) >= 1$. The form ω can be rewritten as

$$\omega = \mathrm{res}_{z=0} \, \mathrm{Tr} \left(\hat{C}^{-1}(z)\delta L(z) \wedge \delta \hat{C}(z) - \hat{C}^{-1}(z)\delta \hat{C}(z) \wedge \delta \hat{k} \right) dz, \tag{3.2}$$

where $\hat{C}(z)$ is a matrix with columns $C(z, k_j)$; $k_j = k_j(z)$ are different eigenvalues of $L(z)$ and $\hat{k}(z)$ is the diagonal matrix $k_j(z)\delta_{ij}$.

Remark 1. The right hand side of formula (3.1) is not gauge invariant. In [25] a gauge $(C_1 = gC, \ L_1 = gLg^{-1})$ was chosen in such a way that ω is equal to the sum of residues of the differential $< C_1^* \delta L_1 \wedge \delta C_1 > dz$.

Note that C^* are rows of the matrix $\hat{C}^{-1}$. That implies that C^* as a function on the spectral curve is: meromorphic outside the punctures; has poles at the branching points of the spectral curve, and zeros at the poles γ_s of C. These analytical properties are used in the proof of the following theorem.

Theorem 3.1 *The two-form ω equals*

$$\omega = 2 \sum_{s=1}^{g+l-1} \delta z(\gamma_s) \wedge \delta k(\gamma_s). \tag{3.3}$$

The meaning of the right hand side of this formula is as follows. The spectral curve by definition arises with the meromorphic function $k(Q)$ and multi-valued holomorphic function $z(Q)$. Their evaluations $k(\gamma_s)$, $z(\gamma_s)$ at the points γ_s define functions on the space $\mathcal{M}$, and the wedge product of their external differentials is a two-form on $\mathcal{M}$. Formula (3.3) identifies $k_s = k(\gamma_s)$, $z_s = z(\gamma_s)$ as Darboux coordinates for ω.

Remark 2. The right hand side of (3.3) can be identified with a particular case of universal algebraic-geometric symplectic forms proposed in [25]. They are defined on the generalized Jacobian bundles over a proper subspaces of the moduli spaces of Riemann surfaces with punctures. In the case of families of hyperelliptic curves that form was pioneered by Novikov and Veselov [27].

Remark 3. Equations (2.28), (2.29), (2.35) are linearized by generalized Abel transform

$$A : S^{g+l-1}\Gamma \longmapsto J(\Gamma) \times C^{l-1}. \tag{3.4}$$

This transform is defined with the help of a basis of the normalized holomorphic differentials $d\omega_i$, $i \le g$, and with the help of normalized meromorphic differentials $d\Omega_j$, $j \le (l-1)$ of the third kind with residues 1 and -1 at the points P_j and P_l, respectively. Let Q be a point of Γ. Then we define $(g + l - 1)$-dimensional vector $A_k(Q)$ with the coordinates

$$A_i(Q) = \int^Q d\omega_i, \quad A_{g+j}(Q) = \int^Q d\Omega_j.$$

The isomorphism (3.4) is given by

$$\phi_k = \sum_{s=1}^{g+l-1} A_k(\gamma_s).$$

The action variables I_k canonically conjugated to ϕ, i.e. such that

$$\omega = \sum_{k=1}^{g+l-1} \delta\phi_k \wedge \delta I_k,$$

are equal to

$$I_i = \oint_{a_i^0} k\,dz, \quad I_{g+j} = \operatorname{res}_{P_j} k\,dz = -\nu_j,$$

where a_i^0 are a-cycles of the basis of cycles on Γ with the canonical matrix of intersections.

Let us outline the proof of Theorem 3.1. The differential $\Omega = \langle C^*\delta L \wedge \delta C \rangle\, dz$ is a meromorphic differential on the spectral curve (the essential singularities of the factors cancel each other at the punctures). Therefore, the sum of its residues at the punctures is equal to the sum of other residues with negative sign. There are poles of two types. First of all, Ω has poles at the poles γ_s of C. Note that δC has pole of the second order at γ_s. Taking into account that C^* has zero at γ_s we obtain

$$\operatorname{res}_{\gamma_s}\Omega = \langle C^*\delta LC \rangle \wedge \delta z(\gamma_s) = \delta k(\gamma_s) \wedge \delta z(\gamma_s). \tag{3.5}$$

The last equality follows from the standard formula for a variation of the eigenvalue of an operator. The second term in (3.1) has the same residue at γ_s.

The second set of poles of Ω is a set of the branching points q_i of the cover. The pole of C^* at q_i cancels with the zero of the differential dz, $dz(q_i) = 0$, considered as differential on Γ. The function C is holomorphic at q_i. If we take an expansion of C in the local coordinate $(z - z(q_i))^{1/2}$ (in general position when the branching point is simple) and consider its variation we get that

$$\delta C = -\frac{dC}{dz}\delta z(q_i) + O(1). \tag{3.6}$$

Therefore, δC has simple pole at q_i. In the similar way we obtain

$$\delta k = -\frac{dk}{dz}\delta z(q_i). \tag{3.7}$$

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Equalities (3.6) and (3.7) imply that

$$\operatorname{res}_{q_i}\Omega = \operatorname{res}_{q_i}\left[< C^*\delta L dC > \wedge\frac{\delta k dz}{dk}\right].\tag{3.8}$$

Due to skew-symmetry of the wedge product we we may replace δL in (3.8) by $(\delta L - \delta k)$. Then using identities $C^*(\delta L - \delta k) = \delta C^*(k - L)$ and $(k - L)dC = (dL - dk)C$ we obtain

$$\operatorname{res}_{q_i}\Omega = -\operatorname{res}_{q_i} < \delta C^*C > \wedge\delta k dz = \operatorname{res}_{q_i} < C^*\delta C > \wedge\delta k dz.\tag{3.9}$$

(Note, that the term with dL does not contributes to the residue, because $dL(q_i) = 0$.) The right hand side of (3.9) cancels with a residue of the second term in the sum (3.1). The theorem is proved.

Now let us express ω in terms of the coordinates (2.39) on $\mathcal{M}$. Using the gauge transformation

$$L = G\,\widetilde{L}\,G^{-1},\quad \hat{C} = G\,\widetilde{C}\tag{3.10}$$

where G is given by (2.18), we obtain

$$\omega = \sum_j \operatorname{res}_{z=0} \operatorname{Tr}\left[\delta\widetilde{L}\wedge\delta h + \widetilde{C}^{-1}\left(\delta\widetilde{L}\wedge\delta\widetilde{C} + [\delta h, \widetilde{L}]\wedge\delta\widetilde{C} - \delta h\,\widetilde{C}\wedge\delta\hat{k}\right)\right],\tag{3.11}$$

where $\delta h = \delta G G^{-1}$, $\delta h = diag\left(\delta x_i \zeta(z)\right)$. From the relation $\widetilde{L}\widetilde{C} = \widetilde{C}\hat{k}$ it follows that:

$$\operatorname{Tr}\left(\widetilde{C}^{-1}[\delta h, \widetilde{L}]\wedge\delta\widetilde{C}\right) = \operatorname{Tr}\left(\widetilde{C}^{-1}\delta h\wedge(\widetilde{L}\delta\widetilde{C} - \delta\widetilde{C}\hat{k})\right) = -\operatorname{Tr}\left(\widetilde{C}^{-1}\delta h\wedge(\delta\widetilde{L}\widetilde{C} - \widetilde{C}\delta\hat{k})\right).$$

Therefore,

$$\omega = \operatorname{res}_{z=0}\operatorname{Tr}\left(2\delta\widetilde{L}\wedge\delta h + \widetilde{C}^{-1}\delta L\wedge\delta\widetilde{C}\right).\tag{3.12}$$

The first term equals $\sum_i \delta x_i\wedge\delta p_i$. The last term equals

$$\frac{1}{2}\operatorname{Tr} C_0^{-1}\delta(B_i^+ A_j)\wedge\delta C_0 = \frac{1}{2}\operatorname{Tr}\left(C_0^{-1}(\delta B_j A_j)\wedge\delta C_0 + \delta C_0^{-1}\wedge(B_j^+\delta A_j)C_0\right),\tag{3.13}$$

where C_0 is the matrix c_i^j of leading coefficients of the expansions (2.45). From (2.46) it follows that

$$\sum_{j=1}^N\left(A_{j\alpha}\delta c_j^k + \delta A_{j,\alpha}c_j^k\right) = 0,\quad \sum_{j=1}^N\left(c_{ki}^*\delta B_j^\alpha + \delta c_{ki}^* B_j^\alpha\right) = 0,$$

where c_{jk}^* are matrix elements of C_0^{-1}. Substitution of the last formulae into (3.13) completes the proof of the following theorem.

Theorem 3.2 *The symplectic form ω given by (3.1) equals*

$$\omega = -\sum_{i=1}^N\left(\delta p_i\wedge\delta x_i + \sum_{\alpha=1}^l\delta B_i^\alpha\wedge\delta A_{i\alpha}\right).\tag{3.14}$$

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When the symplectic structure ω is identified with a standard, it can be directly checked that equations (2.28), (2.35) are Hamiltonian with respect to ω and with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}\sum_{i\neq j}(b_i^+ a_j)(b_j^+ a_i)\wp(x_i - x_j). \tag{3.15}$$

Nevertheless, we would like to show that the existence of a Hamiltonian for system (2.28), (2.35) can be proved in the framework of our approach without use of the explicit formula for the symplectic structure.

By definition a vector field ∂_t on a symplectic manifold is Hamiltonian, if the contraction $i_{\partial_t}\omega(X) = \omega(X, \partial_t)$ of the symplectic form is an exact one-form $dH(X)$. The function H is the Hamiltonian corresponding to the vector field ∂_t. The equations $\partial_t L = [M, L]$, $\partial_t k = 0$, and the equation

$$\partial_t C(t, P) = MC(t, P) + \mu(P, t)C(t, P), \tag{3.16}$$

where $\mu(P, t)$ is a scalar function, imply

$$i_{\partial_t}\omega = \sum_{i=1}^{N} \mathrm{res}_{P_i}\left(< C^*(\delta L + \delta k(z))(M + \mu)C(z, k) > - < C^*(z, k)[M, L]\delta C >\right)dz, \tag{3.17}$$

Note, that if a matrix $\Lambda(z)$ is holomorphic outside P_j, then the differential $< C^*\Lambda C > dz$ is holomorphic outside P_j, as well. Therefore, the sum of its residues at P_j is equal to zero. Using that and the relations

$$< C^*[M, L]\delta C > = < C^*M(L - k)\delta C > = < C^*(z, k)M(\delta k - \delta L)C >,$$

we get

$$i_{\partial_t}\omega = \sum_{i=1}^{N} \mathrm{res}_{P_i}\left(< C^*(\delta L + \delta k)C > \mu(P, t)\right)dz = 2\sum_{i=1}^{N} \mathrm{res}_{P_i}\delta k\ \mu(P, t)dz. \tag{3.18}$$

The singular part of the function $\mu(P, t)$ is equal to the singular part of the eigenvalues of the second Lax operator

$$\mu_j(z) = -z^{-2} + z^{-1}2(k_j(z)) + O(1), \tag{3.19}$$

where $k_j(z)$ is the expansion of $k(z)$ at P_j (see (4.8) in [15]). Hence,

$$i_{\partial_t}\omega = 2\ \mathrm{res}_{z=0}\ \mathrm{Tr}\left(z^{-2}\delta\hat{k} + z^{-1}\delta\hat{k}^2\right)dz = 2\ \delta\mathrm{Tr}\ \hat{k}^2 = 2\delta\ \mathrm{Tr}\ L^2 = \delta H. \tag{3.20}$$

Now, let us define a symplectic structure for spin generalization of the elliptic RS system by the formula

$$\omega = \sum_{i=1}^{N} \mathrm{res}_{P_i}\left(< C^*(z, k)(\delta L(z)L^{-1} + \delta\ln k) \wedge \delta C(z) >\right)dz, \tag{3.21}$$

where L and C are the Lax operator (2.56) and its eigenvector.

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Theorem 3.3 *The two-form (3.21) equals*

$$\omega = 2 \sum_{s=1}^{g+l-1} \delta z(\gamma_s) \wedge \delta \ln k(\gamma_s). \tag{3.22}$$

Equation (2.50)-(2.52) are Hamiltonian with respect to this symplectic structure with the Hamiltonian

$$H = \sum_{i=1}^{N} (b_i^+ a_i). \tag{3.23}$$

Note, that (3.22) implies that ω is closed and does define a symplectic structure on $\mathcal{M}$. The proof of Theorem 3.3 goes almost identically to the previous case.

At this moment, we do not know for $l > 1$ an explicit expression for ω in the original coordinates on $\mathcal{M}$. Formula (3.21) contains the inverse matrix L^{-1}. For $l = 1$ it can be written explicitly. Namely, let $\hat{L}$ be the matrix

$$\hat{L}_{ij} = f_i \, \frac{\sigma(z + x_i - x_j)}{\sigma(x_i - x_j - \eta)},$$

which is (up to gauge transformation (2.18) and a scalar factor) equal to (2.56) for $l = 1$. Then the entries of the inverse matrix equal:

$$(\hat{L}^{-1})_{jm} = f_m^{-1} \frac{\sigma(z + x_j - x_m - (N-2)\eta)}{\sigma(z + \eta)\sigma(z - (N-1)\eta)} \, \frac{\prod_{k \neq m} \sigma(x_j - x_k + \eta) \prod_k \sigma(x_j - x_k - \eta)}{\prod_{k \neq j} \sigma(x_j - x_k) \prod_{k \neq m} \sigma(x_m - x_k)} \, .$$

This formula has been used for the proof of the following theorem.

Theorem 3.4 *Let L be the matrix $L_{ij} = f_i \Phi(x_i - x_j - \eta, z)$, where Φ is given by (2.53). Then the form ω defined by (3.21) equals*

$$\omega = \sum_i \delta \ln f_i \wedge \delta x_i + \sum_{i,j} V(x_i - x_j) \delta x_i \wedge \delta x_j, \tag{3.24}$$

where $V(x)$ is defined in (2.49).

Our approach is evenly applicable to the nested Bethe ansatz equations. It gives Hamiltonian version of the proof that discrete evolution x_i^n is a canonical transform of the RS symplectic structure 3.24). This result was obtained for the first time in [12] with the help of Lagrangian interpretation of (1.1).

Acknowledgements

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