Functional Analysis and Its Applications, Vol. 31, No. 1, 1997

Algebraic-Geometric n-Orthogonal Curvilinear Coordinate Systems and Solutions of the Associativity Equations

I. M. Krichever

UDC 517.9

§1. Introduction

The problem of constructing n-orthogonal curvilinear coordinate systems, or *flat* diagonal metrics

$$ds^2 = \sum_{i=1}^{n} H_i^2(u)(du^i)^2, \qquad u = (u^1, \dots, u^n), \tag{1.1}$$

for more than a century since the famous work of Dupin and Binet published in 1810 was one of the most important problems of differential geometry. Treated as a classification problem, it was mainly solved in the beginning of the 20th century. The crucial contribution here was due to G. Darboux [1].

In the beginning of the 1980s, it was found that this classical problem has deep connections with the modern theory of integrable quasilinear hydrodynamic type systems in $(1+1)$-dimensions [2–4]. This theory was proposed by B. Dubrovin and S. Novikov as a Hamiltonian theory of the averaged (Whitham) equations for periodic solutions of integrable soliton equations in $(1+1)$-dimensions. Later, it was noticed [5] that the classification of Egoroff metrics, i.e., flat diagonal metrics such that

$$\partial_j H_i^2 = \partial_i H_j^2, \qquad \partial_i = \partial/\partial u^i, \tag{1.2}$$

is equivalent to the classification problem for massive topological field theories. Note that (1.2) implies that there exists a function $\Phi(u)$, called a *potential* of the corresponding metric, such that $H_i^2(u) = \partial_i \Phi(u)$. We must point out that the "classical" results in the theory of n-orthogonal curvilinear systems are mainly of classification nature. It was shown that locally the general solution of the Lamé equations

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj}, \qquad i \neq j \neq k, \tag{1.3}$$

$$\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{m \neq i,j} \beta_{mi}\beta_{mj} = 0, \qquad i \neq j, \tag{1.4}$$

for the *rotation coefficients*

$$\beta_{ij} = \partial_i H_j / H_i, \qquad i \neq j, \tag{1.5}$$

depends on $n(n-1)/2$ arbitrary functions of two variables. System (1.3), (1.4) is equivalent to the vanishing of all a priori nontrivial components of the curvature tensor. (Equations (1.3) imply that $R_{ij,ik} = 0$, and Eqs. (1.4) imply $R_{ij,ij} = 0$ for all other coefficients.)

If we know a solution of (1.3), (1.4), then the Lamé coefficients H_i can be found from the linear equations (1.5), whose consistency is equivalent to (1.3). The Lamé coefficients depend on n functions of one variable, namely, on the Cauchy data

$$f_i(u^i) = H_i(0, \dots, 0, u^i, 0, \dots, 0) \tag{1.6}$$

for system (1.5). Then we can find flat coordinates $x^k(u)$ by solving the linear system

$$\partial_{ij}^2 x^k = \Gamma_{ij}^i \partial_i x^k + \Gamma_{ji}^j \partial_j x^k, \tag{1.7}$$

$$\partial_{ii}^2 x^k = \sum_{j=1}^{n} \Gamma_{ii}^j \partial_j x^k, \tag{1.8}$$

Department of Mathematics of Columbia University and Landau Institute for Theoretical Physics. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 31, No. 1, pp. 32–50, January–March, 1997. Original article submitted December 23, 1996.

0016-2663/97/3101-0025 \$18.00

where the Γ^k_{ij} are the Christoffel coefficients of the metric (1.1),

$$\Gamma^i_{ik} = \partial_k H_i / H_i , \qquad \Gamma^j_{ii} = -H_i \partial_i H_i / H_j^2 , \quad i \neq j . \tag{1.9}$$

This scheme is not very effective in constructing n-orthogonal coordinate systems explicitly, and so the list of known exact examples had been relatively short until a number of new examples were recently obtained from the Whitham theory. In particular, the author [6] showed that the moduli spaces of algebraic curves with given jets of local coordinates at the punctures generate flat diagonal metrics.

Quite recently, solutions of (1.3) and (1.4) have been constructed by V. Zakharov [7] with the help of the "dressing procedure" within the framework of the inverse problem method. Equations (1.3) are equivalent to the consistency conditions for the auxiliary linear system $\partial_i \Psi_j = \beta_{ij} \Psi_i$, $i \neq k$. Therefore, any inverse method scheme can be relatively easily adapted to the construction of various classes of exact solutions of Eq. (1.3). For example, one can use the dressing scheme or the algebraic-geometric constructions of the theory of finite-gap solutions of nonlinear equations. The crucial step is to select solutions that satisfy the constraints (1.4). As was shown in [8], the *differential reduction* proposed in [7] for solving this problem in the case of the dressing scheme admits a natural interpretation in terms of the so-called $\bar{\partial}$-problem.

The main goal of this paper is not merely constructing finite-gap or algebraic-geometric solutions of the Lamé equations (1.3), (1.4) but proposing a scheme that simultaneously solves the complete system (1.3)–(1.9), i.e., gives both the Lamé coefficients H_i and the flat coordinates $x^i(u)$.

At first glance, it seems that our approach is completely different from that proposed in [7, 8]. We consider the basic *multi-point* Baker–Akhiezer functions $\psi(u, Q)$, which are uniquely determined by their analytic properties on auxiliary Riemann surfaces Γ, $Q \in \Gamma$, and directly prove (without any use of differential equations!) that under certain constraints on the corresponding set of algebraic-geometric data, the values $x^k(u) = \psi(u, Q_k)$ of ψ on the set of punctures on Γ satisfy the equations

$$\sum_{k,l} \eta_{kl} \partial_i x^k(u) \partial_j x^l(u) = H_i^2(u) \delta_{ij} , \tag{1.10}$$

where η_{kl} is a constant matrix. Therefore, the $x^k(u)$ are flat coordinates for the diagonal metric (1.1) with coefficients $H_i^2(u)$. It turns out that, up to constant factors, the Lamé coefficients $H_i(u)$ are equal to the leading terms of the expansion of the same function ψ at the points P_i on Γ where ψ has exponential type singularities. We must point out that our constraints on the algebraic-geometric data that lead to (1.10) are a generalization of the conditions proposed in [15] for the description of potential two-dimensional Schrödinger operators (see also [16]).

In §3, we relate our results to the approach of [7, 8] and show that ψ is a *generating* function,

$$\partial_i \psi(u, Q) = h_i(u) \Psi_i^0(u, Q), \quad H_i = \varepsilon_i h_i(u), \quad \varepsilon_i = \text{const},$$

for the solutions of the system

$$\partial_i \Psi_j^0 = \beta_{ji} \Psi_i^0 , \quad \partial_i \Psi_j^1 = \beta_{ij} \Psi_i^1 , \quad \partial_j \Psi_j^0 = \Psi_j^1 - \sum_{m \neq j} \beta_{mj} \Psi_m^0 . \tag{1.11}$$

Note that the consistency conditions for this extended linear system are equivalent to (1.3) and (1.4).

In §4, we specify the algebraic-geometric data corresponding to Egoroff metrics and obtain an exact formula in terms of Riemann theta functions for the potentials $\Phi(u)$ of such metrics.

As was mentioned above, the relationship between the classification problem for Egoroff metrics and that for topological field theories was found in [5]. The latter problem for a theory with n primary fields $\phi_1, \ldots, \phi_n$ can be stated in terms of the *associativity* equations for the partition function $F(x_1, \ldots, x_n)$ of the deformed theory [9, 10]. These equations are the conditions that the commutative algebra with generators ϕ_k and structure constants defined by the third derivatives of F,

$$c_{klm}(x) = \frac{\partial^3 F(x)}{\partial x^k \partial x^l \partial x^m} , \tag{1.12}$$

$$\phi_k \phi_l = c^m_{kl}(x) \phi_m , \quad c^m_{kl} = c_{kli} \eta^{im} , \quad \eta_{ki} \eta^{im} = \delta^m_k , \tag{1.13}$$

is an associative algebra, that is, satisfies

$$c_{ij}^k(x)c_{km}^l(x) = c_{jm}^k(x)c_{ik}^l(x) . \tag{1.14}$$

In addition, it is required that there exist constants r^m such that the entries of the constant matrix η in (1.13) are equal to

$$\eta_{kl} = r^m c_{klm}(x) . \tag{1.15}$$

Conditions (1.14) form an overdetermined nonlinear system for the unknown function F. It turns out that for any solution of system (1.14), (1.15) for the case in which the algebra (1.13) is semisimple, there exists an Egoroff metric such that the third derivatives of the partition function have the form

$$c_{klm} = \sum_{i=1}^{n} H_i^2 \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l} \frac{\partial u^i}{\partial x^m} . \tag{1.16}$$

The converse is also true. Namely, for any set of rotation coefficients $\beta_{ij} = \beta_{ji}$ satisfying (1.3), (1.4), there exists an n-parameter family of Egoroff metrics such that the functions defined by (1.16) are the third derivatives of some function F. (Recall that for any given rotation coefficients there are infinitely many corresponding flat diagonal metrics.)

In the last section, for each algebraic-geometric Egoroff metric we define a function F such that its third derivatives have the form (1.16) and satisfy (1.14). Equations (1.14) are a truncated set of the associativity conditions. At the next stage, we select metrics that additionally satisfy (1.15).

§2. Bilinear Relations for the Baker–Akhiezer Functions and Flat Diagonal Metrics

First, let us present some facts from the general algebraic-geometric integration scheme proposed by the author [11, 12]. This scheme is based on the notion of the *Baker–Akhiezer* functions, which are determined by their analytic properties on auxiliary Riemann surfaces.

Let Γ be a smooth genus g algebraic curve with fixed local coordinates $w_i(Q)$ in neighborhoods of n punctures P_i, $i = 1, \dots, n$, on Γ, $w_i(P_i) = 0$. Then for any set R of l points R_α, $\alpha = 1, \dots, l$, and for any set D of $g + l - 1$ points $\gamma_1, \dots, \gamma_{g+l-1}$ in general position there exists a unique function $\psi(u, Q \,|\, D, R)$, $u = (u_1, \dots, u_n)$, $Q \in \Gamma$, such that:

(1^0) $\psi(u, Q \,|\, D, R)$ treated as a function of the variable $Q \in \Gamma$ is meromorphic outside the punctures P_j and has at most simple poles at the points γ_s (if they all are distinct);

(2^0) in a neighborhood of each P_j, the function ψ has the form

$$\psi = e^{u^j w_j^{-1}} \left(\sum_{s=0}^{\infty} \xi_s^j(u) w_j^s \right), \qquad w_j = w_j(Q); \tag{2.1}$$

(3^0) ψ satisfies the normalization conditions

$$\psi(u, R_\alpha) = 1 . \tag{2.2}$$

In the following we often denote the Baker–Akhiezer function by $\psi(u, Q)$ without explicitly indicating the divisors $D = \gamma_1 + \dots + \gamma_{g+l-1}$ and $R = R_1 + \dots + R_l$.

Explicit expressions of the Baker–Akhiezer functions via the Riemann theta functions were proposed in [12] as a generalization of the formula found in [13] for the Bloch solutions of ordinary finite-gap Schrödinger operators.

The Riemann theta function corresponding to an algebraic genus g curve Γ is the entire function of g complex variables $z = (z_1, \dots, z_g)$ defined by the Fourier series

$$\theta(z_1, \dots, z_g) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i (m, z) + \pi i (Bm, m)} ,$$

where $B=(B_{ij})$ is the matrix of b-periods of the normalized holomorphic differentials $\omega_j(P)$ on Γ,

$$B_{ij}=\oint_{b_i}\omega_j, \qquad \oint_{a_j}\omega_i=\delta_{ij}.$$

Here a_i, b_i is a basis of cycles on Γ with canonical matrix of intersections given by $a_i\cdot a_j=b_i\cdot b_j=0$, $a_i\cdot b_j=\delta_{ij}$.

The vector $A(P)$ with coordinates $A_k(Q)=\int_{q_0}^{Q}\omega_k$ defines the *Abel map.*

By the Riemann–Roch theorem, for any divisors $D=\gamma_1+\dots+\gamma_{g+l-1}$ and $R=R_1+\dots+R_l$ in general position there exists a unique meromorphic function $r_\alpha(Q)$ such that the divisor of its poles coincides with D and $r_\alpha(R_\beta)=\delta_{\alpha,\beta}$. This function can be represented in the form (see [14] for details)

$$r_\alpha(Q)=\frac{f_\alpha(Q)}{f_\alpha(R_\alpha)}, \qquad f_\alpha(Q)=\theta(A(Q)+Z_\alpha)\frac{\prod_{\beta\neq\alpha}\theta(A(Q)+F_\beta)}{\prod_{m=1}^{l}\theta(A(Q)+S_m)}, \tag{2.3}$$

where

$$F_\beta=-\mathscr{K}-A(R_\beta)-\sum_{s=1}^{g-1}A(\gamma_s), \qquad S_m=-\mathscr{K}-A(\gamma_{g-1+m})-\sum_{s=1}^{g-1}A(\gamma_s),$$

$$Z_\alpha=Z_0-A(R_\alpha), \qquad Z_0=-\mathscr{K}-\sum_{s=1}^{g+l-1}A(\gamma_s)+\sum_{\alpha=1}^{l}A(R_\alpha),$$

and $\mathscr{K}$ is the vector of the Riemann constants.

Let $d\Omega_j$ be the unique normalized meromorphic differential that is holomorphic on Γ outside P_j and has the form $d\Omega_j=d(w_j^{-1}+O(w_j))$ in a neighborhood of P_j. This differential specifies the vector $V^{(j)}$ with coordinates $V_k^{(j)}=\frac{1}{2\pi i}\oint_{b_k}d\Omega_j$.

Theorem 2.1. *The Baker-Akhiezer function $\psi(u,Q\,|\,D,R)$ has the form*

$$\psi=\sum_{\alpha=1}^{l}r_\alpha(Q)\frac{\theta(A(Q)+\sum_{i=1}^{n}(u^iV^{(i)})+Z_\alpha)\theta(Z_0)}{\theta(A(Q)+Z_\alpha)\theta(\sum_{i=1}^{n}(u^iV^{(i)})+Z_0)}\exp\left(\sum_{i=1}^{n}u^i\int_{R_\alpha}^{Q}d\Omega_i\right).$$

Admissible curves. We shall show that algebraic-geometric flat diagonal metrics can be constructed with the help of the Baker–Akhiezer functions corresponding to algebraic-geometric data of a special class, which will be referred to as *admissible.*

An admissible algebraic curve Γ must be a curve with a holomorphic involution $\sigma\colon\Gamma\to\Gamma$ that has $2m\geq n$ fixed points $P_1,\dots,P_n,Q_1,\dots,Q_{2m-n}$, $m\leq n$. The local coordinates $w_j(Q)$ in neighborhoods of $P_1,\dots,P_n$ must be odd with respect to σ,

$$w_j(Q)=-w_j(\sigma(Q)).$$

The factor curve $\Gamma_0=\Gamma/\sigma$ is a smooth algebraic curve. The projection $\pi\colon\Gamma\to\Gamma_0=\Gamma/\sigma$ represents Γ as a two-sheet covering of Γ_0 with $2m$ branching points P_j, Q_s. In this representation, the involution σ is the permutation of the sheets. For $Q\in\Gamma$, we write $\sigma(Q)=Q^\sigma$.

It follows from the Riemann–Hurwitz formula that $g=2g_0-1+m$, where g_0 is the genus of Γ_0.

Admissible divisors. Let us choose $n-m$ additional punctures $\widehat{Q}_1,\dots,\widehat{Q}_{n-m}$ on Γ_0. A pair (D,R) of divisors on Γ is said to be *admissible* if there exists a meromorphic differential $d\Omega_0$ on Γ_0 such that

(a) $d\Omega_0(P)$, $P\in\Gamma_0$, has $m+l$ simple poles at the points $Q_1,\dots,Q_{2m-n}$, $\widehat{Q}_1,\dots,\widehat{Q}_{n-m}$ and at the points $\widehat{R}_\alpha=\pi(R_\alpha)$;

(b) $d\Omega_0$ is zero at the projection $\widehat{\gamma}_s$ of the points of D,

$$d\Omega_0(\widehat{\gamma}_s)=0, \qquad \widehat{\gamma}_s=\pi(\gamma_s).$$

The differential $d\Omega_0$ can be treated as an even (with respect to σ) meromorphic differential on Γ, where it has $n+2l$ simple poles at the branching points $Q_1, \ldots, Q_{2m-n}$ and at the preimages of the other poles of $d\Omega_0$ on Γ_0. Let us denote the preimages of the points $\widehat{Q}_k$ by $Q_{2m-n+1}, \ldots, Q_{2m}$,

$$\pi(Q_{2m-n+i}) = \pi(Q_{n-i+1}) = \widehat{Q}_i, \qquad i = 1, \ldots, n-m.$$

The involution σ induces an involution $\sigma(k)$ on the set of indices numbering the punctures Q_k so that $\sigma(Q_k) = Q_{\sigma(k)}$. We have

$$\sigma(k) = k, \quad k = 1, \ldots, 2m-n, \qquad \sigma(k) = 2m-k+1, \quad k = 2m-n+1, \ldots, n.$$

In terms of equivalence classes, admissible pairs (D, R) of divisors can be described as those satisfying the condition

$$D + D^\sigma - R - R^\sigma = K + \sum_{j=1}^{n} (Q_j - P_j).$$

Example. *Hyperelliptic curves.* The simplest example of an admissible curve is the hyperelliptic curve Γ defined by the equation

$$\lambda^2 = \frac{\prod_{j=1}^{2m-n}(E - Q_j)\prod_{k=1}^{n-m}(E - \widehat{Q}_k)^2}{\prod_{i=1}^{n}(E - P_i)}, \qquad m \leq n. \tag{2.4}$$

Here the P_i, Q_j, and $\widehat{Q}_k$ are complex numbers. The genus of Γ is $g = m-1$. Any set of $m+l-2$ points $\gamma_s \neq \gamma_{s'}$ and any set of l points form an admissible pair of divisors. The corresponding differential is equal to

$$d\Omega_0 = \frac{\prod_{s=1}^{m+l-2}(E - \gamma_s)}{\prod_{j=1}^{2m-n}(E - Q_j)\prod_{k=1}^{n-m}(E - \widehat{Q}_k)\prod_{\alpha=1}^{l}(E - R_\alpha)}\, dE.$$

As we shall see in the following, the flat diagonal metrics corresponding to hyperelliptic curves are Egoroff metrics. Moreover, it will be shown in the last section that hyperelliptic curves correspond to the simplest solutions of the associativity equations.

Important remark. Unless otherwise specified, in the following main part of the paper, we assume for simplicity of the formulas that the divisors R and $\{Q_j\}$ are in general position and do not intersect each other. We consider the special case $R = \{Q_j\}$ at the end of the last section.

Theorem 2.2. *Let $\psi(u, Q \mid D, R)$ be the Baker–Akhiezer function corresponding to an admissible algebraic curve and an admissible pair (D, R) of divisors. Then the functions $x^j(u) = \psi(u, Q_j)$, $j = 1, \ldots, n$, satisfy the equations*

$$\sum_{k,l} \eta_{kl}\partial_i x^k \partial_j x^l = \varepsilon_i^2 h_i^2 \delta_{ij}, \tag{2.5}$$

where the $h_i = \xi_0^i(u)$ are the first coefficients in the expansions (2.1) of ψ at the punctures P_i; the constants ε_i^2 are defined by the expansion

$$d\Omega_0 = \tfrac{1}{2}(\varepsilon_i^2 + O(w_i^0))\, dw_i^0 = w_i(\varepsilon_i^2 + O(w_i^2))\, dw_i \tag{2.6}$$

of $d\Omega_0$ at P_i, and the constants η_{kl} are given by

$$\eta_{kl} = \eta_k \delta_{k,\sigma(l)}, \qquad \eta_k = \operatorname*{res}_{Q_k} d\Omega_0. \tag{2.7}$$

Proof. Let us consider the differential

$$d\Omega_{ij}^{(1)}(u, Q) = \partial_i \psi(u, Q)\partial_j \psi(u, \sigma(Q))\, d\Omega_0(\pi(Q)).$$

It follows from the definition of the admissible data that this differential for $i \neq j$ is a meromorphic differential with poles only at the points $Q_1, \dots, Q_n$. Indeed, the poles of the first two factors $\partial_i \psi_i(u, Q)$ and $\partial_j \psi(u, \sigma(Q))$ at the points γ_s and $\sigma(\gamma_s)$ are canceled by zeros of $d\Omega_0$. The essential singularities of these factors at P_k cancel each other. The simple poles of the product of these factors at P_i and P_j are canceled by zeros of $d\Omega_0$ treated as a differential on Γ (see (2.6)). Finally, $d\Omega^{(1)}$ has no poles at the points R_α and R_α^σ by virtue of the normalization conditions (2.2). The sum of all the residues of a meromorphic differential on a compact Riemann surface is equal to zero. Therefore,

$$\sum_{k=1}^{n} \operatorname*{res}_{Q_k} d\Omega_{ij}^{(1)} = 0, \qquad i \neq j.$$

The left-hand side of this equation coincides with the left-hand side of (2.5).

In the case $i = j$, the differential $d\Omega_{ii}^{(1)}$ has an additional pole at P_i with residue $\operatorname{res}_{P_i} d\Omega_{ii}^{(1)} = -\varepsilon_i^2 h_i^2$. That implies (2.5) for $i = j$ and completes the proof of the theorem.

Corollary 2.1. *Let* $\{\Gamma, P_i, Q_j, D, R\}$ *be a set of admissible data. Then the formula*

$$H_i(u) = \varepsilon_i \sum_{\alpha=1}^{l} r_\alpha(P_i) \frac{\theta(A(P_i) + \sum_{i=1}^{n}(u^i V^{(i)}) + Z_\alpha)\theta(Z_0)}{\theta(A(P_i) + Z_\alpha)\theta(\sum_{i=1}^{n}(u^i V^{(i)}) + Z_0)} \exp\left(\sum_{j=1}^{n} \omega_{ij}^\alpha u^j\right), \tag{2.8}$$

where $r_\alpha(Q)$ *is the function defined in* (2.3) *and*

$$\omega_{ij}^\alpha = \int_{R_\alpha}^{P_i} d\Omega_j, \quad i \neq j, \qquad \omega_{ii}^\alpha = \lim_{Q \to P_i} \left(\int_{R_\alpha}^{Q} d\Omega_i - w_i^{-1}(Q)\right),$$

defines the coefficients of a flat diagonal metric. The corresponding flat coordinates are given by the formulas

$$x^k(u) = \sum_{\alpha=1}^{l} r_\alpha(Q_k) x_\alpha^k(u),$$

$$x_\alpha^k(u) = \frac{\theta(A(Q_k) + \sum_{i=1}^{n}(u^i V^{(i)}) + Z_\alpha)\theta(Z_0)}{\theta(A(Q_k) + Z_\alpha)\theta(\sum_{i=1}^{n}(u^i V^{(i)}) + Z_0)} \exp\left(\sum_{i=1}^{n} u^i \int_{R_\alpha}^{Q_k} d\Omega_i\right).$$

Conditions for the metric coefficients to be real-valued. In the general case, the above-constructed flat diagonal metrics $H_i(u)$ and their flat coordinates are complex meromorphic functions of the variables u^i. Let us find conditions on the algebraic-geometric data such that the coefficients of the corresponding metrics are *real* functions of the *real* variables u^i.

Let Γ_0 be a real algebraic curve, i.e., a curve with an antiholomorphic involution $\tau_0 \colon \Gamma_0 \to \Gamma_0$, and let the punctures $\{P_1, \dots, P_n\}$ and $\{Q_1, \dots, Q_{2m-n}\}$ be fixed points of τ_0. Then τ_0 induces an antiholomorphic involution τ on Γ. We assume that the local coordinates w_j at P_j satisfy $w_j(\tau(Q)) = \overline{w_j(Q)}$. Let us assume that the set $\{\widehat{Q}_k\}$ and the divisors D and R are invariant with respect to τ, i.e.,

$$\tau(Q_j) = Q_{\kappa(j)}, \quad \tau(R_\alpha) = R_{\kappa_1(\alpha)}, \quad \tau(\gamma_s) = \gamma_{\kappa_2(s)},$$

where the $\kappa_i(\cdot)$ are the corresponding permutations of indices.

Theorem 2.3. *Let the set of admissible data be real. Then the Baker–Akhiezer function satisfies the relation*

$$\psi(u, Q \mid D, R) = \overline{\psi(u, \tau(Q) \mid D, R)},$$

and formula (2.8) *defines a real flat diagonal metric.*

The signature of the corresponding metric depends on the involutions $\kappa(j)$ and $\kappa_1(\alpha)$. By varying the initial data, one can obtain flat diagonal metrics in any pseudo-Euclidean spaces $R^{p,q}$. In general, these metrics are singular for some values of the variables u^i. To obtain smooth metrics for all u, we have to impose additional constraints on the initial data. This procedure is quite standard in the finite-gap theory and will be considered elsewhere.

§3. Differential Equations for the Baker–Akhiezer Function

In this section, we are going to clarify the meaning of our constraints on the algebraic-geometric data in terms of differential equations for the Baker–Akhiezer functions.

The following statement is a simple generalization of the results of [17], where it was shown in the case $n = 2$ that the corresponding Baker–Akhiezer function satisfies a two-dimensional Schrödinger type equation.

Lemma 3.1. *The Baker-Akhiezer function $\psi(u, Q \mid D, R)$ satisfies the equation*

$$\partial_i \partial_j \psi = c^i_{ij} \partial_i \psi + c^j_{ij} \partial_j \psi, \qquad i \neq j, \tag{3.1}$$

where

$$c^i_{ij}(u) = \partial_j h_i / h_i, \qquad c^j_{ij}(u) = \partial_i h_j / h_j,$$

and the $h_i(u) = \xi^i_0(u)$ are the first coefficients in the expansion (2.1).

Equations (3.1) have the form of Eqs. (1.7), which are part of the equations defining the flat coordinates for the diagonal metric with coefficients $H_i(u) = \varepsilon_i h_i(u)$, where the ε_i are constants. Let us now present additional equations that are satisfied by the Baker–Akhiezer functions and are reduced to (1.8) in the case of admissible algebraic-geometric data.

Let $\{\Gamma, P_j, w_j, \gamma_s, R_\alpha\}$ be the set of data that defines a Baker–Akhiezer function $\psi(u, Q \mid D, R)$. Let us fix a set of n additional points $Q_1, \ldots, Q_n$. Then in the generic case there exists a unique function $\psi^1 = \psi^1(u, Q \mid D, R)$ such that

(1^1) $\psi^1(u, Q)$, treated as a function of $Q \in \Gamma$, is meromorphic outside the punctures P_j, has at most simple poles at the points γ_s, and is zero at the punctures $Q_1, \ldots, Q_n$,

$$\psi^1(u, Q_k) = 0;$$

(2^1) in a neighborhood of P_j, the function ψ^1 has the form

$$\psi^1 = w_j^{-1} e^{u^j w_j^{-1}} \left(\sum_{s=0}^{\infty} \xi^j_{1,s}(u) w_j^s \right), \qquad w_j = w_j(Q); \tag{3.2}$$

(3^1) $\psi^1(u, R_\alpha \mid D, R) = 1$.

Lemma 3.2. *The functions $\psi(u, Q \mid D, R)$ and $\psi^1(u, Q \mid D, R)$ satisfy the equations*

$$\partial_i^2 \psi - c^1_i \partial_i \psi^1 + \sum_{j=1}^{n} v_{ij} \partial_j \psi = 0, \tag{3.3}$$

where

$$c^1_i = \frac{h_i}{h^1_i}, \qquad v_{ii} = \frac{\partial_i h^1_i}{h^1_i} - 2 \frac{\partial_i h_i}{h_i} + \frac{g^1_i}{h^1_i} - \frac{g_i}{h_i}, \tag{3.4}$$

$$v_{ij} = \frac{h_i}{h_j} \frac{\partial_i h^1_j}{h^1_i}, \qquad i \neq j, \tag{3.5}$$

and the functions $h_i = \xi^i_0$, $h^1_i = \xi^i_{1,0}$, $g_i = \xi^i_1$, and $g^1_i = \xi^i_{1,1}$ are the first coefficients in the expansions (2.1) *and* (3.2).

The proof is standard. Consider the function defined by the left-hand side of (3.3). Equations (3.4) and (3.5) imply that this function satisfies the first two conditions in the definition of ψ and is zero at each R_α. Therefore, it is equal to zero.

Now consider the case of admissible algebraic-geometric data. (In that case, the set of the punctures in the definition of ψ^1 is the same set as in the definition of the admissible curves and divisors; i.e., $Q_1, \ldots, Q_{2m-n}$ are the branching points and $Q_{2m-n+1}, \ldots, Q_{2n}$ are the preimages of the $\widehat{Q}_k$.)

Theorem 3.1. *The Baker–Akhiezer functions $\psi(u, Q | D, R)$ and $\psi^1(u, Q | D, R)$ corresponding to an admissible set of algebraic-geometric data satisfy Eqs.* (3.1) *and the equations*

$$\partial_i^2 \psi = c_i^1 \partial_i \psi^1 + \sum_{j=1}^{n} \Gamma_{ii}^j \partial_j \psi . \tag{3.6}$$

Here the Γ_{ii}^j are the Christoffel symbols (1.9) *of the metric $H_i(u) = \varepsilon_i h_i(u)$.*

Proof. The differential

$$d\Omega_{ij}^{(2)} = \partial_i \psi^1(u, Q) \partial_j \psi(u, Q^\sigma)\, d\Omega_0(\pi(Q))$$

is holomorphic everywhere outside P_i and P_j. The residues of $d\Omega_{ij}^{(2)}$ at these points are given by

$$\operatorname*{res}_{P_i} d\Omega_{ij}^{(2)} = \varepsilon_i^2 h_i^1 \partial_j h_i, \qquad \operatorname*{res}_{P_j} d\Omega_{ij}^{(2)} = -\varepsilon_j^2 h_j \partial_i h_j^1 .$$

Therefore, $\varepsilon_i^2 h_i^1 \partial_j h_i = \varepsilon_j^2 h_j \partial_i h_j^1$. The last formula implies that the coefficients v_{ij} defined in (3.5) are equal to Γ_{ii}^j for $i \neq j$.

The differential $d\Omega_{ii}^1$ has the only pole at P_i. Therefore, its residue at this point is zero,

$$\operatorname*{res}_{P_i} d\Omega_{ii}^1 = h_i^1(g_i + \partial_i h_i) - h_i(g_i^1 + \partial_i h_i^1) = 0 . \tag{3.7}$$

It follows from (3.7) that the v_{ii} given by (3.4) is equal to Γ_{ii}^i.

Note that Eqs. (3.6) coincide with (1.8) at the points Q_j.

Corollary 3.1. *The functions*

$$\Psi_i^0(u, Q) = \frac{1}{h_i(u)} \partial_i \psi(u, Q), \qquad \Psi_i^1(u, Q) = \frac{1}{h_i^1(u)} \partial_i \psi^1(u, Q) \tag{3.8}$$

satisfy Eqs. (1.11), *where the $\beta_{ij}(u)$ are the rotation coefficients* (1.5) *of the metric $H_i(u)$.*

The proof of the corollary follows by straightforward substitution of (3.8) into (3.1) and (3.6).

Consider the analytical properties of $\Psi_i^0(u, Q)$ and $\Psi_i^1(u, Q)$ viewed as functions on the algebraic curve Γ. It follows from the definition of Baker–Akhiezer functions that

(1^2) the $\Psi_i^N(u, Q)$, $N = 0, 1$, are meromorphic outside the punctures P_j and have at most simple poles at the points $\gamma_1, \ldots, \gamma_{g+l-1}$;

(2^2) in a neighborhood of P_j, the function Ψ_i^N has the form

$$\Psi_i^N = w_j^{-N-1} e^{u_j w_j^{-1}} \left(\delta_{ij} + \sum_{s=1}^{\infty} \zeta_{s,N}^{ij}(u) w_j^s \right), \qquad w_j = w_j(Q); \tag{3.9}$$

(3^2) the functions Ψ_i^N vanish at the punctures R_α, and the functions Ψ_i^1 vanish also at the punctures Q_j,

$$\Psi_i^N(u, R_\alpha) = 0, \qquad \Psi_i^1(u, Q_j) = 0 .$$

Lemma 3.3. *Let Γ be a smooth genus g algebraic curve with $2n$ punctures P_j, Q_j and with given local coordinates $w_j(Q)$ in neighborhoods of the punctures P_j. Then for any set of $g + l - 1$ points γ_s in general position there exist unique functions $\Psi_i^0(u, Q)$ and $\Psi_i^1(u, Q)$ satisfying conditions* (1^2)–(3^2).

For a given admissible curve Γ with punctures P_i, Q_i and local coordinates w_i, the Baker–Akhiezer functions and the coefficients $H_i(u | D, R)$ of the corresponding diagonal flat metric depend on the admissible pair (D, R) of divisors. Two pairs (D, R) and (D', R') of divisors are said to be equivalent if the differences $D - R$ and $D' - R'$ are linearly equivalent, i.e., if there exists a meromorphic function $f(Q)$ on Γ such the divisor $(f)_\infty$ of its poles and the divisor $(f)_0$ of its zeros satisfy

$$(f)_\infty = D + R', \qquad (f)_0 = D' + R. \tag{3.10}$$

Lemma 3.3 implies that the following statement is valid.

Corollary 3.2. *The rotation coefficients $\beta_{ij}(u\,|\,D,R)$ and $\beta_{ij}(u\,|\,D',R')$ corresponding to equivalent pairs of the divisors satisfy the relation*

$$f(P_i)\beta_{ij}(u\,|\,D,R) = f(P_j)\beta_{ij}(u\,|\,D',R'),$$

where $f(Q)$ is a function satisfying (3.10).

Let us express the rotation coefficients in terms of the functions $\Psi_i^1(u, Q\,|\,D,R)$ alone.

Theorem 3.2. *The rotation coefficients $\beta_{ij}(u)$ of the algebraic-geometric flat diagonal metric with coefficients $H_i(u\,|\,D,R)$ are equal to*

$$\beta_{ij}(u\,|\,D,R) = \zeta_{1,1}^{ji}(u\,|\,D,R), \tag{3.11}$$

where the $\zeta_{1,1}^{ji}$ are the first coefficients in the expansions (3.9) *of the functions $\Psi_i^1(u,Q\,|\,D,R)$. The Lamé coefficients $H_i(u\,|\,D,R,r')$ are equal to*

$$H_i(u\,|\,D,R) = -\sum_\alpha d_\alpha \Psi_i^1(u, R_\alpha^\sigma\,|\,D,R), \tag{3.12}$$

where

$$d_\alpha = \operatorname*{res}_{R_\alpha} d\Omega_0\,. \tag{3.13}$$

Proof. It follows from (1.11) that the functions Ψ_i^1 satisfy the equation

$$\partial_i \Psi_j^1 = \beta_{ij}\Psi_i^1, \qquad i \neq j. \tag{3.14}$$

Equation (3.11) readily follows from (3.9) and (3.14). To prove (3.12), let us consider the differential

$$d\Omega_i^{(3)}(u,Q) = \Psi_i^1(u,Q)\psi(u,Q^\sigma)\,d\Omega_0\,.$$

This differential is meromorphic with poles at the points P_i and R_α^σ, and

$$\operatorname*{res}_{P_i} d\Omega_i^{(3)} = H_i(u)\,.$$

The residues of this differential at the points R_α^σ are equal to the corresponding terms in the sum on the right-hand side of (3.12). The sum of all these residues is zero, which completes the proof of theorem.

§4. Egoroff Metrics

In this section we describe the algebraic-geometric data corresponding to Egoroff metrics, i.e., metrics with symmetric rotation coefficients $\beta_{ij} = \beta_{ji}$.

Let $E(P)$ be a meromorphic function on a smooth genus g_0 algebraic curve Γ_0 with n simple poles at the points P_i, $2m-n$ simple zeros at the points $Q_1, \ldots, Q_{2m-n}$, and $n-m$ double zeros at the points $\widehat{Q}_1, \ldots, \widehat{Q}_{n-m}$. The Riemann surface Γ of the function $\lambda = \sqrt{E(P)}$ is an admissible curve in the sense of the definitions in §3. The function $\lambda = \lambda(Q)$ is an odd function with respect to the involution of Γ. Viewed as a function on Γ, it has simple poles at the points P_i and simple zeros at the points Q_j, $j = 1, \ldots, n$. The function λ^{-1} defines local coordinates $w_j(Q) = \lambda^{-1}(Q)$ in neighborhoods of the punctures P_i.

Theorem 4.1. *Let* (D, R) *be an admissible pair of divisors on the Riemann surface* Γ *of the function* $\lambda(Q)$. *Then*

$$\beta_{ij}(u \mid D, R) = \beta_{ji}(u \mid D, R). \tag{4.1}$$

The potential of the Egoroff metric $H_i(u \mid D, R)$ *is given by*

$$\Phi(u \mid D, R) = \sum_{\alpha=1}^{n} \lambda(R_\alpha)\, d_\alpha \psi(u, R_\alpha^\sigma), \tag{4.2}$$

where the d_α *are the residues of* $d\Omega_0$ *at* R_α (3.13).

Proof. To obtain (4.1), it suffices to consider the differential $\lambda(Q)\,\Psi_i^0(u, Q)\,\Psi_j^0(u, \sigma(Q))\, d\Omega_0$, which has poles only at the points P_i and P_j. The residues at these points are equal to β_{ji} and $-\beta_{ij}$, respectively. To prove (4.2), consider the differential

$$d\Omega_i^{(4)} = \lambda(Q)\,\psi(u, Q)\,\partial_i \psi(u, \sigma(Q))\, d\Omega_0 .$$

This differential has poles at the points P_i and R_α with residues

$$\operatorname*{res}_{P_i} d\Omega_i^{(4)} = -H_i^2, \qquad \operatorname*{res}_{R_\alpha} d\Omega_i^{(3)} = d_\alpha \lambda(R_\alpha)\,\partial_i \psi(u, R_\alpha^\sigma).$$

The sum of these residues is zero, which proves (4.2).

§5. Solutions to the Associativity Equations

The equivalence [5] of the classification problem for the rotation coefficients of Egoroff metrics and the classification problem for massive topological fields does not provide explicit solutions of the associativity equations. In this section, we obtain explicit expressions for the partition functions of the models corresponding to the above-constructed symmetric rotation coefficients.

Theorem 5.1. *Let* $\psi(u, Q \mid D, R)$ *be the Baker-Akhiezer function defined on the Riemann surface* Γ *of the function* $\lambda(Q)$ *and corresponding to an admissible pair* (D, R) *of divisors. Then the function* $F(x) = F(u(x))$ *defined by the formula*

$$F(u) = \frac{1}{2}\Big(\sum_{k,l=1}^{n} \eta_{kl} x^k(u) y^l(u) - \sum_\alpha \frac{d_\alpha}{\lambda(R_\alpha)} \psi(u, R_\alpha^\sigma) \Big),$$

where (η_{kl}) *is the constant matrix defined in* (2.7), *the constants* d_α *are defined in* (3.13), *and*

$$x^k(u) = \psi(u, Q_k), \qquad y^k = d\psi(u, Q_k)/d\lambda,$$

satisfies the equation

$$\frac{\partial^3 F(x)}{\partial x^k \partial x^l \partial x^m} = c_{klm} = \sum_{i=1}^{n} H_i^2 \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l} \frac{\partial u^i}{\partial x^m}. \tag{5.1}$$

Moreover, the functions

$$c_{kl}^m = \sum_{i=1}^{n} \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l} \frac{\partial x^m}{\partial u^i} \tag{5.2}$$

satisfy the associativity equations (1.14).

Proof. Consider the functions

$$\phi_k = \frac{\partial \psi}{\partial x^k}, \qquad \phi_{kl} = \frac{\partial^2 \psi}{\partial x^k \partial x^l}.$$

In a neighborhood of P_i, they have the form

$$\phi_k = \frac{\partial u^i}{\partial x^k}\,\lambda e^{\lambda u^i}(h_i + O(\lambda^{-1})), \qquad \phi_{kl} = \frac{\partial u^i}{\partial x^k}\,\frac{\partial u^i}{\partial x^k}\,\lambda^2 e^{\lambda u^i}(h_i + O(\lambda^{-1})).$$

Therefore,

$$c_{klm} = \sum_{i=1}^{n} \operatorname*{res}_{P_i} d\Omega_{k;\,lm}, \qquad d\Omega_{k;\,lm} = \phi_k(u, Q)\,\phi_{lm}(u, Q^\sigma)\,\frac{d\Omega_0}{\lambda(Q)}.$$

By the definition of x^k, we have $\phi_k(u, Q_m) = \delta_{km}$, $\phi_{kl}(u, Q_m) = 0$. Therefore, the differential $d\Omega_{k;\,lm}$ outside the punctures P_i has a pole only at Q_k. Hence,

$$c_{klm} = -\operatorname*{res}_{Q_k} d\Omega_{k;\,lm} = -\operatorname*{res}_{Q_k} \phi_{lm}(u, \sigma(Q_k))\frac{d\Omega_0}{\lambda(Q)} = -\frac{\partial^2}{\partial x^l \partial x^m}\left(\operatorname*{res}_{Q_k} \psi(u, \sigma(Q_k))\right)\frac{d\Omega_0}{\lambda(Q)}. \tag{5.3}$$

Near Q_k, we have

$$\psi(u, \sigma(Q)) = x^{\sigma(k)} - y^{\sigma(k)}\lambda + O(\lambda^2)$$

and

$$d\Omega_0 = \frac{d\lambda}{\lambda}\,(\eta_k + \eta_k^1\lambda + O(\lambda^2)).$$

Therefore,

$$\operatorname*{res}_{Q_k} \psi(u, \sigma(Q_k))\,\frac{d\Omega_0}{\lambda} = \eta_k^1 x^{\sigma(k)} - \eta_k y^{\sigma(k)}. \tag{5.4}$$

It follows from the definition of F that

$$2\,\frac{\partial}{\partial x^k}\,F = \eta_k y^{\sigma(k)} + \sum_{l}^{n} \eta_l x^l\,\frac{\partial y^{\sigma(l)}}{\partial x^k} - \frac{d_\alpha}{\lambda(R_\alpha)}\,\frac{\partial \psi(u, R_\alpha)}{\partial x^k}.$$

Consider the differential

$$d\Omega_k^{(5)} = \frac{\partial \psi(u, Q)}{\partial x^k}\,\psi(u, Q^\sigma)\,\frac{d\Omega_0}{\lambda(Q)}.$$

It has poles at Q_l and R_α^σ with residues

$$\operatorname*{res}_{Q_l} d\Omega_k^{(5)} = \eta_l x^{\sigma(l)}\,\frac{\partial y^l}{\partial x^k} + \delta_{l,\sigma(k)}(\eta_k^1 x^{\sigma(k)} - \eta_k y^{\sigma(k)}), \qquad \operatorname*{res}_{R_\alpha^\sigma} d\Omega_k^{(5)} = -\frac{d_\alpha}{\lambda(R_\alpha)}\,\frac{\partial \psi(u, R_\alpha^\sigma)}{\partial x^k}.$$

Therefore,

$$\sum_{l=1}^{n} \eta_l x^l\,\frac{\partial y^{\sigma(l)}}{\partial x^k} - \sum_{\alpha} \frac{d_\alpha}{\lambda(R_\alpha)}\,\frac{\partial \psi(u, R_\alpha)}{\partial x^k} = \eta_k y^{\sigma(k)} - \eta_k^1 x^{\sigma(k)}.$$

Finally,

$$\frac{\partial}{\partial x^k}\,F = \eta_k y^{\sigma(k)} - \frac{1}{2}\,\eta_k^1 x^{\sigma(k)}.$$

The latter equality and Eqs. (5.3) and (5.4) imply (5.1).

Lemma 5.1. *The Baker–Akhiezer function $\psi(u, Q\,|\,D, R)$ defined on the Riemann surface Γ of the function $\lambda(Q)$ and corresponding to an admissible pair (D, R) of divisors satisfies the equations*

$$\frac{\partial^2}{\partial x^k \partial x^l}\,\psi - \lambda \sum_{m=1}^{n} c_{kl}^m\,\frac{\partial}{\partial x^m}\,\psi_m = 0. \tag{5.5}$$

Proof. Consider the function $\widetilde{\psi}$ defined by the left-hand side of (5.5). Outside of the punctures P_j it has poles only at the points of the divisor D and is equal to zero at the points Q_j. It follows from the definition of c_{kl}^m that in the expansion of $\widetilde{\psi}$ at the points P_i, the meromorphic factor is $O(\lambda^{-1})$. Therefore, it follows from the uniqueness of the Baker–Akhiezer function that $\widetilde{\psi} = 0$.

The associativity equations (1.14) for the functions c_{kl}^m are the consistency conditions for the system (5.5). The proof of the theorem is complete.

Remark. Equations (5.5) can be rewritten in the vector form:

$$\frac{\partial}{\partial x^k} \widetilde{\Psi}_l = \lambda \sum_{m=1}^{n} c_{kl}^m \widetilde{\Psi}_m, \qquad \widetilde{\Psi}_k = \frac{\partial \psi}{\partial x^k}. \tag{5.6}$$

System (5.6) with symmetric coefficients $c_{kl}^m = c_{lk}^m$ was introduced in [5] as an auxiliary linear system for the associativity equations (1.14).

Now we shall consider the special case of our construction in which the divisor R coincides with the divisor $\mathcal{Q}$ of the punctures Q_j. It was mentioned in the remark preceding Theorem 2.2 that the assumption that R does not intersect $\mathcal{Q}$ was adopted only for simplicity of the formulas.

For the case in which $R = \mathcal{Q}$, admissible divisors D are defined as follows. A divisor $D = \gamma_1 + \cdots + \gamma_{g+n-1}$ is said to be admissible if there exists a meromorphic differential $d\Omega_0$ on Γ_0 with poles of the order 2 at the points $Q_1, \ldots, Q_{2m-n}$ and poles of the order 3 at the double zeros of $E(P)$ such that $d\Omega_0(\pi(\gamma_s)) = 0$. The differential $d\Omega_0$ treated as an odd differential on Γ has the form

$$d\Omega_0 = \frac{d\lambda}{\lambda^3(P)} (\eta_k + O(\lambda))$$

at the punctures Q_k, $k = 1, \ldots, n$ (where $\lambda(Q_k) = 0$). In our special case, the flat coordinates are no longer defined by the values $\psi(Q_k)$ (which are all equal to 1). Instead, we must use the subsequent terms of the expansion.

Theorem 5.2. *Let $\psi(x, Q \,|\, D, \mathcal{Q})$ be the Baker–Akhiezer function defined by an admissible divisor D on the Riemann surface of $\lambda(Q)$. Then the function $F(x) = \widetilde{F}(u(x))$, where*

$$\widetilde{F}(u) = \frac{1}{2} \sum_{k=1}^{n} \eta_k x^k(u) y^{\sigma(k)}(u),$$

$\eta_k = \operatorname{res}_{Q_k} \lambda^2 \, d\Omega_0$, *and the $x^k(u)$ and $y^k(u)$ are defined from the expansion*

$$\psi = 1 + x^k(u)\lambda + y^k(u)\lambda^2 + O(\lambda^3),$$

is a solution of the associativity equations (1.12)–(1.15), *i.e., satisfies Eqs.* (5.1); *the functions c_{kl}^m defined in* (5.2) *satisfy* (1.14) *and the additional relation*

$$\sum_{m=1}^{n} c_{klm}(u) = \eta_{kl}. \tag{5.7}$$

Proof. The proof of the fact that the functions x^k are flat coordinates for the diagonal metric with Lamé coefficients $H_i = \varepsilon_i h_i(u)$, where $h_i(u)$ is the leading term in the expansion of the corresponding Baker–Akhiezer function at the puncture P_i, is just the same as in the general case. The proof of the other statements of the theorem but the last one is also almost identical to that in Theorem 5.1. Equation (5.7) is a consequence of the following statement.

Lemma 5.2. *The Baker–Akhiezer function ψ corresponding to the data specified by the assumptions of Theorem 4.3 satisfies the equation*

$$\sum_{s=1}^{n} \frac{\partial}{\partial u^s} \psi = \lambda \psi . \tag{5.8}$$

The two sides of (5.8) are regular outside the punctures P_k and have the same leading terms in their expansions at P_k. Therefore, they are identically equal to each other by virtue of the uniqueness of the Baker–Akhiezer function. Equation (5.8) at Q_m gives

$$\sum_{s=1}^{n} \frac{\partial x^m}{\partial u^s} = 1 .$$

Hence,

$$\sum_{m=1}^{n} c_{klm}(u) = \sum_{i=1}^{n} H_i^2 \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l} = \eta_{kl} .$$

The proof of the theorem is complete.

Exact theta function formulas for the partition function F can be obtained by substituting the corresponding expressions for the Baker–Akhiezer function.

Example. *Elliptic solutions.* Let us consider the simplest elliptic curvilinear coordinates and solutions of the associativity equations that correspond to $n = l = 3$, $m = 2$ in the example of §2.

Consider the elliptic curve Γ with periods 2ω and $2\omega'$, $\operatorname{Im} \omega'/\omega > 0$. In this representation, we identify the punctures P_i with the half-periods ω_i; i.e.,

$$P_1 = \omega_1 = \omega, \quad P_2 = \omega_2 = \omega', \quad P_3 = \omega_3 = -\omega - \omega'.$$

The punctures Q_j are the points of the fundamental parallelogram $Q_1 = 0$, $Q_2 = z_0$, $Q_3 = -z_0$. In the case $g = 1$, any divisors D and R form an admissible pair. The corresponding differential $d\Omega_0$ has the form

$$d\Omega_0 = \eta_0 \frac{\sigma(z-\omega)\sigma(z-\omega')\sigma(z+\omega+\omega')}{\sigma(z)\sigma(z+z_0)\sigma(z-z_0)} \prod_{s=1}^{l} \frac{\sigma(z-\gamma_s)\sigma(z+\gamma_s)}{\sigma(z-R_s)\sigma(z+R_s)} dz,$$

where $\sigma(z) = \sigma(z \,|\, \omega, \omega')$ is the classical Weierstrass σ-function. The residues

$$\operatorname*{res}_{z=0} d\Omega_0 = \eta_1, \qquad \operatorname*{res}_{z=\pm z_0} d\Omega_0 = \eta_2$$

are the coefficients of the flat metric $ds^2 = \eta_1 (dx^1)^2 + \eta_2 (dx^2)(dx^3)$. The Baker–Akhiezer function has the form

$$\psi(u,z) = \prod_{s=1}^{l} \frac{\sigma(z-R_s)}{\sigma(z-\gamma_s)} \left[\sum_{\alpha=1}^{l} r_\alpha \frac{\sigma(z+U-R_\alpha)}{\sigma(z-R_\alpha)\sigma(U)} \exp\left(\Omega(u,z) - \Omega(u,R_\alpha)\right) \right], \tag{5.9}$$

where

$$U = u^1 + u^2 + u^3, \quad \Omega(u,z) = u^1(\zeta(z-\omega)+\eta) + u^2(\zeta(z-\omega')+\eta') + u^3(\zeta(z+\omega+\omega') - \eta - \eta'),$$

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \quad \eta = \zeta(\omega), \quad \eta' = \zeta(\omega'), \quad r_\alpha = \frac{\prod_{s=1}^{l} \sigma(R_\alpha - \gamma_s)}{\prod_{s \neq \alpha} \sigma(R_\alpha - R_s)} .$$

In the general case, where $R_\alpha \neq Q_j$, the values of ψ at Q_j give an expression of the flat coordinates:

$$x^1 = \psi(u,0), \quad x^2 = \psi(u,z_0), \quad x^3 = \psi(u,-z_0) .$$

The corresponding Lamé coefficients are given by

$$H_1(u)=\varepsilon_1\sum_{\alpha=1}^{n} r_\alpha \frac{\sigma(U-R_\alpha)}{\sigma(\omega-R_\alpha)\sigma(U)}\,e^{U\eta},$$
$$H_2(u)=\varepsilon_2\sum_{\alpha=1}^{n} r_\alpha \frac{\sigma(\omega'+U-R_\alpha)}{\sigma(\omega'-R_\alpha)\sigma(U)}\,e^{U\eta'},$$
$$H_3(u)=\varepsilon_3\sum_{\alpha=1}^{n} r_\alpha \frac{\sigma(\omega+\omega'+U-R_\alpha)}{\sigma(\omega+\omega'-R_\alpha)\sigma(U)}\,e^{(-U\eta-U\eta')}.$$

The elliptic solutions of the associativity equations correspond to the Baker–Akhiezer function given by (5.9) with $l=3$ and $R_1=0$, $R_2=z_0$, $R_3=-z_0$. Let us give the corresponding formulas for the simplest Baker-Akhiezer function

$$\psi(u,z)=\frac{\sigma(z+s)}{\sigma(z)\sigma(s)}\,e^{\Omega(u,z)}.$$

The coefficients of the expansions

$$\psi=1/z+x^1(u)+y^1(u)z+O(z^2),$$
$$\psi=x^2+y^2(u)(z-z_0)+O((z-z_0)^2),\qquad \psi=x^3+y^3(u)(z+z_0)+O((z+z_0)^2)$$

define the solution

$$F=x^1y^1-\tfrac{1}{2}(x^2y^3+x^3y^2) \tag{5.10}$$

of the associativity equations. We have

$$x^1=\zeta(U)-\wp(\omega)u^1-\wp(\omega')u^2-\wp(\omega+\omega')u^3,$$
$$x^2=\frac{\sigma(z_0+U)}{\sigma(z_0)\sigma(U)}\exp\Omega(u,z_0),\qquad x^3=\frac{\sigma(U-z_0)}{\sigma(-z_0)\sigma(U)}\exp\Omega(u,-z_0)$$

and

$$y^1=\frac{\sigma''(U)}{2\sigma(U)}-\zeta(U)\sum_{i=1}^{3}(\wp(\omega_i)u^i)+\frac{1}{2}\Big(\sum_{i=1}^{3}\wp(\omega_i)u^i\Big)^2,$$
$$y^2=x^2(u)\Big(\zeta(z_0+U)-\zeta(z_0)-\sum_{i=1}^{3}\wp(z_0-\omega_i)u^i\Big),$$
$$y^3=x^3(u)\Big(\zeta(-z_0+U)+\zeta(z_0)-\sum_{i=1}^{3}\wp(z_0-\omega_i)u^i\Big).$$

The function $\widehat{F}=F-\frac{1}{2}(x^1)^2$ has the same third derivatives as F. Therefore, on substituting the expressions for x^i and y^i into (5.10), we obtain the following formula for the simplest elliptic solution of the associativity equations:

$$\widehat{F}=-\frac{1}{2}\wp(U)-\frac{1}{2}(\wp(U)-\wp(z_0))\Big(\zeta(z_0-U)-\zeta(z_0+U)-\sum_{i=1}^{3}\wp(z_0-\omega_i)u^i\Big).$$

Acknowledgements. The author thanks V. E. Zakharov and S. V. Manakov for their explanation of the ideas of their remarkable works that were an inspiration for this paper.

References

1. G. Darboux, Leçons sur le systèmes orthogonaux et les coordonnées curvilignes, Paris, 1910.
2. B. A. Dubrovin and S. P. Novikov, "The Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method," Dokl. Akad. Nauk SSSR, **27**, 665–654 (1983).
3. B. A. Dubrovin and S. P. Novikov, "Hydrodynamics of weakly deformed soliton lattices: Differential geometry and Hamiltonian theory," Usp. Mat. Nauk, **44**, 35–124 (1989).
4. S. P. Tsarev, "The geometry of Hamiltonian systems of hydrodynamic type. Generalized hodograph method," Izv. Akad. Nauk SSSR, Ser. Mat., **54**, No. 5, 1048–1068 (1990).
5. B. Dubrovin, "Integrable systems in topological field theory," Nuclear Phys. B, **379**, 627–689 (1992).
6. I. Krichever, "Tau-function of the universal Whitham hierarchy and topological field theories," Comm. Pure Appl. Math., **47**, 1–40 (1994).
7. V. Zakharov, Description of the n-orthogonal curvilinear coordinate systems and hamiltonian integrable systems of hydrodynamic type. Part 1, Integration of the Lamé equations, Preprint (to appear in Duke Math. J.).
8. V. E. Zakharov and S. V. Manakov, Private communication.
9. E. Witten, "The structure of the topological phase of two-dimensional gravity," Nuclear Phys. B, **340**, 281–310 (1990).
10. E. Verlinder and H. Verlinder, A solution of two-dimensional topological quantum gravity, Preprint IASSNS-HEP 90/40, PUPT-1176, 1990.
11. I. M. Krichever, "The algebraic-geometric construction of the Zakharov–Shabat equations and their periodic solutions," Dokl. Akad. Nauk SSSR, **227**, No. 2, 291–294 (1976).
12. I. M. Krichever, "The integration of nonlinear equations with the help of algebraic-geometric methods," Funkts. Anal. Prilozhen., **11**, No. 1, 15–31 (1977).
13. A. R. Its and V. B. Matveev, "On a class of solutions of the Korteweg–de Vries equation," in: Problems of Mathematical Physics, **8**, Leningrad State University, 1976.
14. I. Krichever, O. Babelon, E. Billey, and M. Talon, "Spin generalization of the Calogero–Moser system and the matrix KP equation," Am. Math. Soc. Transl. (2), **170**, 83–119 (1995).
15. S. P. Novikov and A. P. Veselov, "Finite-gap two-dimensional periodic Schrödinger operators: exact formulas and evolution equations," Dokl. Akad. Nauk SSSR, **279**, No. 1, 20–24 (1984).
16. I. M. Krichever, "Algebraic-geometric two-dimensional operators with self-consistent potentials," Funkts. Anal. Prilozhen., **28**, No. 1, 26–40 (1994).
17. B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, "The Schrödinger equation in a periodic field and Riemann surfaces," Dokl. Akad. Nauk SSSR, **229**, No. 1, 15–18 (1976).

Translated by I. M. Krichever