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The effective prepotential of $N = 2$ supersymmetric $SU(N_c)$ gauge theories ★

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Abstract

We determine the effective prepotential for $N = 2$ supersymmetric $SU(N_c)$ gauge theories with an arbitrary number of flavors $N_f < 2N_c$, from the exact solution constructed out of spectral curves. The prepotential is the same for the several models of spectral curves proposed in the literature. It has to all orders the logarithmic singularities of the one-loop perturbative corrections, thus confirming the non-renormalization theorems from supersymmetry. In particular, the renormalized order parameters and their duals have all the correct monodromy transformations prescribed at weak coupling. We evaluate explicitly the contributions of 1- and 2-instanton processes. © 1997 Elsevier Science B.V.

1. Introduction

A recurrent feature in many phenomenologically interesting supersymmetric field as well as string theories is the presence of a flat potential, resulting in a moduli space of inequivalent vacua. For $N = 2$ supersymmetric $SU(2)$ gauge theories, Seiberg and Witten [1] have shown in their 1994 seminal work how to extract the physics from a fibration, over the space of vacua, of *spectral curves*, i.e. Riemann surfaces

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equipped with a meromorphic 1-form $d\lambda$, whose periods determine the spectrum of BPS states. Rapid progress has led since then to an almost complete classification [2] of the spectral curves for $N = 2$ supersymmetric theories with arbitrary gauge groups and number of quark hypermultiplets N_f, and in particular for $SU(N_c)$ theories with $N_f < 2N_c$ [3,4]. Although these curves are strongly suggested, e.g., by consistency requirements in various limits (such as the classical limit or the infinite mass limit as a quark decouples), or by analogies with singularity theory and soliton theory, they are often still conjectural. In particular, in most cases, even a direct verification that the spectral curves do reproduce the known perturbative correction [5] to the prepotential is lacking. Nor has the predictive power of the spectral curves been put to effective use, for example in deriving the contributions to the prepotential of d-instanton processes.

Mathematically, the evaluation of the prepotential presents the unusual feature that the B-periods of the meromorphic form $d\lambda$ have to be expressed in terms of its A-periods, rather than in terms of the moduli parameters which occur explicitly as coefficients in the defining equation of the Riemann surface. It is well known that, as functions of moduli parameters, the vector of both A- and B-periods satisfies Picard–Fuchs differential equations. Picard–Fuchs equations have been applied successfully in the case of $SU(3)$ by Klemm et al. and Ito and Yang [6], but their complexity increases rapidly with the number of colors, and makes it difficult to treat the case of general $SU(N_c)$, even without hypermultiplets.

In this paper, we develop methods for determining the prepotential from the spectral curves for an arbitrary number of colors and flavors in the $N_f < 2N_c$ asymptotically free case, in the regime where the dynamically generated scale Λ of the theory is small. We show that, in this regime, the full expansion of the A-periods can actually been calculated by the method of residues. For the B-periods, we provide a simple algorithm to arbitrary order of multi-instanton processes. This method is based on an analytic continuation in an auxiliary parameter ξ, which allows us to expand the form $d\lambda$ into a series of rational functions that can thus be integrated in closed form. The method is powerful enough to let us identify completely all the logarithmic singularities of the prepotential and, in particular, to confirm the non-renormalization theorems resulting from supersymmetry.

We have worked out explicitly the perturbative corrections, as well as the contributions of up to 2-instanton processes, to the prepotential for any gauge group $SU(N_c)$ and arbitrary $N_f < 2N_c$. The perturbative part does coincide with the one calculated from one-loop effects in field theory [5]. The instanton contributions are in complete agreement with all known expressions, and in particular, with (a) the 1-instanton contribution for pure (with $N_f = 0$) $SU(3)$ obtained in Ref. [6] via Picard–Fuchs equations; (b) more generally, with the 1-instanton contribution for pure $SU(N_c)$ obtained in Ref. [7] via holomorphicity arguments; and (c), with the 2-instanton contribution for $SU(2)$ with $N_f < 4$ hypermultiplets obtained recently in Ref. [8] from first principles.

We note that, although the spectral curves proposed in Refs. [3,4] differ when $N_f \geqslant N_c + 2$, the corresponding prepotentials are the same.

We observe that the final form of the prepotential is very simple, and results from a large number of remarkable cancellations. This suggests the presence of a deeper geometric structure on the moduli space of vacua. Some of it has recently come to light, especially in connection with Calabi-Yau compactifications in string theory [9], topological field theories and WDVV equations [10], symplectic geometry [11,4] and integrable models [12,13] (see especially Ref. [13], where a closed form for the prepotential is written in terms of parameters inherent to Whitham-averaged hierarchies). It seems that further investigation along these lines is warranted.

2. Integrability conditions and model independence

We consider $N=2$ supersymmetric $SU(N_c)$ gauge theories with N_f quark flavors, $N_f < 2N_c$. The field content is an $N=2$ chiral multiplet and N_f hypermultiplets of bare masses m_j. The $N=2$ chiral multiplet contains a complex scalar field ϕ in the adjoint representation. The flat directions in the potential correspond to $[\phi, \phi^\dagger] = 0$, so that the classical moduli space of vacua is $N_c - 1$ dimensional, and can be parametrized by the eigenvalues

$$\bar{a}_k, \quad 1 \leqslant k \leqslant N_c, \quad \sum_{k=1}^{N_c} \bar{a}_k = 0$$

of ϕ. For generic $\bar{a}_k$, the $SU(N_c)$ gauge symmetry is broken down to $U(1)^{N_c-1}$. In the $N=1$ formalism, the Wilson effective Lagrangian of the quantum theory to leading order in the low momentum expansion is of the form

$$\mathcal{L} = \mathrm{Im}\frac{1}{4\pi}\left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^i}\overline{A^i} + \frac{1}{2}\int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W^i W^j\right],$$

where the A^i's are $N=1$ chiral superfields whose scalar components correspond to the $\bar{a}_i$'s, and $\mathcal{F}$ is the holomorphic prepotential. For $SU(N_c)$ gauge theories with $N_f < 2N_c$ flavors, general arguments, based on holomorphicity of $\mathcal{F}$, perturbative non-renormalization beyond 1-loop order, the nature of instanton corrections and the restrictions of $U(1)_R$ invariance, suggest that $\mathcal{F}$ should have the following form:

$$\begin{aligned}\mathcal{F}(A) = &\frac{1}{2\pi i}(2N_c - N_f)\sum_{i=1}^{N_c} A_i^2 \\ &-\frac{1}{8\pi i}\left(\sum_{k,l=1}^{N_c}(A_k - A_l)^2 \log\frac{(A_k - A_l)^2}{\Lambda^2}\right. \\ &\left.-\sum_{k=1}^{N_c}\sum_{j=1}^{N_f}(A_k + m_j)^2 \log\frac{(A_k + m_j)^2}{\Lambda^2}\right) + \sum_{d=1}^{\infty}\mathcal{F}_d(A_k)\Lambda^{(2N_c - N_f)d}. \end{aligned} \tag{2.1}$$

The terms on the right-hand side are respectively the classical prepotential, the contribution of perturbative one-loop effects (higher loops do not contribute in view of non-renormalization theorems), and the contributions of d-instantons processes.

2.1. The spectral curves of the effective theory

The Seiberg-Witten ansatz for determining the full prepotential $\mathcal{F}$ (as well as the spectrum of BPS states) is based on the choice of a fibration of spectral curves over the space of vacua, and of a meromorphic 1-form $\mathrm{d}\lambda$ over each of these curves. The renormalized order parameters a_k of the theory, their duals $a_{D,k}$, and the prepotential $\mathcal{F}$ are then given by

$$2\pi i a_k = \oint_{A_k} \mathrm{d}\lambda, \qquad 2\pi i a_{D,k} = \oint_{B_k} \mathrm{d}\lambda, \qquad a_{D,k} = \frac{\partial \mathcal{F}}{\partial a_k}, \tag{2.2}$$

with A_k, B_k a canonical basis of homology cycles on the spectral curves.

For $\mathrm{SU}(N_c)$ theories with N_f hypermultiplets, the following candidate spectral curves [3,4] and meromorphic forms $\mathrm{d}\lambda$ have been proposed

$$\begin{aligned} y^2 &= A^2(x) - B(x), \\ \mathrm{d}\lambda &= \frac{x}{y}\left(A' - \frac{1}{2}(A-y)\frac{B'}{B}\right)\mathrm{d}x, \end{aligned} \tag{2.3}$$

where $A(x)$ and $B(x)$ are polynomials in x of respective degrees N_c and N_f, whose coefficients vary with the physical parameters of the theory. More specifically, let Λ be the dynamically generated scale of the theory, $\bar{s}_i$, $0 \leqslant i \leqslant N_c$, and $t_p(m)$, $1 \leqslant p \leqslant N_f$, be the ith and pth symmetric polynomials in $\bar{a}_k$ and m_j

$$\bar{s}_i = (-1)^i \sum_{k_1<\ldots<k_i} \bar{a}_{k_1}\ldots\bar{a}_{k_i}, \qquad t_p(m) = \sum_{j_1<\ldots<j_p} m_{j_1}\ldots m_{j_p}, \tag{2.4}$$

and let $\bar{\sigma}_i$, $0 \leqslant i \leqslant N_c$, and $\tilde{t}_q$, be defined by $\bar{\sigma}_0 = 1$, $\bar{\sigma}_1 = 0$, and

$$\begin{aligned} \delta_{p,0} &= \sum_{i+j=p} \bar{s}_i\bar{\sigma}_j, \quad 0 \leqslant p \leqslant N_c, \\ \tilde{t}_q &= \sum_{p+i=q} t_p(m)\bar{\sigma}_i. \end{aligned} \tag{2.5}$$

Then the polynomials $A(x)$ and $B(x)$ are given by

$$\begin{aligned} A(x) &= C(x) + \Lambda^{2N_c-N_f}T(x), \\ B(x) &= \Lambda^{2N_c-N_f}\prod_{j=1}^{N_f}(x+m_j), \end{aligned} \tag{2.6}$$

where

$$C(x)=\prod_{k=1}^{N_c}(x-\bar{a}_k)=x^{N_c}+\sum_{i=2}^{N_c}\bar{s}_i x^{N_c-i} \tag{2.7}$$

and $T(x)$ is a polynomial of degree N_f-N_c when $N_f-N_c\geqslant 0$, and is zero when $N_f-N_c<0$,

$$T(x)=\frac{1}{4}\sum_{p=0}^{N_f-N_c}t_p x^{N_f-N_c-p}\qquad\text{or}\qquad T(x)=\frac{1}{4}\sum_{p=0}^{N_f-N_c}\tilde{t}_p x^{N_f-N_c-p}, \tag{2.8}$$

depending on whether we consider the model of Ref. [3] or of Ref. [4], respectively. Notice that $T(x)$ is independent of $\bar{s}_i$ for the model of Ref. [3], but does depend on $\bar{s}_i$ for the model of Ref. [4]. Evidently, the two choices for $T(x)$ agree when $N_f\leqslant N_c+1$.

It has become customary to describe the moduli space of supersymmetric vacua in terms of $SU(N_c)$ invariant polynomials in $\bar{a}_k$, such as the symmetric polynomials $\bar{s}_i$ or $\bar{\sigma}_i$ of (2.4) and (2.5). In particular, the approaches to calculating the prepotential using the Picard–Fuchs equations [6] as well as the calculations of strong coupling monodromies make use of such variables. Within the context of our own calculations, which are performed for arbitrary N_c and $N_f<2N_c$, the use of the polynomials $\bar{s}_i$ and $\bar{\sigma}_i$ does not appear to be as fruitful. Instead, we shall parametrize supersymmetric vacua by $\bar{a}_k$ at the classical level and by a_k at the quantum level. In doing so, we keep in mind that

$$\sum_k \bar{a}_k=\sum_k a_k=0$$

and that the space of all such $\bar{a}_k$ must be coseted out by the permutation group (i.e. the Weyl group of $SU(N_c)$). Our final result for the prepotential will be expressed in terms of invariant functions as well, but their translation into $\bar{s}_i$ variables is cumbersome and unnecessary. This is perhaps not so surprising, since even the one-loop answer is not simply rewritten in terms of the functions $\bar{s}_i$.

2.2. *Integrability conditions for the prepotential*

The key features of the meromorphic 1-form $d\lambda$ are the following.

(i) The residue of $d\lambda$ at the pole $x=-m_j$ (and the lower sheet of the Riemann surface) is clearly equal to $-m_j$, as required by the linearity of the BPS mass formula in terms of the quark masses.

(ii) The derivatives of $d\lambda$ with respect to the moduli parameters $\bar{a}_k$ are holomorphic. Actually, as discussed in Ref. [4], there are two natural connections with which $d\lambda$ can be differentiated along the $\bar{a}_k$ directions. With respect to the first, $\nabla^{(E)}$, the values of the Abelian integral $\log(y+A(x))$ are kept fixed, and $\nabla^{(E)}d\lambda$ is the holomorphic 1-form

$$\nabla^{(E)}(d\lambda)=-\frac{1}{y}\left(\delta A-\frac{1}{2}\frac{\delta B}{B}(A-y)\right)dx. \tag{2.9}$$

With respect to the second, $\nabla^{(x)}$, the values of x are kept fixed, and $\nabla^{(x)}\,\mathrm{d}\lambda$ differs from the above holomorphic form by the differential of a meromorphic function

$$\nabla^{(x)}\,\mathrm{d}\lambda = -\frac{1}{y}\left(\delta A - \frac{1}{2}\frac{\delta B}{B}(A-y)\right)\mathrm{d}x + \mathrm{d}\left(-\frac{x}{y}\left(\delta A - \frac{1}{2}\frac{\delta B}{B}(A-y)\right)\right). \tag{2.10}$$

Since exact differentials do not contribute to the periods (2.2), we can ignore this ambiguity and adopt the right-hand side of (2.9) for the derivatives of $\mathrm{d}\lambda$. We also note that we may simplify (2.9) to $-\frac{1}{y}(\delta A - \frac{1}{2}\frac{\delta B}{B}A)$, since the term $\frac{\delta B}{B}A$ is y-independent, and does not contribute to the periods. Similarly, the corresponding term $x(B'/B)A$ in (2.3) may be ignored.

It follows from (ii) that the derivatives $\nabla_{\bar{a}_k}\mathrm{d}\lambda$ of $\mathrm{d}\lambda$ with respect to $\bar{a}_k$, say for $k = 2,\ldots,N_c$, form a basis ω_k of holomorphic 1-forms on the spectral curve, at least for Λ small. (This is evident if we differentiate with respect to the $\bar{s}_i$'s. The derivatives with respect to the $\bar{a}_k$ amount then to a change of basis.) An important observation is that this implies that the dual periods $a_{D,k}$, given by integrals over the B_k cycles, do form a gradient, i.e. that the ansatz (2.2) does guarantee the existence of a prepotential. In fact, for each fixed Λ, we can view $(\bar{a}_k) \to (a_k)$ as a change of variables. Thus the dual variables $a_{D,l}$, which depend originally on $\bar{a}_k$, m_j, and Λ, can be thought of as functions $a_{D,l}(a,m,\Lambda)$ of a_k, m_j, and Λ. By the chain rule,

$$\frac{\partial a_{D,l}}{\partial a_m} = \sum_{k=2}^{N_c}\frac{\partial a_{D,l}}{\partial \bar{a}_k}\frac{\partial \bar{a}_k}{\partial a_m} = \sum_{k=2}^{N_c}\left(\frac{\partial a_D}{\partial \bar{a}}\right)_{lk}\left(\frac{\partial a}{\partial \bar{a}}\right)^{-1}_{km}.$$

Since $(\partial a_D/\partial\bar{a})_{lk}$ and $(\partial a/\partial\bar{a})_{lk}$ are respectively the matrices of periods of the Abelian differentials ω_k over the cycles B_l and A_l of the canonical homology basis, we recognize the matrix ratio above as just the period matrix τ_{lm} of the spectral curve. In view of the symmetry of period matrices, we have then established the integrability condition

$$\frac{\partial a_{D,l}}{\partial a_m} = \tau_{lm} = \tau_{ml} = \frac{\partial a_{D,m}}{\partial a_l} \tag{2.11}$$

and thus the existence of a prepotential function $\mathcal{F}$.

2.3. Model independence of the prepotential

The functional form of the renormalized order parameters a_k and their duals $a_{D,k}$ as functions of the classical parameters $\bar{a}_k$ is different for the two models listed in (2.8). Both a_k and $a_{D,k}$, as functions of $\bar{a}_k$, depend non-trivially upon the parameters t_p or $\tilde{t}_p$ that specify the function $T(x)$. This dependence will be exhibited explicitly in Section 3.1, where an exact expression for $a_k(\bar{a}_k, t_p, m_j; \bar{\Lambda})$ will be given. We shall now establish that the prepotential $\mathcal{F}$, as well as the renormalized dual order parameters $a_{D,k}$, expressed as functions of the renormalized order parameters a_k, are *independent* of t_p or $\tilde{t}_p$. Thus, the prepotential $\mathcal{F}$, expressed in terms of a_k is independent of the models in (2.8), and both models yield the same prepotential $\mathcal{F}$, and thus the same low energy physics.

To establish independence of $T(x)$, we notice that all $\bar{a}_k$-dependence of $\mathrm{d}\lambda$, and thus of a_k and $a_{D,k}$, resides in the function $A(x)$ of (2.3) and (2.6). Also, $A(x)$ is the only place where the dependence on $T(x)$ enters. Now, all dependence on $T(x)$ may be absorbed in a redefinition of the classical order parameters $\bar{a}_k$, since the addition of $T(x)$ just modifies the bare parameters $\bar{s}_i$ in (2.7) to parameters $\tilde{s}_i$, as follows:

$$A(x) = x^{N_c} + \sum_{i=2}^{N_c} \tilde{s}_i x^{N_c-i} = \prod_{k=1}^{N_c} (x - \tilde{a}_k),$$

$$\tilde{s}_i = \bar{s}_i + \frac{1}{4}\Lambda^{2N_c-N_f} t_i. \tag{2.12}$$

Thus, the addition of $T(x)$ may be absorbed by defining new classical order parameters $\tilde{a}_k$, as shown above, and we have

$$a_k = a_k(\bar{a}_l, t_p, m_j; \Lambda) = a_k(\tilde{a}_l, 0, m_j; \Lambda),$$

$$a_{D,k} = a_{D,k}(\bar{a}_l, t_p, m_j; \Lambda) = a_{D,k}(\tilde{a}_l, 0, m_j; \Lambda).$$

Eliminating $\tilde{a}_l$ between a_k and $a_{D,k}$ or $\mathcal{F}$ precisely amounts to eliminating $\bar{a}_l$ when $T(x) = 0$, which demonstrates that $a_{D,k}(a_l)$ and $\mathcal{F}(a_l)$ are independent of $T(x)$.

In view of this model independence of $\mathcal{F}$, we may set $\tilde{a}_k = \bar{a}_k$ and $T = 0$ when calculating $\mathcal{F}$, or when calculating $a_{D,k}$ as functions of a_k.

3. Logarithms and non-renormalization

In this section, we shall describe an algorithm for calculating the renormalized order parameters a_k and their duals $a_{D,k}$ to any order of perturbation theory, in the regime where Λ is small, and the variables a_k (equivalently, their classical counterparts $\bar{a}_k$) are well separated. In particular, we establish a general non-renormalization theorem of the prepotential $\mathcal{F}$, expressed in terms of the renormalized a_k, $\mathcal{F}$ contains only those logarithmic terms appearing in the expression (2.1).

We begin by describing more precisely the representation of the Riemann surface (2.3) as a double cover of the complex plane, and our choice of homology cycles. It is convenient to set

$$\bar{\Lambda} = \Lambda^{N_c-N_f/2}. \tag{3.1}$$

The branch points $x_k^{\pm}$, $1 \leqslant k \leqslant N_c$, of the surface are defined then by

$$A(x_k^{\pm})^2 - B(x_k^{\pm}) = 0.$$

For $\bar{\Lambda}$ small, $x_k^{\pm}$ are just perturbations of the $\bar{a}_k$'s. We view the Riemann surface as two copies of the complex plane, cut and joined along slits from x_k^- to x_k^+. A canonical homology basis A_k, B_k, $2 \leqslant k \leqslant N_c$, is obtained by choosing A_k to be a simple contour enclosing the slit from x_k^- to x_k^+, and B_k to consist of the curves going from x_k^- to x_1^- on each sheet. With this choice, we do have $\#(A_j \cap A_k) = \#(B_j \cap B_k) = 0$, $\#(A_j \cap B_k) = \delta_{jk}$.

3.1. Expansion of the order parameters a_k

First, we recall that the renormalized order parameters a_k are given by combining (2.2) and (2.3)

$$2\pi i a_k = \oint_{A_k} \mathrm{d}\lambda = \oint_{A_k} \mathrm{d}x \frac{x\left(\frac{A'}{A} - \frac{1}{2}\frac{B'}{B}\right)}{\sqrt{1 - \frac{B}{A^2}}}. \tag{3.2}$$

We can now reduce the evaluation of this integral to a set of residue calculations in the following way. We fix the location of the contour A_k, such that its distance away from the branch cut between x_k^- and x_k^+ is much larger than $\bar{\Lambda}$. This can always be achieved since we assumed that the points $\bar{a}_k$ are well separated. Having fixed A_k, we consider a_k in a power series in $\bar{\Lambda}^2$, i.e. this allows us to expand the denominator in a convergent power series in $B(x)$,

$$2\pi i a_k = \oint_{A_k} \mathrm{d}x\, x\frac{A'}{A} + \sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \oint_{A_k} \mathrm{d}x\, x\left(\frac{A'}{A} - \frac{1}{2}\frac{B'}{B}\right)\left(\frac{B}{A^2}\right)^m. \tag{3.3}$$

Similarly, powers and derivatives of $A(x)$ may be expanded in a series in powers of $\bar{\Lambda}^2 T(x)$, leaving integrals of rational functions in x, which can be evaluated using residue methods. It is very convenient to introduce the residue functions $R_k(x)$ and $S_k(x)$, defined by

$$\frac{B(x)}{C(x)^2} = \frac{\bar{\Lambda}^2}{(x-\bar{a}_k)^2} S_k(x) \tag{3.4}$$

and

$$\frac{T(x)}{C(x)} = \frac{1}{x-\bar{a}_k} R_k(x). \tag{3.5}$$

The first term on the right-hand side of (3.3) is evaluated by expressing $\frac{A'}{A}$ as

$$\frac{A'(x)}{A(x)} = \frac{C'(x)}{C(x)} + \frac{\mathrm{d}}{\mathrm{d}x}\log\left(1 + \bar{\Lambda}^2 \frac{T(x)}{C(x)}\right)$$

and expanding the logarithm in a power series in $\bar{\Lambda}^2$

$$\oint_{A_k} \mathrm{d}x\, x\frac{A'}{A} = 2\pi i \bar{a}_k + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \bar{\Lambda}^{2n} \oint_{A_k} \mathrm{d}x \left(\frac{T(x)}{C(x)}\right)^n. \tag{3.6}$$

These line integrals can be easily evaluated by the method of residues, since the cycle A_k is a contour enclosing the pole $\bar{a}_k$, and we obtain at once

$$\oint_{A_k} dx\, x\frac{A'}{A} = 2\pi i\bar{a}_k + 2\pi i\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\bar{\Lambda}^{2n}\left(\frac{\partial}{\partial\bar{a}_k}\right)^{n-1}R_k(\bar{a}_k)^n. \tag{3.7}$$

The second term on the right-hand side of (3.3) is evaluated using the identity

$$x\left(\frac{A'}{A}-\frac{1}{2}\frac{B'}{B}\right)\left(\frac{B}{A^2}\right)^m = -\frac{d}{dx}\left[\frac{x}{2m}\left(\frac{B}{A^2}\right)^m\right]+\frac{1}{2m}\left(\frac{B}{A^2}\right)^m \tag{3.8}$$

and the following Taylor expansion in powers of $T(x)$:

$$\frac{B^m(x)}{A^{2m}(x)} = \sum_{n=0}^{\infty}\frac{\Gamma(2m+n)}{\Gamma(2m)\Gamma(n+1)}\frac{B(x)^m T(x)^n}{C(x)^{2m+n}}(-\bar{\Lambda}^2)^n.$$

In terms of the functions $R_k(x)$ and $S_k(x)$, we readily get

$$\oint_{A_k} dx\frac{B(x)^m}{A(x)^{2m}} = 2\pi i\sum_{n=0}^{\infty}\frac{(-1)^n\bar{\Lambda}^{2m+2n}}{\Gamma(2m)\Gamma(n+1)}\left(\frac{\partial}{\partial\bar{a}_k}\right)^{2m+n-1}(R_k(\bar{a}_k)^n S_k(\bar{a}_k)^m). \tag{3.9}$$

Assembling (3.7) and (3.9), and using the Γ-function identity

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2m+1) = 2^{2m}\Gamma\left(m+\frac{1}{2}\right)\Gamma(m+1),$$

we obtain the following expansion to all orders in $\bar{\Lambda}$ for the renormalized order parameter a_k:

$$a_k = \bar{a}_k + \sum_{\substack{m,n\geqslant 0\\ m+n>0}}\frac{(-1)^n\bar{\Lambda}^{2m+2n}}{(m!)^2 n! 2^{2m}}\left(\frac{\partial}{\partial\bar{a}_k}\right)^{2m+n-1}(R_k(\bar{a}_k)^n S_k(\bar{a}_k)^m). \tag{3.10}$$

As we already noted above, the special case $T(x) = 0$ is of particular interest, since it suffices for us to determine the prepotential in all generality. For later reference, we list here the corresponding expansion for a_k,

$$a_k = \bar{a}_k + \sum_{m=1}^{\infty}\frac{\bar{\Lambda}^{2m}}{2^{2m}(m!)^2}\left(\frac{\partial}{\partial\bar{a}_k}\right)^{2m-1}S_k(\bar{a}_k)^m. \tag{3.11}$$

Notice that this expansion is analytic in powers of $\bar{\Lambda}^2$, as expected from general arguments.

3.2. *Expansion of the branch points $x_k^{\pm}$ in powers of $\bar{\Lambda}$*

We shall now solve the branch point equation $A(x_k^{\pm})^2 - B(x_k^{\pm}) = 0$ in a power series in $\bar{\Lambda}$, around a solution $\bar{a}_k$ of $C(x) = 0$. The problem is conveniently reformulated in terms of the functions R_k and S_k introduced in (3.4) and (3.5),

$$x_k^{\pm} = \bar{a}_k \pm \bar{\Lambda} S(x_k^{\pm})^{\frac{1}{2}} - \bar{\Lambda}^2 R_k(x_k^{\pm}). \tag{3.12}$$

Since $S_k^{\frac{1}{2}}$ and R_k are analytic for $\bar{a}_k - \bar{a}_l$ and $\bar{a}_k + m_j$ away from 0, it follows that $x_k^{\pm}$ is an analytic function in $\bar{\Lambda}$, and admits a Taylor series expansion of the form

$$x_k^{\pm} = \bar{a}_k + \sum_{m=1}^{\infty} (\pm)^m \bar{\Lambda}^m \delta_k^{(m)}. \tag{3.13}$$

We begin by deriving an explicit formula for $\delta_k^{(m)}$ in the special case where $R_k(x) = 0$, i.e. $T(x) = 0$, first. Then, viewing x_k^- as a function of $x(\bar{\Lambda})$, and using the defining equation (3.12), we find

$$\delta_k^{(m)} = \frac{(-1)^m}{m!} \frac{\partial^m}{\partial \bar{\Lambda}^m} x(\bar{\Lambda}) \bigg|_{\bar{\Lambda}=0} = \frac{(-1)^{m-1}}{(m-1)!} \frac{\partial^{m-1}}{\partial \bar{\Lambda}^{m-1}} S_k(x(\bar{\Lambda}))^{\frac{1}{2}} \bigg|_{\bar{\Lambda}=0} .$$

The last expression can be put under the form of a contour integral

$$\delta_k^{(m)} = \frac{(-1)^{m-1}}{2\pi i} \oint_{C_0} \frac{d\bar{\Lambda}}{\bar{\Lambda}^m} \left(S_k x(\bar{\Lambda})\right)^{\frac{1}{2}},$$

where C_0 is a small contour in the $\bar{\Lambda}$ plane, enclosing the point $\bar{\Lambda} = 0$ once. Making the change of variables $\bar{\Lambda} \to x = x(\bar{\Lambda})$ and using $\bar{\Lambda}(x) = -(x - \bar{a}_k) S_k(x)^{-\frac{1}{2}}$, the above contour integral becomes

$$\delta_k^{(m)} = -\frac{1}{2\pi i} \oint_{C_{\bar{a}_k}} dx \left[\frac{(S_k(x)^{\frac{1}{2}})' S_k(x)^{\frac{m-1}{2}}}{(x - \bar{a}_k)^{m-1}} - \frac{S_k(x)^{\frac{m}{2}}}{(x - \bar{a}_k)^m} \right],$$

where $C_{\bar{a}_k}$ is now a contour enclosing $x = \bar{a}_k$ once. Integrating by parts gives at once the following key formula for $\delta_k^{(m)}$:

$$\delta_k^{(m)} = \frac{1}{m!} \left(\frac{\partial}{\partial \bar{a}_k} \right)^{m-1} S_k(\bar{a}_k)^{\frac{m}{2}}. \tag{3.14}$$

The expression for the general case, where $R_k(x) \neq 0$, is easily read off from (3.14) by performing the substitution $S_k(x)^{\frac{1}{2}} \longrightarrow S_k(x)^{\frac{1}{2}} \mp \bar{\Lambda} R_k(x)$, which gives for (3.13)

$$x_k^{\pm} = \bar{a}_k + \sum_{m=1}^{\infty} \bar{\Lambda}^m \frac{1}{m!} \left(\frac{\partial}{\partial \bar{a}_k} \right)^{m-1} \left[\pm S_k(\bar{a}_k)^{\frac{1}{2}} - \bar{\Lambda} R_k(\bar{a}_k) \right]^m . \tag{3.15}$$

Identifying powers in $\bar{\Lambda}$ yields the coefficients $\delta_k^{(m)}$ for general non-zero S_k and R_k,

$$\delta_k^{(m)} = \sum_{n=0}^{m-1} \frac{(-1)^n}{(m-2n)!n!} \left(\frac{\partial}{\partial \bar{a}_k} \right)^{m-n-1} \left(S_k(\bar{a}_k)^{\frac{m}{2}-n} R_k(\bar{a}_k)^n \right). \tag{3.16}$$

3.3. Method of analytic continuation

The complete determination of both the quantum order parameters a_k and the branch points $x_k^{\pm}$ in terms of exact Taylor series expansions in $\bar{\Lambda}$ was possible with the use of residue calculations. Clearly, it would be desirable to carry over these methods, as much as possible, to the calculation of the dual order parameters $a_{D,k}$, (and thus to the prepotential $\mathcal{F}$) given by

$$2\pi i a_{D,k} = \oint_{B_k} \mathrm{d}\lambda = 2\int_{x_1^-}^{x_k^-} \mathrm{d}x \frac{x\left(\frac{A'}{A} - \frac{1}{2}\frac{B'}{B}\right)}{\sqrt{1-\frac{B}{A^2}}} \tag{3.17}$$

(note that $\sqrt{A^2} = -A$ for x on the path from x_1^- to x_k^-). Evaluation of these integrals is more delicate than for the A-periods, since the end point x_k^- of the integration is within a distance of order $\bar{\Lambda}$ from the points $\bar{a}_k$ and a_k. Thus, naively expanding the denominator square root in a power series may run into convergence problems.

This difficulty may be circumvented with the use of the following analytic continuation prescription. We introduce an auxiliary complex parameter ξ, with $|\xi| \leqslant 1$, and we define

$$2\pi i a_{D,k}(\xi) = \oint_{B_k} \mathrm{d}\lambda = 2\int_{x_1^-}^{x_k^-} \mathrm{d}x \frac{x\left(\frac{A'}{A} - \frac{1}{2}\frac{B'}{B}\right)}{\sqrt{1-\xi^2\frac{B}{A^2}}}. \tag{3.18}$$

We consider this expression throughout the unit disc $|\xi| \leqslant 1$, and use analyticity at the origin to expand the denominator square root in a power series in ξ^2,

$$2\pi i a_{D,k}(\xi) = 2\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)}\xi^{2m} \int_{x_1^-}^{x_k^-} \mathrm{d}x\, x\left(\frac{A'}{A} - \frac{1}{2}\frac{B'}{B}\right)\left(\frac{B}{A^2}\right)^m. \tag{3.19}$$

The original quantity $a_{D,k}$ is then obtained by analytic continuation of the function $a_{D,k}$ to $\xi = 1$. (In practice, we shall find that singularities may occur at $\xi = 1$ at intermediate stages of the calculations, but cancel out in the final form for $a_{D,k}$; thus, the final analytic continuations are trivial.)

3.4. Non-renormalization theorems

We are now in a position to prove that the only logarithmic contributions, both in the dynamical scale $\bar{\Lambda}$ and in the differences $a_k - a_l$ are those generated by 1-loop perturbation theory, as given by the second line in (2.1). As our starting point, we use the expansion (3.19) for $a_{D,k}$, and further expand the denominators in $A(x)$ in a power series in $T(x)$. As a result, each term in this double series is now a rational function in x: a polynomial in B and in T, divided by a power of $C(x)$, and for $m = 0$, a single term of the form B'/B. Again, from residue calculus, we have a general expression for the *residues at the simple poles*: they are given by contour integrals around that pole

$$\frac{x\left(\frac{A'}{A}-\frac{1}{2}\frac{B'}{B}\right)}{\sqrt{1-\xi^2\frac{B}{A^2}}}=\frac{1}{2}\sum_{j=1}^{N_f}\frac{m_j}{x+m_j}+\sum_{l=1}^{N_c}\frac{1}{x-\bar{a}_l}\frac{1}{2\pi i}\oint_{A_l}dy\frac{y\left(\frac{A'}{A}-\frac{1}{2}\frac{B'}{B}\right)}{\sqrt{1-\xi^2\frac{B}{A^2}}}$$
$$+\left(N_c-\frac{1}{2}N_f\right)+\sum_{l=1}^{N_c}\sum_{p=2}^{\infty}\frac{M_l^{(p)}(\xi)}{(x-\bar{a}_l)^p}. \tag{3.20}$$

Letting $\xi \longrightarrow 1$, we recognize the residues of the simple poles at $\bar{a}_l$ as the quantum order parameters a_l, so that

$$2\pi i a_{D,k}=\sum_{j=1}^{N_f}m_j\log(x_k^-+m_j)+2\sum_{l=1}^{N_c}a_l\log(x_k^--\bar{a}_l)$$
$$+2\left(N_c-\frac{1}{2}N_f\right)x_k^- -2\sum_{l=1}^{N_c}\sum_{p=2}^{\infty}\frac{1}{p-1}\frac{M_l^{(p)}(1)}{(x_k^--\bar{a}_l)^{p-1}}. \tag{3.21}$$

Here, as in the remainder of the paper, we have written explicitly in the expression for $2\pi i a_{D,k}$ only the contributions of the upper integration bound x_k^- in the integral (3.19). Evidently, the lower bound x_1^- contributes a similar term with k replaced by 1, and with opposite sign. In particular, we observe that we are free to add k-independent constants to (3.21), since such constants will cancel between the two bounds of integration.

Using the exact result for $x_k^- - \bar{a}_k$, from (3.12), and the fact that x_k^- and a_k are analytic functions of $\bar{\Lambda}$, with Taylor series coefficients that are rational functions of the parameters $\bar{a}_k$, it is clear that

$$2\pi i a_{D,k}=2\log\bar{\Lambda}\ a_k+\sum_{j=1}^{N_f}(a_k+m_j)\log(a_k+m_j)-\sum_{l\neq k}^{N_c}(a_k-a_l)\log(a_k-a_l)^2$$
$$+\ \text{power series in } \bar{\Lambda} \text{ with coefficients rational in } a_l. \tag{3.22}$$

The first three terms on the right-hand side arise from the 1-loop contribution to the effective prepotential.

To see this, we need to clarify the implications of the constraint $\sum_{k=1}^{N_c}a_k=0$. The prepotential $\mathcal{F}$ of (2.1) is a function $\mathcal{F}(a_1,\dots,a_{N_c})$ of *all* variables a_k, whose restriction to the constraint hyperplane $\sum_{k=1}^{N_c}a_k=0$ is the prepotential appearing in (2.2). Written in terms of $\mathcal{F}(a_1,\dots,a_{N_c})$, the condition (2.2) becomes

$$a_{D,k}=\frac{\partial}{\partial a_k}\mathcal{F}(a_1,\dots,a_{N_c})-\frac{\partial}{\partial a_1}\mathcal{F}(a_1,\dots,a_{N_c}),\quad 2\leqslant k\leqslant N_c. \tag{3.23}$$

Conveniently, such differences occur automatically in expressions of the form (3.17), if we take into account the contributions of the lower integration bound x_1^-. Thus, it suffices to identify the gradient of the prepotential function $\mathcal{F}$, with *all* variables $a_1,\dots,a_k$, viewed as independent. Doing so, the first line of (3.22) is readily recognized as a gradient

$$-\frac{1}{4}\frac{\partial}{\partial a_k}\left[\sum_{l,m=1}^{N_c}(a_l-a_m)^2\log\frac{(a_l-a_m)^2}{\Lambda^2}-\sum_{l=1}^{N_c}\sum_{j=1}^{N_f}(a_l+m_j)^2\log\frac{(a_l+m_j)^2}{\Lambda^2}\right] \tag{3.24}$$

up to power series in Λ with coefficients rational in a_k. These are exactly the logarithmic singularities indicated in (2.1), due to perturbative corrections. As for the instanton contributions themselves, the d-instanton contribution $\mathcal{F}_d$ is homogeneous of degree d in the a_l's, and thus the Euler relation implies

$$d\mathcal{F}_d=\sum_{l=1}^{N_c}a_l\frac{\partial\mathcal{F}_d}{\partial a_l}.$$

Since $\partial\mathcal{F}_d/\partial a_l$ is logarithm-free, so is $\mathcal{F}_d$. This completes the proof of the non-renormalization theorem.

As we worked it out above, the dual order parameters $a_{D,k}$ and thus the prepotential $\mathcal{F}$ are a sum of the classical and 1-loop perturbative contributions, plus a power series expansion in powers of $\bar{\Lambda}$. Actually, general arguments suggest that the power series should involve only even powers of $\bar{\Lambda}$. This fact is not completely obvious from the construction of $a_{D,k}$ in (3.17): while the integrand is even in $\bar{\Lambda}$, the integration limit x_k^- is not even. However, it is clear from the form of the integral in (3.17), that the part of $a_{D,k}$ which is odd in Λ is obtained by taking the following difference:

$$\begin{aligned}2\pi i\left(a_{D,k}(\bar{\Lambda})-a_{D,k}(-\bar{\Lambda})\right)&=2\int_{x_k^+}^{x_k^-}dx\frac{x\left(\frac{A'}{A}-\frac{1}{2}\frac{B'}{B}\right)}{\sqrt{1-\frac{B}{A^2}}}-2\int_{x_1^+}^{x_1^-}dx\frac{x\left(\frac{A'}{A}-\frac{1}{2}\frac{B'}{B}\right)}{\sqrt{1-\frac{B}{A^2}}}\\&=-2\pi i(a_k-a_1).\end{aligned}$$

This linear addition is merely a modification of the classical value of the prepotential, and is innocuous. Thus, we see that the instanton contributions are purely even in $\bar{\Lambda}$, as required by general arguments.

3.5. Expansion of the dual order parameter $a_{D,k}$

In this subsection, we shall work out an explicit formula for the dual order parameters $a_{D,k}$, in a series expansion in powers of $\bar{\Lambda}$. In view of the general arguments[4] advanced in Section 2.3, the dual order parameters $a_{D,k}$ and the prepotential $\mathcal{F}$, when expressed as functions of the renormalized order parameters a_k, are independent of the function $T(x)$. Thus, henceforth, we shall set $T(x)=0$ without loss of generality.

[4] We have checked the validity of these general arguments by explicit calculation to 1- and 2-instanton orders; these calculations will not be presented here.

When $T(x)=0$, the function $A(x)$ simplifies and is given by

$$A(x)=C(x).$$

Our starting point for the calculation of the dual order parameters $a_{D,k}$ is then the series expansion of (3.19), which for $T(x)=0$ becomes

$$2\pi i a_{D,k}(\xi)=2\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)}\xi^{2m}\int_{x_1^-}^{x_k^-}\mathrm{d}x\, x\left(\frac{C'}{C}-\frac{1}{2}\frac{B'}{B}\right)\left(\frac{B}{C^2}\right)^m. \tag{3.25}$$

The $m=0$ terms in (3.25) can be easily integrated, and we find

$$\begin{aligned}2\int_{x_1^-}^{x_k^-}\mathrm{d}x\, x\left(\frac{C'}{C}-\frac{1}{2}\frac{B'}{B}\right)&=(2N_c-N_f)x_k^-+2\sum_{l=1}^{N_c}\bar{a}_l\log(x_k^--\bar{a}_l)\\&+\sum_{m=1}^{N_f}m_j\log(x_k^-+m_j).\end{aligned} \tag{3.26}$$

As for the remaining $m>0$ terms, it is convenient to perform an initial integration by parts using the identity (3.8), and rewrite them as

$$\int_{x_1^-}^{x_k^-}\mathrm{d}x\, x\left(\frac{C'}{C}-\frac{1}{2}\frac{B'}{B}\right)\left(\frac{B}{C^2}\right)^m=-\frac{1}{2m}x_k^-+\frac{1}{2m}\int_{x_1^-}^{x_k^-}\mathrm{d}x\left(\frac{B}{C^2}\right)^m. \tag{3.27}$$

The contribution of the first term on the right-hand side of (3.27) can be summed explicitly, and we find (cf. Appendix A)

$$\begin{aligned}2\pi i a_{D,k}&=(2N_c-N_f-2\log 2)x_k^-+2\sum_{l=1}^{N_c}\bar{a}_l\log(x_k^--\bar{a}_l)+\sum_{j=1}^{N_f}m_j\log(x_k^-+m_j)\\&+2\sum_{m=1}^{\infty}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)}\frac{\xi^{2m}}{2m}\int_{x_1^-}^{x_k^-}\mathrm{d}x\left(\frac{B}{C^2}\right)^m.\end{aligned} \tag{3.28}$$

Next, we expand the rational function B^m/C^{2m} into partial fractions. Since $N_f<2N_c$, the function B^m/C^{2m} vanishes as $x\to\infty$, so that, with suitable coefficients $Q_{l;p}^{(2m)}$, we have the expansion

$$\left(\frac{B}{C^2}\right)^m=\sum_{l=1}^{N_c}\sum_{p=1}^{2m}\bar{\Lambda}^{2m}Q_{l;p}^{(2m)}\frac{1}{(x-\bar{a}_l)^p}. \tag{3.29}$$

We now make us of the observation of Section 3.4 on the structure of the logarithmic terms. For $p=1$ in (3.29), the coefficient $Q_{l;1}^{(2m)}$ is just the residue at the pole $\bar{a}_l$ of B^m/A^{2m}, and can thus be expressed as a contour integral around $\bar{a}_l$, so that

$$\bar{\Lambda}^{2m} Q_{l;1}^{(2m)} = \frac{2m}{2\pi i} \oint_{A_l} dx\, x \left(\frac{C'}{C} - \frac{1}{2}\frac{B'}{B} \right) \left(\frac{B}{C^2} \right)^m .$$

The contributions of these terms to $a_{D,k}$ are now easily recognizable,

$$\sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{\xi^{2m}\bar{\Lambda}^{2m}}{2m} Q_{l;1}^{(2m)} = \frac{1}{2\pi i} \oint_{A_l} dx\, x \left(\frac{C'}{C} - \frac{1}{2}\frac{B'}{B} \right) \times \left(\frac{1}{\sqrt{1-\xi^2 \frac{B}{C^2}}} - 1 \right) = a_l - \bar{a}_l. \quad (3.30)$$

Substituting in (3.28), and carrying out the dx integrations, we arrive at the following formula, basic for our subsequent calculations:

$$2\pi i a_{D,k} = (2N_c - N_f - 2\log 2) x_k^- + 2\sum_{l=1}^{N_c} a_l \log(x_k^- - \bar{a}_l) + \sum_{j=1}^{N_f} m_j \log(x_k^- + m_j)$$
$$-2\sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{\xi^{2m}}{2m} \sum_{l=1}^{N_c} \sum_{p=2}^{2m} \frac{Q_{l;p}^{(2m)}}{p-1} \frac{\bar{\Lambda}^{2m}}{(x_k^- - \bar{a}_l)^{p-1}} . \quad (3.31)$$

The coefficients $Q_{l;p}^{(2m)}$ can be obtained from a Taylor expansion of the function $S_k(x)$ combined with (3.4), and we obtain the simple expressions

$$Q_{l;p}^{(2m)} = \frac{1}{(2m-p)!} \left(\frac{\partial}{\partial \bar{a}_l} \right)^{2m-p} S_l(\bar{a}_l)^m . \quad (3.32)$$

It remains to express the quantities x_l^- and $\bar{a}_l$, as well as the coefficients $Q_{l;p}^{(2m)}$ in terms of the renormalized order parameters a_l. The relevant formulas were already derived in (3.11), (3.13) and (3.14). Carrying out these substitutions in practice is rather cumbersome however, and will be done in Section 4 for 1- and 2-instanton contributions, i.e. up to and including corrections of order $\bar{\Lambda}^4$.

4. The 1- and 2-instanton contributions

In this section, we derive explicit expressions for the prepotential, including perturbative (1-loop), 1- and 2-instanton contributions (i.e. up to and including order $\bar{\Lambda}^4$). As was argued in Section 2.3, without loss of generality, the prepotential can be evaluated from the curve with $T(x) = 0$, which we assume from now on. Thus, our starting point is the expansion of (3.31), which we now wish to express in terms of a_l, up to and including order $\bar{\Lambda}^4$. To avoid any confusion of orders of expansion, it is most systematic (though perhaps somewhat lengthier) to first express all quantities in terms of the parameters $\bar{a}_l$, and then convert the expression into a function solely of the renormalized order parameters a_l.

4.1. Expansion of the branch points x_k^-

Our calculations can be carried out either in terms of the derivatives at $\bar{a}_k$ of the functions $S_k(x)$, or, equivalently, in terms of the quantities $\delta_k^{(m)}$, defined in (3.13) and expressed in terms of $S_k(x)$ in (3.14). It turns out that the calculations can be presented most concisely in terms of $\delta_k^{(m)}$. From (3.13), we recall that, to order $\bar{\Lambda}^4$, we have

$$\begin{aligned} x_k^- &= \bar{a}_k - \bar{\Lambda}\delta_k^{(1)} + \bar{\Lambda}^2\delta_k^{(2)} - \bar{\Lambda}^3\delta_k^{(3)} + \bar{\Lambda}^4\delta_k^{(4)}, \\ a_k &= \bar{a}_k + \frac{1}{2}\bar{\Lambda}^2\delta_k^{(2)} + \frac{3}{8}\bar{\Lambda}^4\delta_k^{(4)}. \end{aligned} \tag{4.1}$$

Also, the expansion up to order $\bar{\Lambda}^3$ of powers of $x_k^- - \bar{a}_k$

$$\begin{aligned} (x_k^- - \bar{a}_k)^{-p} = &\Bigg[1 + \bar{\Lambda}p\frac{\delta_k^{(2)}}{\delta_k^{(1)}} + \bar{\Lambda}^2\left\{-p\frac{\delta_k^{(3)}}{\delta_k^{(1)}} + \frac{1}{2}p(p+1)\left(\frac{\delta_k^{(2)}}{\delta_k^{(1)}}\right)^2\right\} \\ &+\bar{\Lambda}^3\left\{p\frac{\delta_k^{(4)}}{\delta_k^{(1)}} - p(p+1)\frac{\delta_k^{(2)}\delta_k^{(3)}}{(\delta_k^{(1)})^2} + \frac{p(p+1)(p+2)}{6}\frac{\delta_k^{(2)}}{\delta_k^{(1)}}\right\}\Bigg] \\ &\times(-1)^p\bar{\Lambda}^{-p}(\delta_k^{(1)})^{-p} \end{aligned} \tag{4.2}$$

as well as the expansion up to order $\bar{\Lambda}^2$ of powers of $x_k - \bar{a}_l$, for $k \neq l$,

$$\begin{aligned} (x_k^- - \bar{a}_l)^{-p} = (\bar{a}_k - \bar{a}_l)^{-p}&\Bigg[1 + \bar{\Lambda}p\frac{\delta_k^{(1)}}{\bar{a}_k - \bar{a}_l} + \bar{\Lambda}^2\left\{-p\frac{\delta_k^{(2)}}{\bar{a}_k - \bar{a}_l}\right. \\ &\left.+\frac{1}{2}p(p+1)\frac{(\delta_k^{(1)})^2}{(\bar{a}_k - \bar{a}_l)^2}\right\}\Bigg] \end{aligned} \tag{4.3}$$

will be of particular use.

4.2. Evaluation of the coefficients $Q_{l;p}^{(2m)}$

A general expression for the residue functions $Q_{l;p}^{(2m)}$ in terms of $S_k(x)$ was given in (3.32). In terms of the functions $\delta_k^{(m)}$, they are easily worked out to low order as follows:

$$\begin{aligned} Q_{l;2m}^{(2m)} &= (\delta_l^{(1)})^{2m}, \\ Q_{l;2m-1}^{(2m)} &= 2m\delta_l^{(2)}(\delta_l^{(1)})^{2m-2}, \\ Q_{l;2m-2}^{(2m)} &= 2m\delta_l^{(3)}(\delta_l^{(1)})^{2m-3} + m(2m-3)(\delta_l^{(2)})^2(\delta_l^{(1)})^{2m-4}, \\ Q_{l;2m-3}^{(2m)} &= 2m\delta_l^{(4)}(\delta_l^{(1)})^{2m-4} + 2m(2m-4)\delta_l^{(2)}\delta_l^{(3)}(\delta_l^{(1)})^{2m-5} \\ &\quad+\frac{1}{3}m(2m-4)(2m-5)(\delta_l^{(2)})^3(\delta_l^{(1)})^{2m-6}. \end{aligned} \tag{4.4}$$

4.3. Summation of the series in m

It is convenient to separate the series in m in (3.31) into two terms (I) and (II), corresponding respectively to the contributions of the poles at $\bar{a}_l$, $l \neq k$ and at $\bar{a}_k$.

$$(\mathrm{I}) = 2\sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{\xi^{2m}}{2m} \sum_{l\neq k}\sum_{p=2}^{2m} \frac{Q_{l;p}^{(2m)}}{p-1} \frac{\bar{\Lambda}^{2m}}{(x_k^- - \bar{a}_l)^{p-1}},$$

$$(\mathrm{II}) = 2\sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{\xi^{2m}}{2m} \sum_{p=2}^{2m} \frac{Q_{k;p}^{(2m)}}{p-1} \frac{\bar{\Lambda}^{2m}}{(x_k^- - \bar{a}_k)^{p-1}}. \tag{4.5}$$

We concentrate first on (I). In this case $x_k^- - \bar{a}_l \sim \bar{\Lambda}^0$, the contributions up to and including terms of order $\bar{\Lambda}^4$ arise only from $m \leqslant 2$, and so we may set $\xi = 1$ immediately. We obtain

$$(\mathrm{I}) = \sum_{l\neq k}\left[\frac{1}{2}\bar{\Lambda}^2 \frac{Q_{l;2}^{(2)}}{x_k^- - \bar{a}_l} + \frac{1}{16}\bar{\Lambda}^4 \frac{Q_{l;4}^{(4)}}{(x_k^- - \bar{a}_l)^3} + \frac{3}{32}\bar{\Lambda}^4 \frac{Q_{l;3}^{(4)}}{(x_k^- - \bar{a}_l)^2} \right.$$
$$\left. + \frac{3}{16}\bar{\Lambda}^4 \frac{Q_{l;2}^{(4)}}{x_k^- - \bar{a}_l}\right]. \tag{4.6}$$

We can now substitute in the expression (4.6) for $Q_{l;p}^{(2m)}$ in terms of the $\delta_k^{(n)}$, and expand $(x_k^- - \bar{a}_l)^{-p}$ as in (4.3). The result is

$$(\mathrm{I}) = \sum_{l\neq k}\left[\frac{1}{2}\bar{\Lambda}^2 \frac{(\delta_l^{(1)})^2}{\bar{a}_k - \bar{a}_l} + \frac{1}{2}\bar{\Lambda}^3 \frac{(\delta_l^{(1)})^2\delta_k^{(1)}}{(\bar{a}_k - \bar{a}_l)^2} - \frac{1}{2}\bar{\Lambda}^4 \frac{(\delta_l^{(1)})^2\delta_k^{(2)}}{(\bar{a}_k - \bar{a}_l)^2} \right.$$
$$+ \frac{1}{2}\bar{\Lambda}^4 \frac{(\delta_l^{(1)})^2(\delta_k^{(1)})^2}{(\bar{a}_k - \bar{a}_l)^3} + \frac{1}{16}\bar{\Lambda}^4 \frac{(\delta_l^{(1)})^4}{(\bar{a}_k - \bar{a}_l)^3} + \frac{3}{8}\bar{\Lambda}^4 \frac{(\delta_l^{(1)})^2\delta_l^{(2)}}{(\bar{a}_k - \bar{a}_l)^2} + \frac{3}{4}\bar{\Lambda}^4 \frac{\delta_l^{(1)}\delta_l^{(3)}}{\bar{a}_k - \bar{a}_l}$$
$$\left. + \frac{3}{8}\bar{\Lambda}^4 \frac{(\delta_l^{(2)})^2}{\bar{a}_k - \bar{a}_l}\right]. \tag{4.7}$$

We turn next to contribution (II). Here $\bar{\Lambda}^{2m}(x_k^- - \bar{a}_k)^{-p+1} \sim \bar{\Lambda}^{2m-p+1}$, so that contributions to 2-instanton order are received from all terms satisfying $2m-3 \leqslant p \leqslant 2m$. (We note that, in particular, all m contribute to any instanton order.) Thus, we have

$$(\mathrm{II}) = 2\sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{\xi^{2m}}{2m}\bar{\Lambda}^{2m}$$
$$\times\left[\frac{Q_{k;2m}^{(2m)}}{(2m-1)(x_k^- - \bar{a}_k)^{2m-1}}\theta_{m-1} + \frac{Q_{k;2m-1}^{(2m)}}{(2m-2)(x_k^- - \bar{a}_k)^{2m-2}}\theta_{m-2} \right.$$
$$\left. + \frac{Q_{k;2m-2}^{(2m)}}{(2m-3)(x_k^- - \bar{a}_k)^{2m-3}}\theta_{m-2} + \frac{Q_{k;2m-3}^{(2m)}}{(2m-4)(x_k^- - \bar{a}_k)^{2m-4}}\theta_{m-3}\right]. \tag{4.8}$$

Here, we have denoted by θ_m the Heaviside function defined by $\theta_m = 1$ for $m \geqslant 0$, and $\theta_m = 0$ otherwise. We may now use the values for $Q_{k;p}^{(2m)}$ found earlier (4.4), and the expansions for $(x_k^- - \bar{a}_k)^{-p}$ of (4.2) to arrive at the following formula, valid to order $\bar{\Lambda}^4$ included:

$$
\begin{aligned}
(\mathrm{II}) = & -2\bar{\Lambda}\delta_k^{(1)} \sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{1}{2m(2m-1)} \\
& +4\bar{\Lambda}^2\delta_k^{(2)} \sum_{m=2}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{1}{2m(2m-2)} \\
& -6\bar{\Lambda}^3\delta_k^{(3)} \sum_{m=2}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{1}{2m(2m-3)} \\
& +8\bar{\Lambda}^4\delta_k^{(4)} \sum_{m=3}^{\infty} \frac{G(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{1}{2m(2m-4)} \\
& -\frac{1}{2}\bar{\Lambda}^2\delta_k^{(2)} + \frac{1}{2}\bar{\Lambda}^3\delta_k^{(3)} - \frac{11}{16}\bar{\Lambda}^4\delta_k^{(4)} - \frac{1}{2}\bar{\Lambda}^3\frac{(\delta_k^{(2)})^2}{\delta_k^{(1)}} \\
& +\frac{1}{4}\bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}} - \frac{3}{8}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}.
\end{aligned}
\tag{4.9}
$$

We note that all series in m are now convergent, which is why ξ has simply been set to 1. The values of the series that occur in (4.9) have been worked out in Appendix A. Inserting them in (4.9), we find

$$
\begin{aligned}
(\mathrm{II}) = & \bar{\Lambda}\delta_k^{(1)}(2\log 2 - 2) + \bar{\Lambda}^2\delta_k^{(2)}\left(\frac{1}{2} - \log 2\right) + \bar{\Lambda}^3\left[\delta_k^{(3)}\left(-\frac{5}{3} + 2\log 2\right)\right. \\
& \left.-\frac{1}{2}\frac{(\delta_k^{(2)})^2}{\delta_k^{(1)}}\right] + \bar{\Lambda}^4\left[\delta_k^{(4)}\left(\frac{9}{16} - \frac{5}{4}\log 2\right) + \frac{1}{4}\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}} - \frac{3}{8}\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}\right].
\end{aligned}
\tag{4.10}
$$

4.4. Rearranging logarithmic contributions

Our next task is to rewrite the logarithmic terms, which are so far expressed in terms of $x_k^- - \bar{a}_l$, in terms of the renormalized order parameters a_k. We shall do so in two stages, first replacing x_k^- by $\bar{a}_k$, and then replacing both $\bar{a}_k$ and $\bar{a}_l$ by a_k and a_l.

It is convenient to exploit the fact that x_k^- satisfies the branch point Eq. (3.3), and to rewrite the logarithm terms in (3.19) first as

$$
\begin{aligned}
& 2\sum_{l=1}^{N_c} a_l \log(x_k^- - \bar{a}_l) + \sum_{j=1}^{N_f} m_j \log(x_k^- + m_j) - 2(\log\bar{\Lambda})x_k^- \\
& \quad = -(\mathrm{III}) - (\mathrm{IV}) - (\mathrm{V}), \\
& (\mathrm{III}) = 2\sum_{l\neq k}^{N_c}(x_k^- - \bar{a}_l)\log(x_k^- - \bar{a}_l) - \sum_{j=1}^{N_f}(x_k^- + m_j)\log(x_k^- + m_j),
\end{aligned}
$$

$$(\mathrm{IV}) = 2\sum_{l\neq k}^{N_c}(\bar{a}_l - a_l)\log(x_k^- - \bar{a}_l),$$
$$(\mathrm{V}) = 2(x_k^- - a_k)\log(x_k^- - \bar{a}_k). \tag{4.11}$$

To evaluate (III), we expand the functions $\log(x_k^- - \bar{a}_l)$ and $\log(x_k + m_j)$ in power series around $\log(\bar{a}_k - \bar{a}_l)$ and $\log(\bar{a}_k + m_j)$, respectively. We obtain in this way

$$\begin{aligned}(\mathrm{III}) = &(2N_c - 2 - N_f)(x_k^- - \bar{a}_k) + 2\sum_{l\neq k}^{N_c}(\bar{a}_k - \bar{a}_l)\log(\bar{a}_k - \bar{a}_l)\\ &-\sum_{j=1}^{N_f}(\bar{a}_k + m_j)\log(\bar{a}_k + m_j) - (x_k^- - \bar{a}_k)\log S_k(\bar{a}_k)\\ &-\frac{1}{2}(x_k^- - \bar{a}_k)^2\frac{\partial}{\partial\bar{a}_k}\log S_k(\bar{a}_k) - \frac{1}{6}(x_k^- - \bar{a}_k)^3\frac{\partial^2}{\partial\bar{a}_k^2}\log S_k(\bar{a}_k)\\ &-\frac{1}{24}(x_k^- - \bar{a}_k)^4\frac{\partial^3}{\partial\bar{a}_k^3}\log S_k(\bar{a}_k).\end{aligned} \tag{4.12}$$

As before, the derivatives of $S_k(\bar{a}_k)$ can be readily expressed in terms of the $\delta_k^{(q)}$,

$$\begin{aligned}\frac{\partial}{\partial\bar{a}_k}\log S_k(\bar{a}_k) &= 2\frac{\delta_k^{(2)}}{(\delta_k^{(1)})^2},\\ \frac{\partial^2}{\partial\bar{a}_k^2}\log S_k(\bar{a}_k) &= 4\frac{\delta_k^{(3)}}{(\delta_k^{(1)})^3} - 6\frac{(\delta_k^{(2)})^2}{(\delta_k^{(1)})^4},\\ \frac{\partial^3}{\partial\bar{a}_k^3}\log S_k(\bar{a}_k) &= 12\frac{\delta_k^{(4)}}{(\delta_k^{(1)})^4} - 48\frac{\delta_k^{(2)}\delta_k^{(3)}}{(\delta_k^{(1)})^5} + 40\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^6}.\end{aligned} \tag{4.13}$$

Expanding as well the powers of $x_k^- - \bar{a}_k$ in $\bar{\Lambda}$, we obtain the following expression for (III):

$$\begin{aligned}(\mathrm{III}) = &(2N_c - 2 - N_f)(x_k^- - \bar{a}_k) + 2\sum_{l\neq k}^{N_c}(\bar{a}_k - \bar{a}_l)\log(\bar{a}_k - \bar{a}_l)\\ &-\sum_{j=1}^{N_f}(\bar{a}_k + m_j)\log(\bar{a}_k + m_j) - 2(x_k^- - \bar{a}_k)\log\delta_k^{(1)} - \bar{\Lambda}^2\delta_k^{(2)} + \frac{2}{3}\bar{\Lambda}^3\delta_k^{(3)}\\ &+\bar{\Lambda}^3\frac{(\delta_k^{(2)})^2}{\delta_k^{(1)}} - \frac{1}{2}\bar{\Lambda}^4\delta_k^{(4)} + \frac{1}{3}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2} - 2\bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}}.\end{aligned} \tag{4.14}$$

Similarly, expansions for (IV) and (V) in terms of $\bar{\Lambda}$ are easily determined using the expansions for a_k and x_k^- given in (3.12) and (4.1-2). We find for (IV)

$$(\mathrm{IV}) = -\bar{\Lambda}^2\sum_{l\neq k}\delta_l^{(2)}\log(\bar{a}_k - \bar{a}_l) - \frac{3}{4}\bar{\Lambda}^4\sum_{l\neq k}\delta_l^{(4)}\log(\bar{a}_k - \bar{a}_l)$$

$$+\bar{\Lambda}^3 \sum_{l\neq k} \frac{\delta_k^{(1)}\delta_l^{(2)}}{\bar{a}_k-\bar{a}_l} - \bar{\Lambda}^4 \sum_{l\neq k} \frac{\delta_l^{(2)}\delta_k^{(2)}}{\bar{a}_k-\bar{a}_l} + \frac{1}{2}\bar{\Lambda}^4 \sum_{l\neq k} \frac{\delta_l^{(2)}(\delta_k^{(1)})^2}{(\bar{a}_k-\bar{a}_l)^2} \tag{4.15}$$

and for (V)

$$\begin{aligned}(\mathrm{V}) = {}& 2\left(-\bar{\Lambda}\delta_k^{(1)} + \frac{1}{2}\bar{\Lambda}^2\delta_k^{(2)} - \bar{\Lambda}^3\delta_k^{(3)} + \frac{5}{8}\bar{\Lambda}^4\delta_k^{(4)}\right)(\log\bar{\Lambda} + \log\delta_k^{(1)}) \\ & +2\bar{\Lambda}^2\delta_k^{(2)} - 2\bar{\Lambda}^3\delta_k^{(3)} + 2\bar{\Lambda}^4\delta_k^{(4)} + \bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}} + \frac{1}{6}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}.\end{aligned} \tag{4.16}$$

For computational purposes, we had replaced the coefficients a_k in front of the log terms in (3.19) by $\bar{a}_k$. Upon assembling the expressions we just obtained for (III)-(V), and recognizing the combinations making up a_k from $\bar{a}_k$, we obtain

$$\begin{aligned}&\left(-2\log\frac{\bar{\Lambda}}{2} - 2N_c + N_f\right)x_k^- + 2\sum_{l=1}^{N_c}(x_k^- - a_l)\log(x_k^- - \bar{a}_l) \\ &-\sum_{j=1}^{N_f}(x_k^- + m_j)\log(x_k^- + m_j) \\ &= 2\log 2\, x_k^- - 2\log\bar{\Lambda}\, a_k + (N_f - 2N_c)\bar{a}_k - 2(x_k^- - \bar{a}_k) \\ &\quad + 2\sum_{l\neq k}(\bar{a}_k - a_l)\log(\bar{a}_k - \bar{a}_l) - \sum_{j=1}^{N_f}(\bar{a}_k + m_j)\log(\bar{a}_k + m_j) \\ &\quad + \bar{\Lambda}^2\delta_k^{(2)} - \bar{\Lambda}^2\delta_k^{(2)}\log\delta_k^{(1)} - \frac{3}{4}\bar{\Lambda}^4\delta_k^{(4)}\log\delta_k^{(1)} - \frac{4}{3}\bar{\Lambda}^3\delta_k^{(3)} + \frac{3}{2}\bar{\Lambda}^4\delta_k^{(4)} \\ &\quad + \bar{\Lambda}^3\frac{(\delta_k^{(2)})^2}{\delta_k^{(1)}} + \frac{1}{2}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2} - \bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}} \\ &\quad + \bar{\Lambda}^3\sum_{l\neq k}\frac{\delta_k^{(1)}\delta_l^{(2)}}{\bar{a}_k-\bar{a}_l} - \bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}\delta_l^{(2)}}{\bar{a}_k-\bar{a}_l} + \frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2\delta_l^{(2)}}{(\bar{a}_k-\bar{a}_l)^2}.\end{aligned} \tag{4.17}$$

In view of the identity

$$\begin{aligned}-\bar{\Lambda}^2\delta_k^{(2)}\log\delta_k^{(1)} - \frac{3}{4}\bar{\Lambda}^4\delta_k^{(4)}\log\delta_k^{(1)} &= -2(\bar{a}_k - a_k)\sum_{l\neq k}\log(\bar{a}_k - \bar{a}_l) \\ &\quad +(\bar{a}_k - a_k)\sum_{j=1}^{N_f}\log(\bar{a}_k + m_j)\end{aligned}$$

the preceding expression simplifies to

$$\left(-2\log\frac{\bar{\Lambda}}{2} - 2N_c + N_f\right)x_k^- + 2\sum_{l=1}^{N_c}(x_k^- - a_l)\log(x_k^- - \bar{a}_l)$$

$$
\begin{aligned}
&-\sum_{j=1}^{N_f}(x_k^- + m_j)\log(x_k^- + m_j) \\
&= 2\log 2x_k^- - 2\log\bar{\Lambda}\; a_k + (N_f - 2N_c)\bar{a}_k - 2(x_k^- - \bar{a}_k) \\
&\quad + 2\sum_{l\neq k}(a_k - a_l)\log(\bar{a}_k - \bar{a}_l) - \sum_{j=1}^{N_f}(a_k + m_j)\log(\bar{a}_k + m_j) \\
&\quad + \bar{\Lambda}^2\delta_k^{(2)} - \frac{4}{3}\bar{\Lambda}^3\delta_k^{(3)} + \frac{3}{2}\bar{\Lambda}^4\delta_k^{(4)} + \bar{\Lambda}^3\frac{(\delta_k^{(2)})^2}{\delta_k^{(1)}} + \frac{1}{2}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2} - \bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}} \\
&\quad + \bar{\Lambda}^3\sum_{l\neq k}\frac{\delta_k^{(1)}\delta_l^{(2)}}{\bar{a}_k - \bar{a}_l} - \bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}\delta_l^{(2)}}{\bar{a}_k - \bar{a}_l} + \frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2\delta_l^{(2)}}{(\bar{a}_k - \bar{a}_l)^2}.
\end{aligned}
\tag{4.18}
$$

We now replace $\bar{a}_k$ by a_k, using the expansion (3.12). The logarithms expand as

$$
\begin{aligned}
&2\sum_{l\neq k}(a_k - a_l)\log(\bar{a}_k - \bar{a}_l) - \sum_{j=1}^{N_f}(a_k + m_j)\log(\bar{a}_k + m_j) \\
&= 2\sum_{l\neq k}(a_k - a_l)\log(a_k - a_l) - \sum_{j=1}^{N_f}(a_k + m_j)\log(a_k + m_j) \\
&\quad - \frac{1}{2}\bar{\Lambda}^2\delta_k^{(2)}(2N_c - N_f) - \frac{3}{8}\bar{\Lambda}^4\delta_k^{(4)}(2N_c - N_f) \\
&\quad - \frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(2)} - \delta_l^{(2)})^2}{a_k - a_l} + \frac{1}{8}\bar{\Lambda}^4\sum_{j=1}^{N_f}\frac{(\delta_k^{(2)})^2}{a_k + m_j}.
\end{aligned}
\tag{4.19}
$$

Altogether, we have

$$
\begin{aligned}
&\left(-2\log\frac{\bar{\Lambda}}{2} - 2N_c + N_f\right)x_k^- + 2\sum_{l=1}^{N_c}(x_k^- - a_l)\log(x_k^- - \bar{a}_l) \\
&- \sum_{j=1}^{N_f}(x_k^- + m_j)\log(x_k^- + m_j) \\
&= 2(\log 2)x_k^- + (-2\log\bar{\Lambda} + N_f - 2N_c)a_k \\
&\quad + 2\sum_{l\neq k}(a_k - a_l)\log(a_k - a_l) - \sum_{j=1}^{N_f}(a_k + m_j)\log(a_k + m_j) \\
&\quad + 2\bar{\Lambda}\delta_k^{(1)} - \bar{\Lambda}^2\delta_k^{(2)} + \frac{2}{3}\bar{\Lambda}^3\delta_k^{(3)} - \frac{1}{2}\bar{\Lambda}^4\delta_k^{(4)} + \frac{3}{4}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2} + \bar{\Lambda}^3\frac{(\delta_k^{(2)})^2}{\delta_k^{(1)}} \\
&\quad - \bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}} - \frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}\delta_l^{(2)}}{\bar{a}_k - \bar{a}_l} - \frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(2)})^2}{\bar{a}_k - \bar{a}_l} + \bar{\Lambda}^3\sum_{l\neq k}\frac{\delta_k^{(1)}\delta_l^{(2)}}{\bar{a}_k - \bar{a}_l}
\end{aligned}
$$

$$+\frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2\delta_l^{(2)}}{\bar{a}_k-\bar{a}_l}. \tag{4.20}$$

4.5. Assembling all contributions

We are now ready to assemble all the components (4.7), (4.10) and (4.20) of $a_{D,k}$. Several cancellations take place in view of the first group of resummation identities in Appendix B, and we obtain

$$\begin{aligned}
2\pi i a_{D,k} = & -(2\log 2 - 2\log\bar{\Lambda} + N_f - 2N_c)a_k \\
& -2\sum_{l\neq k}(a_k-a_l)\log(a_k-a_l) + \sum_{j=1}^{N_f}(a_k+m_j)\log(a_k+m_j) \\
& +\frac{1}{2}\bar{\Lambda}^2\delta_k^{(2)} - \frac{1}{2}\bar{\Lambda}^2\sum_{l\neq k}\frac{(\delta_l^{(1)})^2}{\bar{a}_k-\bar{a}_l} - \frac{1}{8}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(2)})^2}{\bar{a}_k-\bar{a}_l} \\
& +\frac{7}{16}\bar{\Lambda}^4\delta_k^{(4)} - \frac{1}{8}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2} + \frac{1}{4}\bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}} \\
& +\frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}(\delta_l^{(1)})^2}{(\bar{a}_k-\bar{a}_l)^2} - \frac{1}{16}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(1)})^4}{(\bar{a}_k-\bar{a}_l)^3} \\
& -\frac{3}{8}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(1)})^2\delta_l^{(2)}}{(\bar{a}_k-\bar{a}_l)^2} - \frac{3}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_l^{(1)}\delta_l^{(3)}}{\bar{a}_k-\bar{a}_l}.
\end{aligned} \tag{4.21}$$

Our penultimate task, before we are in position to identify the prepotential, is to eliminate the $\bar{a}_l$'s, and rewrite the whole expression for $a_{D,k}$ entirely in terms of the renormalized a_l's. We observe in the above that the terms $\delta_l^{(p)}$ are written in terms of the $\bar{a}_m$'s. We temporarily stress this feature by denoting them by $\delta_l^{(p)}(\bar{a})$, while letting the simpler notation $\delta_l^{(p)}$ stand rather for their counterparts $\delta_l^{(p)}(a)$. The passage from $\delta_l^{(p)}(\bar{a})$ to $\delta_l^{(p)}(a) \equiv \delta_l^{(p)}$ is easily worked out for $p = 1, 2$, using the explicit expressions (B.10) and (B.11) for $\delta_l^{(p)}$,

$$\begin{aligned}
(\delta_l^{(1)}(\bar{a}))^2 &= (\delta_l^{(1)})^2 - \bar{\Lambda}^2(\delta_l^{(2)})^2 - \bar{\Lambda}^2\sum_{m\neq l}\frac{\delta_m^{(2)}(\delta_l^{(1)})^2}{a_l-a_m}, \\
\delta_l^{(2)}(\bar{a}) &= \delta_l^{(2)} + \frac{1}{2}\bar{\Lambda}^2\frac{(\delta_l^{(2)})^3}{(\delta_l^{(1)})^2} - \bar{\Lambda}^2\frac{\delta_l^{(2)}\delta_l^{(3)}}{\delta_l^{(1)}} - \bar{\Lambda}^2\sum_{m\neq l}\frac{\delta_l^{(2)}\delta_m^{(2)}}{a_l-a_m} \\
&\quad +\frac{1}{2}\bar{\Lambda}^2\sum_{m\neq l}\frac{(\delta_l^{(1)})^2\delta_m^{(2)}}{(a_l-a_m)^2}.
\end{aligned} \tag{4.22}$$

Applying (4.22), we can now express $a_{D,k}$ in terms of the $\delta_l^{(p)}(a)$'s,

$$2\pi i a_{D,k} = (2\log\bar{\Lambda} - 2\log 2 + 2N_c - N_f)a_k$$

$$-2\sum_{l\neq k}(a_k-a_l)\log(a_k-a_l)+\sum_{j=1}^{N_f}(a_k+m_j)\log(a_k+m_j)$$
$$-\frac{1}{2}\bar{\Lambda}^2\sum_{l\neq k}\frac{(\delta_l^{(1)})^2}{a_k-a_l}+\frac{1}{2}\bar{\Lambda}^2\delta_l^{(2)}+\frac{1}{8}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}+\frac{7}{16}\bar{\Lambda}^4\delta_k^{(4)}$$
$$-\frac{1}{4}\bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}}+\frac{3}{8}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(2)})^2}{a_k-a_l}-\frac{1}{8}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(1)})^2\delta_l^{(2)}}{(a_k-a_l)^2}$$
$$+\frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\sum_{m\neq l}\frac{\delta_m^{(2)}(\delta_l^{(1)})^2}{(a_l-a_m)(a_k-a_l)}-\frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}\delta_l^{(2)}}{a_k-a_l}$$
$$+\frac{1}{4}\sum_{l\neq k}\frac{(\delta_k^{(1)})^2\delta_l^{(2)}}{(a_k-a_l)^2}-\frac{1}{16}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(1)})^4}{(a_k-a_l)^3}-\frac{3}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_l^{(1)}\delta_l^{(2)}}{a_k-a_l}. \tag{4.23}$$

As was emphasized after Eq. (3.21), we recall that (4.23) incorporates implicitly a similar, opposite, contribution with the index k replaced by the index 1.

4.6. Integrability conditions and the prepotential

We show now how to use the summation identities in Appendix B and rearrange the terms so as to exhibit the prepotential. The fact that this can at all be done provides a powerful check on the final answer. As noted already before (cf. (3.22)) the classical part together with perturbative corrections $\mathcal{F}^{(0)}$ is easy to identify,

$$\mathcal{F}^{(0)}=\frac{1}{2\pi i}(2N_c-N_f-\log 2)\sum_{l=1}^{N_c}a_k^2-\frac{1}{8\pi i}\Bigg(\sum_{l,m=1}^{N_c}(a_l-a_m)^2\log\frac{(a_l-a_m)^2}{\Lambda^2}$$
$$-\sum_{l=1}^{N_c}\sum_{j=1}^{N_f}(a_l+m_j)^2\log\frac{(a_l+m_j)^2}{\Lambda^2}\Bigg). \tag{4.24}$$

Next, it follows immediately from the definition of $\delta_k^{(2)}$ and the summation identity (B.3) that the 1-instanton contribution to the prepotential is given by

$$\mathcal{F}^{(1)}=\frac{1}{8\pi i}\bar{\Lambda}^2\sum_{l=1}^{N_c}(\delta_l^{(1)})^2. \tag{4.25}$$

We shall identify the 2-instanton potential $\mathcal{F}^{(2)}$ in several steps. First, we use the identity (B.8) to rewrite the corresponding terms $a_{D,k}^{(2)}$ as

$$2\pi i a_{D,k}^{(2)}=\frac{1}{16}\bar{\Lambda}^4\frac{\partial}{\partial a_k}\left[\sum_{l\neq k}\sum_{m\neq l,k}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2}\right]-\frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2(\delta_l^{(1)})^2}{(a_k-a_l)^3}$$

$$-\frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_l^{(1)}\delta_l^{(3)}}{a_k-a_l}+\frac{1}{8}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(2)})^2}{a_k-a_l}-\frac{1}{8}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(1)})^2\delta_l^{(2)}}{(a_k-a_l)^2}$$
$$-\frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}\delta_l^{(2)}}{a_k-a_l}+\frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2\delta_l^{(2)}}{(a_k-a_l)^2}-\frac{1}{16}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_l^{(1)})^4}{(a_k-a_l)^3}$$
$$+\frac{1}{8}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}+\frac{7}{16}\bar{\Lambda}^4\delta_k^{(4)}-\frac{1}{4}\bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}}. \tag{4.26}$$

Use now the identities (B.5) and (B.6) to exhibit the terms involving $\delta_l^{(3)}$ and $(\delta_l^{(2)})^2$ in terms of a derivative with respect to a_k. We find

$$2\pi i a_{D,k}^{(2)}=\frac{1}{16}\bar{\Lambda}^4\frac{\partial}{\partial a_k}\left[\sum_{l\neq k}\sum_{m\neq l,k}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2}+\sum_{l\neq k}\left(\delta_l^{(1)}\delta_l^{(3)}-\frac{1}{2}(\delta_l^{(2)})^2\right)\right]$$
$$-\frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2(\delta_l^{(1)})^2}{(a_k-a_l)^3}+\frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2\delta_l^{(2)}}{(a_k-a_l)^2}-\frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}\delta_l^{(2)}}{a_k-a_l}$$
$$+\frac{1}{8}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}+\frac{7}{16}\bar{\Lambda}^4\delta_k^{(4)}-\frac{1}{4}\bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}}. \tag{4.27}$$

The identities (B.1) and (B.2) now apply to produce

$$2\pi i a_{D,k}^{(2)}=\frac{1}{16}\bar{\Lambda}^4\frac{\partial}{\partial a_k}\left[\sum_{l\neq k}\sum_{m\neq l,k}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2}+\sum_{l\neq k}\left(\delta_l^{(1)}\delta_l^{(3)}-\frac{1}{2}(\delta_l^{(2)})^2\right)\right]$$
$$-\frac{1}{2}\bar{\Lambda}^4\sum_{l\neq k}\frac{(\delta_k^{(1)})^2(\delta_l^{(1)})^2}{(a_k-a_l)^3}+\frac{1}{4}\bar{\Lambda}^4\sum_{l\neq k}\frac{\delta_k^{(2)}(\delta_l^{(1)})^2}{(a_k-a_l)^2}$$
$$+\frac{1}{8}\bar{\Lambda}^4\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}-\frac{1}{4}\bar{\Lambda}^4\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}}+\frac{3}{16}\bar{\Lambda}^4\delta_k^{(4)}. \tag{4.28}$$

The terms in the second row of (4.28) are just the ones occurring in the identity (B.7). As for the remaining local terms, they are also integrable, in view of the following simple local identities, which are a direct consequence of their definition (4.2):

$$\delta_k^{(4)}=\frac{1}{6}\frac{\partial}{\partial a_k}\left[(\delta_k^{(2)})^2+2\delta_k^{(1)}\delta_k^{(3)}\right], \tag{4.29}$$

$$\frac{\delta_k^{(2)}\delta_k^{(3)}}{\delta_k^{(1)}}-\frac{1}{2}\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^2}=\frac{1}{4}\frac{\partial}{\partial a_k}(\delta_k^{(2)})^2. \tag{4.30}$$

We can now write the 2-instanton contribution to $a_{D,k}$ under the form

$$2\pi i a_{D,k}^{(2)}=\frac{1}{16}\bar{\Lambda}^4\frac{\partial}{\partial a_k}\left[\sum_{l\neq m}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2}+\sum_{l=1}^{N_c}\left(\delta_l^{(1)}\delta_l^{(3)}-\frac{1}{2}(\delta_l^{(2)})^2\right)\right]. \tag{4.31}$$

4.7. Summary of result for the prepotential up to 2instanton contributions

We summarize here the main formulas, written in terms of the functions

$$S_k(x)=\frac{\prod_{j=1}^{N_f}(x+m_j)}{\prod_{l\neq k}(x-a_l)^2}. \tag{4.32}$$

The dual-order parameters $a_{D,k}$ are given by

$$a_{D,k}=\frac{\partial}{\partial a_k}\mathcal{F}(a)$$

with

$$\mathcal{F}=\mathcal{F}^{(0)}+\mathcal{F}^{(1)}+\mathcal{F}^{(2)}+O(\Lambda^{3(2N_c-N_f)})$$

and

$$\begin{aligned}2\pi i\mathcal{F}^{(0)}=&-\frac{1}{4}\sum_{l\neq k}(a_k-a_l)^2\log\frac{(a_k-a_l)^2}{\Lambda^2}\\&+\frac{1}{4}\sum_{j=1}^{N_f}\sum_{k=1}^{N_c}(a_k+m_j)^2\log\frac{(a_k+m_j)^2}{\Lambda^2}-(\log 2-2N_c+N_f)\sum_{k=1}^{N_c}a_k^2,\end{aligned} \tag{4.33a}$$

$$2\pi i\mathcal{F}^{(1)}=\frac{1}{4}\Lambda^{2N_c-N_f}\sum_{l=1}^{N_c}S_l(a_l), \tag{4.33b}$$

$$2\pi i\mathcal{F}^{(2)}=\frac{1}{16}\Lambda^{2(2N_c-N_f)}\left[\sum_{l\neq m}\frac{S_l(a_l)S_m(a_m)}{(a_l-a_m)^2}+\frac{1}{4}\sum_{l=1}^{N_c}S_l(a_l)\frac{\mathrm{d}^2S_l}{\mathrm{d}a_l^2}(a_l)\right]. \tag{4.33c}$$

The log 2 terms can be eliminated by a redefinition of the scale $\frac{\Lambda^{N_c-\frac{1}{2}N_f}}{2}\to\Lambda^{N_c-\frac{1}{2}N_f}$, upon which the prepotential takes the more familiar form

$$\begin{aligned}2\pi i\mathcal{F}^{(0)}=&-\frac{1}{4}\sum_{l\neq k}(a_k-a_l)^2\log\frac{(a_k-a_l)^2}{\Lambda^2}\\&+\frac{1}{4}\sum_{j=1}^{N_f}\sum_{k=1}^{N_c}(a_k+m_j)^2\log\frac{(a_k+m_j)^2}{\Lambda^2}+(2N_c-N_f)\sum_{k=1}^{N_c}a_k^2,\end{aligned} \tag{4.34a}$$

$$2\pi i\mathcal{F}^{(1)}=\Lambda^{2N_c-N_f}\sum_{l=1}^{N_c}S_l(a_l), \tag{4.34b}$$

$$2\pi i\mathcal{F}^{(2)}=\Lambda^{2(2N_c-N_f)}\left[\sum_{l\neq m}\frac{S_l(a_l)S_m(a_m)}{(a_l-a_m)^2}+\frac{1}{4}\sum_{l=1}^{N_c}S_l(a_l)\frac{\mathrm{d}^2S_l}{\mathrm{d}a_l^2}(a_l)\right]. \tag{4.34c}$$

By construction, these contributions to the effective prepotential are invariant under permutations of the variables a_k, and hence are invariant under the Weyl group of

SU(N_c). It is of course possible, though in general very cumbersome, to rewrite these results in terms of symmetric polynomials in the a_k, such as s_i and σ_i, defined in (2.4) and (2.5). However, the above results are perfectly well defined and invariant as they stand.

5. Special cases and discussion

In this section, we evaluate some of our results in various special cases discussed in the literature either directly from the quantum field theory point of view, using instanton calculations or from the Seiberg–Witten type approach.

5.1. Comparisons with quantum field theory instanton calculations

Results directly from instanton calculations in quantum field theory are available as follows. For the simplest case of two colors, $N_c = 2$, and $N_f < 2N_c$, various tests of the exact results of Ref. [1] were performed in Refs. [8,14], while 2-instanton results were obtained in Ref. [8], and are found to be in agreement with Ref. [1]. For the case of a general number of colors, N_c, results appear to be available only for contributions involving just a single instanton, and are found to agree with the exact results of Ref. [2].

5.2. $N_c = 2$ results

A useful check on the correctness of the coefficients in (4.33) is provided by a comparison with the exact results for $N_c = 2$ and $N_f = 0$. We set $\bar{a} = \bar{a}_1 = -\bar{a}_2$, $a = a_1 = -a_2$, $\bar{\Lambda} = \Lambda^2$ and we find from the definition of S_k that

$$\begin{aligned} S_1(a_1) = S_2(a_2) &= \frac{1}{4a^2}, \\ \frac{\partial^2 S_1(a_1)}{\partial a_1^2} = \frac{\partial^2 S_1(a_1)}{\partial a_1^2} &= \frac{3}{8a^6}. \end{aligned} \tag{5.1}$$

Using these results, we find the effective prepotential up to 2-instanton corrections

$$\mathcal{F} = \frac{i}{2\pi}\left[2a^2\log 4a^2 + (2\log 2 - 4\log\Lambda - 4)a^2 - \frac{\Lambda^4}{8a^2} - \frac{5\Lambda^8}{2^{10}a^6} + \mathcal{O}(\Lambda^{12})\right]. \tag{5.2}$$

This result agrees with the expansion of the exact results of Ref. [1] in powers of Λ, as calculated in Ref. [6], provided we make the redefinition $\Lambda^2 \longrightarrow \Lambda^2/2$.

It is not much more difficult to incorporate the effects of N_f matter hypermultiplets. We shall do this here for $N_f = 3$; the result for lower values of N_f may then be obtained by decoupling. We thus find for $N_f = 3$

$$
\begin{aligned}
2\pi i\mathcal{F}^{(1)} &= \frac{\Lambda_3}{8a^2}\left[a^2(m_1+m_2+m_3)+m_1m_2m_3\right],\\
2\pi i\mathcal{F}^{(2)} &= \frac{\Lambda_3^2}{2^{10}a^6}\left[a^6+a^4(m_1^2+m_2^2+m_3^2)-3a^2(m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2)\right.\\
&\quad\left.+5m_1^2m_2^2m_3^2\right]
\end{aligned}
\tag{5.3}
$$

Decoupling the third hypermultiplet, by letting $\Lambda_3 m_3 = \Lambda_2^2$, and sending $m_3 \to \infty$, we obtain for $N_f = 2$

$$
\begin{aligned}
2\pi i\mathcal{F}^{(1)} &= \frac{\Lambda_2^2}{8a^2}\left[a^2+m_1m_2\right],\\
2\pi i\mathcal{F}^{(2)} &= \frac{\Lambda_2^4}{2^{10}a^6}\left[a^4-3a^2(m_1^2+m_2^2)+5m_1^2m_2^2\right]
\end{aligned}
\tag{5.4}
$$

and decoupling also the second hypermultiplet, letting $\Lambda_2 m_2 = \Lambda_1^2$ as $m_2 \to \infty$, we find for $N_f = 1$

$$
\begin{aligned}
2\pi i\mathcal{F}^{(1)} &= \frac{\Lambda_1^3}{8a^2}m_1,\\
2\pi i\mathcal{F}^{(2)} &= \frac{\Lambda_1^6}{2^{10}a^6}\left[-3a^2+5m_1^2\right].
\end{aligned}
\tag{5.5}
$$

From the decoupling of the first hypermultiplet, with $\Lambda_1 m_1 = \Lambda^2$, as $m_1 \to \infty$, we recover the result of (5.2).

We notice that for $N_c = 2$, the expression for the renormalized order parameter a in terms of the classical value $\bar{a}$ is readily worked out to all orders with the help of the expression for $\delta_1^{(m)}$,

$$
\delta_1^{(m)} = (-)^m \frac{\Gamma(m-\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(m+1)}\frac{1}{\bar{a}^{2m-1}},
\tag{5.6}
$$

which yields, using (3.11)

$$
a = \bar{a}\sum_{m=0}^{\infty}\frac{\Gamma(2m-\frac{1}{2})}{2^{2m}\Gamma(-\frac{1}{2})\Gamma(m+1)^2}\left(\frac{\Lambda}{\bar{a}}\right)^{4m}.
\tag{5.7}
$$

This sum is easily recognized as the hypergeometric function $\bar{a}F(-\frac{1}{4},\frac{1}{4};1;\Lambda^4/\bar{a}^4)$, which is the correct expression, as obtained in Ref. [6].

5.3. $N_c = 3$ *results*

We can similarly work out the contributions to the prepotential from 1- and 2-instanton effects for $N_c = 3$. We have $a_1 + a_2 + a_3 = 0$, and it is customary to express the results in terms of the symmetric polynomials, u and v and the discriminant Δ

$$
u = -a_1a_2 - a_2a_3 - a_3a_1, \qquad v = a_1a_2a_3, \qquad \Delta = 4u^3 - 27v^3.
\tag{5.8}
$$

The functions $S_k(a_k)$ are easily evaluated and we have

$$S_k(a_k) = \frac{\prod_{j=1}^{N_f}(a_k + m_j)}{\prod_{l\neq k}(a_k - a_l)^2}. \tag{5.9}$$

Using (4.33), we find the 1-instanton results

$$\begin{aligned}
N_f = 5 \qquad & 2\pi i\mathcal{F}^{(1)} = \frac{\Lambda_5}{4\Delta}\left[-u^2 v + (2u^3 - 9v^2)t_1 - 3uvt_2 + 2u^2 t_3 - 9vt_4 + 6ut_5\right],\\
N_f = 4 \qquad & 2\pi i\mathcal{F}^{(1)} = \frac{\Lambda_4^2}{4\Delta}\left[(2u^3 - 9v^2) - 3uvt_1 + 2u^2 t_2 - 9vt_3 + 6ut_4\right],\\
N_f = 3 \qquad & 2\pi i\mathcal{F}^{(1)} = \frac{\Lambda_3^3}{4\Delta}\left[-3uv + 2u^2 t_1 - 9vt_2 + 6ut_3\right],\\
N_f = 2 \qquad & 2\pi i\mathcal{F}^{(1)} = \frac{\Lambda_2^4}{4\Delta}\left[2u^2 - 9vt_1 + 6ut_2\right],\\
N_f = 1 \qquad & 2\pi i\mathcal{F}^{(1)} = \frac{\Lambda_1^5}{4\Delta}\left[-9v + 6ut_1\right],\\
N_f = 0 \qquad & 2\pi i\mathcal{F}^{(1)} = \frac{6u\Lambda^6}{4\Delta}.
\end{aligned} \tag{5.10}$$

For the case $N_f = 0$, these results are in agreement with those of Ref. [6], derived using Picard-Fuchs equations.

Two-instanton effects may similarly be evaluated. For $N_f = 0$, we find that

$$2\pi i\mathcal{F}^{(2)} = \frac{9\Lambda^{12}}{16\Delta^3}(17u^3 + 189v^2), \tag{5.11}$$

in agreement with Ref. [6]. It is possible to express also the 2-instanton corrections for $N_f > 0$ in terms of the symmetric polynomials u, v and t_p; the expressions become quite cumbersome however, and it is much better to leave the results in the original forms of (4.33).

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Appendix A. Some useful numerical series

In this appendix, we derive the values of the series which are needed in the evaluation of the dual periods $a_{D,k}$. Consider the series

$$f_n^{(q)} = \sum_{m=q}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)}\frac{1}{2m-n}, \tag{A.1}$$

where q and n are some fixed integers satisfying $2q > n \geqslant 0$. Then we have

$$
\begin{aligned}
&f_0^{(1)} = \log 2, \qquad f_0^{(2)} = \log 2 - \frac{1}{4}, \\
&f_1^{(1)} = 1, \qquad f_1^{(2)} = \frac{1}{2}, \\
&f_2^{(2)} = \frac{1}{2}\log 2 + \frac{1}{4}, \\
&f_3^{(2)} = \frac{5}{6}, \\
&f_4^{(3)} = \frac{3}{8}\log 2 + \frac{9}{32}.
\end{aligned}
\tag{A.2}
$$

To see this, we note that $f_n^{(q)}$ is the value at $\xi = 1$ of the corresponding function

$$
f_n^{(q)}(\xi) = \sum_{m=q}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \frac{1}{2m-n} \xi^{2m},
$$

which is easily seen to admit an integral representation. In fact,

$$
\begin{aligned}
\xi^{n+1} \frac{d}{d\xi}\left(\xi^{-n} f_n^{(q)}(\xi)\right) &= \sum_{m=q}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \xi^{2m} \\
&= \frac{1}{\sqrt{1-\xi^2}} - \sum_{m=0}^{q-1} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \xi^{2m}.
\end{aligned}
$$

It follows that

$$
f_n^{(q)}(\xi) = \xi^n \int_0^{\xi} \frac{d\eta}{\eta^{n+1}} \left[\frac{1}{\sqrt{1-\eta^2}} - \sum_{m=0}^{q-1} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \eta^{2m} \right],
\tag{A.3}
$$

which is an absolutely convergent integral for $2q > n$. These integrals for $\xi = 1$ can be evaluated explicitly by a change of variables to $\eta = 2t(1+t^2)^{-1}$. We find, e.g.,

$$
f_n^{(2)}(1) = 2^{-n+1} \int_0^1 dt\, t^{-n+3}(t^2+3)(1+t^2)^{n-3},
$$

which easily leads the values indicated in (A.2) for $f_0^{(2)}$, $f_1^{(2)}$, $f_2^{(2)}$, and $f_3^{(2)}$ (and hence $f_0^{(1)}$ and $f_1^{(1)}$). For $f_4^{(3)}$, it is somewhat faster to make the change of variables $\eta = \sqrt{1-u^2}$, which leads to the integral of a rational fraction

$$
f_4^{(3)}(1) = \int_0^1 \frac{du}{(1-u^2)^3} \left[1 - u - \frac{1}{2}u(1-u^2) - \frac{3}{8}(1-u^2)^2 \right].
$$

This works out to the value given in (A.2).

Appendix B. Summation identities

The identities we need concern the sums over $l \neq k$ of expressions involving $(a_k - a_l)^{-p}$, and are of two types. The first leads to local terms involving only the $\delta_k^{(n)}$'s, while the second leads to total derivatives with respect to a_k of expressions which ultimately make up the prepotential. The identities of the first type of interest to us are the following:

$$\sum_{l\neq k}\left[\frac{(\delta_l^{(1)})^2}{(a_k-a_l)^2}+2\frac{\delta_l^{(2)}}{a_k-a_l}\right]=2\frac{\delta_k^{(3)}}{\delta_k^{(1)}}-\left(\frac{\delta_k^{(2)}}{\delta_k^{(1)}}\right)^2, \tag{B.1}$$

$$\sum_{l\neq k}\left[\frac{(\delta_l^{(1)})^2}{(a_k-a_l)^3}+\frac{\delta_l^{(2)}}{(a_k-a_l)^2}\right]=-\frac{\delta_k^{(4)}}{(\delta_k^{(1)})^2}+2\frac{\delta_k^{(2)}\delta_k^{(3)}}{(\delta_k^{(1)})^3}-\frac{(\delta_k^{(2)})^3}{(\delta_k^{(1)})^4}. \tag{B.2}$$

To show these identities, we observe that the pole expansion for $B(x)/C^2(x)$ can be rewritten as

$$\sum_{l\neq k}\left[\frac{S_l(\bar a_l)}{(x-\bar a_l)^2}+\frac{S_l'(\bar a_l)}{x-\bar a_l}\right]=\frac{1}{(x-\bar a_k)^2}\left(S_k(x)-S_k(\bar a_k)-S_k'(\bar a_k)(x-\bar a_k)\right).$$

Letting $x\to\bar a_k$ gives (B.1). Taking a higher order Taylor expansion for $S_k(x)$ and letting again $x\to\bar a_k$ after differentiation gives (B.2).

The identities of the second type depend on the explicit forms for $\delta_k^{(q)}$. They are

$$\sum_{l\neq k}\frac{(\delta_l^{(1)})^2}{a_k-a_l}=-\frac{1}{2}\frac{\partial}{\partial a_k}\sum_{l\neq k}(\delta_l^{(1)})^2, \tag{B.3}$$

$$\sum_{l\neq k}\frac{(\delta_l^{(1)})^4}{(a_k-a_l)^3}=-\frac{1}{6}\frac{\partial}{\partial a_k}\sum_{l\neq k}\frac{(\delta_l^{(1)})^4}{(a_k-a_l)^2}, \tag{B.4}$$

$$\sum_{l\neq k}\left[\frac{(\delta_l^{(1)})^2\delta_l^{(2)}}{(a_k-a_l)^2}+2\frac{(\delta_l^{(2)})^2}{a_k-a_l}\right]=-\frac{1}{2}\frac{\partial}{\partial a_k}\sum_{l\neq k}(\delta_l^{(2)})^2, \tag{B.5}$$

$$\sum_{l\neq k}\left[\frac{(\delta_l^{(1)})^4}{(a_k-a_l)^3}+3\frac{(\delta_l^{(1)})^2\delta_l^{(2)}}{(a_k-a_l)^2}+4\frac{\delta_l^{(1)}\delta_l^{(3)}}{a_k-a_l}\right]=-\frac{\partial}{\partial a_k}\sum_{l\neq k}\delta_l^{(1)}\delta_l^{(3)}, \tag{B.6}$$

$$\sum_{l\neq k}\left[2\frac{(\delta_l^{(1)})^2\delta_k^{(2)}}{(a_k-a_l)^2}-4\frac{(\delta_l^{(1)})^2(\delta_k^{(1)})^2}{(a_k-a_l)^3}\right]=\frac{\partial}{\partial a_k}\sum_{l\neq k}\frac{(\delta_l^{(1)})^2(\delta_k^{(1)})^2}{(a_k-a_l)^2}, \tag{B.7}$$

$$\sum_{l\neq k}\sum_{m\neq l}\frac{\delta_m^{(2)}(\delta_l^{(1)})^2}{(a_l-a_m)(a_k-a_m)}=\frac{1}{8}\frac{\partial}{\partial a_k}\sum_{l\neq k}\sum_{m\neq l,k}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2}-\frac{1}{2}\sum_{l\neq k}\frac{(\delta_l^{(1)})^2(\delta_k^{(1)})^2}{(a_k-a_l)^3}$$

$$+\sum_{l\neq k}\frac{\delta_l^{(1)}\delta_l^{(3)}}{a_k-a_l}-\frac{1}{2}\sum_{l\neq k}\frac{(\delta_l^{(1)})^2}{a_k-a_l}. \tag{B.8}$$

We begin by establishing (B.4). The left-hand side of the identity can be rewritten as

$$\sum_{l\neq k}\frac{1}{(a_k-a_l)^3}\frac{\prod_{j=1}^{N_f}(a_l+m_j)^2}{\prod_{m\neq l}(a_l-a_m)^4}=\sum_{l\neq k}\frac{1}{(a_k-a_l)^7}\frac{\prod_{j=1}^{N_f}(a_l+m_j)^2}{\prod_{m\neq l,k}(a_l-a_m)^4}.$$

This is the same as

$$-\frac{1}{6}\frac{\partial}{\partial a_k}\sum_{l\neq k}\sum_{l\neq k}\frac{1}{(a_k-a_l)^6}\frac{\prod_{j=1}^{N_f}(a_l+m_j)^2}{\prod_{m\neq l,k}(a_l-a_m)^4}=-\frac{1}{6}\sum_{l\neq k}\frac{1}{(a_k-a_l)^2}(\delta_l^{(1)})^4,$$

as claimed. The identity (B.3) is proven in the same way. Next, we note that for $l\neq k$

$$\frac{\partial}{\partial a_k}\delta_l^{(p)}=\frac{1}{(p-1)!}\frac{\partial^{p-1}}{\partial a_l^{p-1}}\left\{\frac{1}{a_l-a_k}(\delta_l^{(1)})^p\right\}. \tag{B.9}$$

For $p=1$, this follows by direct calculation from the explicit formula for $\delta_l^{(1)}$

$$\delta_l^{(1)}=\frac{1}{a_l-a_k}\frac{\prod_{j=1}^{N_f}(a_l+m_j)^{\frac{1}{2}}}{\prod_{m\neq l,k}(a_l-a_m)}. \tag{B.10}$$

The case of general p follows next from the definition (B.9) for $\delta_l^{(p)}$, which expresses $\delta_l^{(p)}$ in terms of $\delta_l^{(1)}$. The identities (B.5), (B.6), and (B.7) are now easy to establish, simply by differentiating the right-hand side, and applying (B.9).

Finally, we consider (B.8). We may use first the identity to eliminate $\delta_m^{(2)}$ in favor of $\delta_m^{(1)}$,

$$\sum_{l\neq k}\sum_{m\neq l}\frac{\delta_m^{(2)}(\delta_l^{(1)})^2}{(a_l-a_m)(a_k-a_l)}=-\frac{1}{2}\sum_{l\neq k}\sum_{m\neq l}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2(a_k-a_l)}$$
$$+\sum_{l\neq k}\frac{\delta_l^{(1)}\delta_l^{(3)}}{a_k-a_l}-\frac{1}{2}\sum_{l\neq k}\frac{(\delta_l^{(2)})^2}{a_k-a_l}. \tag{B.11}$$

The double sum in the right-hand side can be rewritten as

$$\sum_{l\neq k}\sum_{m\neq l}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2(a_k-a_l)}=\sum_{l\neq k}\frac{(\delta_k^{(1)})^2(\delta_l^{(1)})^2}{(a_k-a_l)^3}$$
$$+\sum_{l\neq k}\sum_{m\neq l,k}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2(a_k-a_l)}.$$

The second term above can now be recognized as a total derivative

$$\sum_{l\neq k}\sum_{m\neq l}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2(a_k-a_l)}=-\frac{1}{4}\frac{\partial}{\partial a_k}\sum_{l\neq k}\sum_{m\neq l}\frac{(\delta_m^{(1)})^2(\delta_l^{(1)})^2}{(a_l-a_m)^2},$$

where we made use of the symmetry in l and m of all the expressions involved. The identity (B.8) follows.

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