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A generating formula for solutions of the associativity equations

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1. Introduction. The associativity equations (or WDVV equations) were introduced at the beginning of the 1990s to describe the free energy in topological quantum models of field theory (see [1], [2]). In the last few years these equations have attracted ever-increasing attention thanks to their connections with Gromov-Witten invariants, quantum cohomoloy, and Whitham theory.

As was noted in [3], the classification problem for topological quantum models of field theory, or the problem of the construction of general solutions of the associativity equation, is equivalent to the classification problem for Egorov metrics of special type. Egorov metrics are *flat* diagonal metrics $ds^2 = \sum_{i=1} h_i^2(u)(du^i)^2$ such that $\partial_i h_j^2(u) = \partial_j h_i^2(u)$, where $\partial I = \partial/\partial u^i$. It turns out that for any Egorov metric satisfying the additional condition $\sum_{j=1} \partial_j h_i = 0$, the functions

$$c_{kl}^m(x) = \sum_{i=1}^n \frac{\partial x^m}{\partial u^i} \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l}, \tag{1}$$

where the $x^k(u)$ are the flat coordinates of the metric, satisfy the equations

$$c_{ij}^k(x) c_{km}^l(x) = c_{jm}^k(x) c_{ik}^l(x), \tag{2}$$

which are equivalent to the associativity condition $\phi_k \phi_l = c_{kl}^m \phi_m$ for the algebra. Moreover, it turns out that there is a function $F(x)$ whose third derivatives are given by

$$\frac{\partial^3 F(x)}{\partial x^k \partial x^l \partial x^m} = c_{klm}(x) = \eta_{mi} c_{kl}^i, \quad \text{where } \eta_{pq} = \sum_{i=1}^n h_i^2(u) \frac{\partial u^i}{\partial x^p} \frac{\partial u^i}{\partial x^q}. \tag{3}$$

In addition, there are constants r^m such that the relation $\eta_{kl} = r^m c_{lm}(x)$ holds for the constant matrix specifying the metric in flat coordinates.

Equations (2) and the condition that a function F exists satisfying equations (3) are equivalent to the consistency conditions for the linear equations (see [3])

$$\partial_k \Phi_l - \lambda c_{kl}^m \Phi_m = 0, \tag{4}$$

where λ is the spectral parameter. We note that this assertion has in a certain sense the character of an existence theorem, since an explicit expression for F in terms of horizontal sections of the flat connection $\nabla_k = \partial/\partial x^k - \lambda c_{kl}^m$ was unknown in the general case (such expressions have been found in a number of special cases in [3]–[5]). The main aim of this note is to obtain an explicit generating formula for F. It was motivated by the results of [6], where a corresponding formula was obtained for algebraic-geometric solutions of the associativity equations. We note that although

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the general formula in [6] (Theorem 5.1) is correct, one of the terms has been omitted in a special case (Theorem 5.2) of particular interest. We take this oppportunity of correcting the mistake.

2. Let us consider a solution $\beta_{ij}(u) = \beta_{ji}(u)$ of the Darboux-Egorov equations: $\partial_k \beta_{ij} = \beta_{ik}\beta_{kj}$; $\sum_{m=1}^n \partial_m \beta_{ij} = 0$, $i \neq j \neq k$. Following [3] we fix a unique Egorov metric by determining the Lamé coefficients $h_i(u)$ by means of the equations $\partial_j h_i(u) = \beta_{ij}(u)h_j(u)$; $\partial_i h_i(u) = -\sum_{j\neq i} \beta_{ij}(u)h_j(u)$, with the initial conditions $h_i(0) = 1$.

The flat coordinates of this metric can be found from the system of linear equations $\partial_i\partial_j x^k = \Gamma^i_{ij}\partial_i x^k + \Gamma^j_{ji}\partial_j x^k$, $i \neq j$; $\partial_i\partial_i x^k = \sum_{j=1}^n \Gamma^j_{ii}\partial_j x^k$, where the Γ^k_{ij} are the Chrisoffel symbols: $\Gamma^i_{ij} = \partial_j h_i/h_i$, $\Gamma^j_{ii} = (2\delta_{ij} - 1)(h_i\partial_j h_i)/(h_j^2)$. We fix a unique solution of this system by the initial conditions: $x^k(0) = 0$, $\sum_{k,l} \eta_{kl}\partial_i x^k(0) = \delta_{ij}$. Here η_{kl} is a given symmetric non-singular matrix.

The Darboux-Egorov system is equivalent to the conditions for the consistency of the system of linear equations

$$\partial_j \Psi_i(u,\lambda) = \beta_{ij}(u)\Psi_j(u,\lambda); \quad \partial_i\Psi_i(u,\lambda) = \lambda\Psi_i(u,\lambda) - \sum_{k\neq i}\beta_{ik}(u)\Psi_k(u,\lambda). \tag{5}$$

We consider the unique solution $\Psi_i^k(0,\lambda) = \lambda\partial_i x^k(0)$. It follows from the system of equations (5) that the expansion of Ψ_i has the form $\Psi_i(u,\lambda) = h_i^{-1}(u)\sum_{s=0}^\infty \xi_s^k(u)\lambda^s$, where $\xi_0^k = r^k$ are constants (which we shall determine later), $\xi_1^k(u) = x^k(u)$ are the flat coordinates, and the ξ_s^k for $s \geq 2$ are found by recurrence from the equations $\partial_i\partial_j\xi_s^k = \Gamma^i_{ij}\partial_i\xi_s^k + \Gamma^j_{ji}\partial_j\xi_s^k$, $i \neq j$; $\partial_i\partial_i\xi_s^k = \sum_{j=1}^n \Gamma^j_{ii}\partial_j\xi_s^k + \partial_i\xi_{s-1}^k$ and the initial conditions $\xi_s^k(0) = 0$, $\partial_i\xi_s^k(0) = 0$. Hence we obtain the equations

$$\frac{\partial^2\xi_s^m}{\partial x^k\partial x^l} = \sum_{p=1}^n c^p_{kl}\frac{\partial\xi^m_{s-1}}{\partial x^p}, \tag{6}$$

where the c^p_{kl} are defined in (1). We denote $\xi_2^k(u)$ and $\xi_3^k(u)$ by $y^k(u)$ and $z^k(u)$, respectively.

It follows from (5) that $\lambda\Psi_i = \sum_{j=1}^n \partial_j\Psi_i$. Hence we have $\sum_{i=1}^n \partial_i\xi_s^k(u) = \xi_{s-1}^k$ for $s \geq 1$. The last equation for $s = 1$ determines the constants r^k.

3. We define a generating vector function ψ by the equation $\lambda\psi(u,\lambda) = \sum_{i=1}^n h_i(u)\Psi_i(u,\lambda)$. It can be immediately verified that $\partial_i\psi(u,\lambda) = h_i(u)\Psi_i(u,\lambda)$. The first few coefficients of the expansion of the kth component of this function in powers of λ have the form: $\psi^k(u,\lambda) = r^k + x^k(u)\lambda + y^k(u)\lambda^2 + z^k(u)\lambda^3 + \sum_{s=4}^\infty \xi_s^k(u)\lambda^s$. We note that it follows from (6) that ψ is also a generating function for the flat sections of the connection ∇_k. More precisely, it follows immediately from (6) that the functions $\Phi_k(x) = \partial\psi(x)/\partial x^k$ satisfy the equations (4). Moreover, $\lambda\psi(x) = \sum_{k=1}^n r^k\Phi_k(x)$.

Lemma 1. *The functions $x^k(u)$, $y^k(u)$, and $z^k(u)$ satisfying the relations*:

$$\sum_{q=1}^n \eta_{kq}y^q = \sum_{p,q=1}^n \eta_{pq}\left(x^q\frac{\partial y^p}{\partial x^k} - r^q\frac{\partial z^p}{\partial x^k}\right).$$

Theorem. *The function $F(x) = F(u(x))$, $F(u) = \frac{1}{2}\sum_{p,q=1}^n \eta_{pq}(x^q(u)y^p(u) - r^q z^p(u))$, satisfies equation* (3).

We note that the equation $\partial F/\partial x^k = \sum_{q=1}^n \eta_{kq}y^q$ follows from the assertion of the lemma. After this, the assertion of the theorem follows immediately from (1) and (6) for $s = 2$.

The equations proved can be represented in the form of an equation of renorm-group type: $F(x) - \sum_{k=1}^n x^k\frac{\partial F}{\partial x^k} = -\sum_{p,q=1}^n \eta_{pq}r^q z^p$. In a paper to follow we plan to obtain a more general equation including in F the dependency on infinitely many variables corresponding to the gravitational descendants of the primary fields.

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