Two-Dimensional Algebraic-Geometric Operators with Self-Consistent Potentials

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Dedicated to I. M. Gel'fand on his 80th birthday

§1. Introduction. Statement of Results

The dynamics of strings in strong gravitational fields and especially in the vicinity of space-time singularities has recently attracted special attention. One of the main goals of this paper is to construct exact solutions of string equations in $(2+1)$-de Sitter space-time. The simplest solutions of this equation were obtained in [1], which stimulated our work and where the physical motivation and bibliographical references can be found.

From the geometric viewpoint the problem is to construct minimal surfaces in a pseudosphere, i.e., if $q = \{q_i(\sigma, \tau)\}$ is a parametrization of such a surface, then

$$\langle q, q\rangle = \sum_{i=1}^{D} \eta_i q_i q_i = 1 . \tag{1.1}$$

Unless otherwise specified, the constants η_i that define the metric are assumed to be given by $\eta = (-1, 1, \dots)$. The equations that define embeddings of minimal surfaces in the quadric (1.1) have the form

$$(\partial_+\partial_- + u)q = 0, \qquad \partial_\pm = \partial/\partial x_\pm,\ x_\pm = (\tau \pm \sigma)/2 . \tag{1.2}$$

As follows from (1.1) and (1.2), the Lagrange multiplier $u = u(x_+, x_-)$ in (1.2) is equal to

$$u = \langle \partial_+ q, \partial_- q\rangle. \tag{1.3}$$

Equations (1.2) and (1.3) form a system of nonlinear equations for the functions $q_i(\sigma, \tau)$. They are a particular case of the general σ-models

$$\partial_+\partial_-\Psi + \langle \partial_+\Psi, \partial_-\Psi\rangle_*\Psi = 0, \tag{1.4}$$

where $\langle\cdot\,,\cdot\rangle_*$ stands for the inner product defined by some matrix (g_{ij}). Equations (1.4) can be considered as *linear equations with self-consistent potentials.* The well-known nonlinear Schrödinger equation can be treated in the same manner. It can be represented as the linear equation

$$(i\partial_t - \partial_x^2 + u(x,t))\psi = 0 \tag{1.5}$$

with the self-consistency condition

$$u = \alpha|\psi|^2. \tag{1.6}$$

Numerous physical models that describe the interactions of long and short waves also have the form of a system comprising linear equation (1.5) and various self-consistency conditions. For example,

$$u_t + u_x = |\psi|_x^2 \quad [2], \tag{1.7}$$

$$u_{tt} - u_{xx} + \alpha u_{xxxx} + \beta(u^2)_{xx} = |\psi|_{xx}^2 \quad [3] . \tag{1.8}$$

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The general construction of exact solutions to the nonstationary Schrödinger equation (1.5) with various types of self-consistency conditions was proposed in [4] (its realization for the construction of soliton solutions was presented in [5]). Such a scheme was originally advanced in [6] for the construction of exact solutions to the nonlinear Schrödinger equation and its vector analogs.

The main goal of this paper is to show that the ideas of this scheme are universal enough to be applied to many other systems. In particular, for σ-models the construction can be considered as an alternative to the inverse transform method. In the author's opinion the essential advantage of the scheme is that it allows us to construct solutions of the σ-model which, in addition, satisfy the so-called *string constraints.* The string constraints follow from the reparametrization invariance of the world surface of the string. They have the form

$$T_{+,+} = \langle \partial_+ q, \partial_+ q\rangle = 0, \qquad T_{-,-} = \langle \partial_- q, \partial_- q\rangle = 0. \tag{1.9}$$

(Equations (1.9) are the classical analogs of the quantum equations

$$L_n|0\rangle = 0, \quad n > -1, \qquad \langle 0|L_n = 0, \quad n < 1,$$

that define the vacuum state in the model of a bosonic string. Here L_n are the generators of the Virasoro algebra.) The basic idea of constructing exact solutions to Eqs. (1.4) can be presented in the following form. First we construct "integrable potentials" $u = u(x_+, x_-)$ of the two-dimensional wave operator

$$(\partial_+\partial_- + u(x_+, x_-))\psi(x_+, x_-, Q) = 0, \tag{1.10}$$

i.e., potentials such that a set of solutions to (1.10) with spectral parameter which is a point Q of an auxiliary Riemann surface of finite genus is known. We shall say that the self-consistency conditions are satisfied if there exist values Q_i, $i = 1, \ldots, 2N$, of the spectral parameter such that

$$u(x_+, x_-) = \sum_{i,j} g_{ij}\partial_+\psi_i\partial_-\psi_j, \tag{1.11}$$

where

$$\psi_i(x_+, x_-) = \psi_i(x_+, x_-, Q_i). \tag{1.12}$$

Linear equation (1.10), in conjunction with the self-consistency conditions, implies nonlinear equations (1.4) for the vector $\Psi(x_+, x_-) = (\psi_1, \ldots, \psi_N)$ with components that are values of the wave function $\psi(x_+, x_-, Q)$ at Q_i.

Thus, the approach described for solving the nonlinear equations (1.4) is to construct integrable linear problems and then to select those of them whose corresponding potentials satisfy the self-consistency conditions.

This scheme is developed for the string equations (1.1)–(1.3) and (1.9) in the next two sections. In the second section the necessary results [8, 9] on the integrable potentials of the two-dimensional Schrödinger operator are presented. In the third section, the constraints on the construction parameters for such potentials are imposed to guarantee the self-consistency conditions (1.11) and the string constraints (1.9).

The algebraic-geometric solutions thus constructed are determined by an auxiliary algebraic curve Γ_0 of genus g_0 with two marked points $P_\pm$ and by a meromorphic function $E(P)$, $P \in \Gamma_0$, with simple poles Q_i^0, $i = 1, \ldots, N$. Let Γ be a double covering of Γ_0 with exactly two branch points $P_\pm$. The matrix of the b-periods of the normalized odd (with respect to the permutation of the sheets) differentials on Γ defines the Prim theta function $\theta_{\mathrm{Pr}}(z)$, $z = (z_1, \ldots, z_{g_0})$. In terms of the Prim theta function the exact solutions have the form

$$\begin{aligned}
\varphi_j(x_+, x_-) &= r_j\,\frac{\theta_{\mathrm{Pr}}(A_j + U^+x_+ + U^-x_- + Z)\,\theta_{\mathrm{Pr}}(Z)}{\theta_{\mathrm{Pr}}(A_j + Z)\,\theta_{\mathrm{Pr}}(U^+x_+ + U^-x_- + Z)}\,e^{ip_j^+x_+ + ip_j^-x_-}, \\
\varphi_j^\sigma(x_+, x_-) &= r_j\,\frac{\theta_{\mathrm{Pr}}(-A_j + U^+x_+ + U^-x_- + Z)\,\theta_{\mathrm{Pr}}(Z)}{\theta_{\mathrm{Pr}}(-A_j + Z)\,\theta_{\mathrm{Pr}}(U^+x_+ + U^-x_- + Z)}\,e^{-(ip_j^+x_+ + ip_j^-x_-)}.
\end{aligned} \tag{1.13}$$

Here Z is an arbitrary vector; the constants r_j, $p_j^\pm$ and the g_0-dimensional vectors $U^\pm$, A_j, $j = 1, \ldots, N$, are expressed in quadratures via Γ, $P_\pm$, and $E(P)$.

The functions (1.13) satisfy Eq. (1.10), where

$$u = \partial_+\partial_-\theta_{\mathrm{Pr}}(U^+x_+ + U^-x_- + Z) + \mathrm{const}\,. \tag{1.14}$$

Furthermore,

$$\sum_{i=1}^{N} \varphi_i\varphi_i^\sigma = 1, \tag{1.15}$$

$$2u = \sum_{i=1}^{N} \partial_+\varphi_i\partial_-\varphi_i^\sigma + \partial_+\varphi_i^\sigma\partial_-\varphi_i\,. \tag{1.16}$$

Hence the functions

$$\begin{gathered} q_1 = (\varphi_1 - \varphi_1^\sigma)/2, \quad q_2 = (\varphi_1 + \varphi_1^\sigma)/2, \\ q_{2j} = (\varphi_j - \varphi_j^\sigma)/2i, \quad q_{2j+1} = (\varphi_j + \varphi_j^\sigma)/2, \qquad j = 2, \ldots, N, \end{gathered} \tag{1.17}$$

satisfy system (1.1)–(1.3).

In terms of the original algebraic-geometric data Γ_0, $P_\pm$, and $E(P)$ the solutions satisfying the string constraints can be distinguished as follows. Suppose that the differential of the function $E(P)$ vanishes at the marked points, i.e., $dE(P_\pm) = 0$. Then

$$\sum_{i=1}^{N} \partial_\pm\varphi_i\partial_\pm\varphi_i^\sigma = 0\,. \tag{1.18}$$

The solutions of string equations for the $(2+1)$-de Sitter space-time correspond to the case $N = 2$. The $(2+1)$-de Sitter space-time can be represented as the one-sheet hyperboloid

$$1 = -q_1^2 + q_2^2 + q_3^2 + q_4^2 \tag{1.19}$$

in the flat Minkowski space with coordinates $q = (q_1, q_2, q_3, q_4)$ and metric

$$ds^2 = H^{-2}(-dq_1^2 + dq_2^2 + dq_3^2 + dq_4^2), \tag{1.20}$$

where H is the Habble constant. This implies that the original curve Γ_0 should be a hyperelliptic curve. This case is considered in detail in §4. All the parameters r_j, $p_j^\pm$, A_j, and $U^\pm$, $j = 1, 2$, occurring in (1.13) are expressed by quadratures via the original data formed by the set $E_- < E_1 < \cdots < E_{2n} < E_+$ $(n = g_0)$ of branching points of the hyperelliptic curve and by an arbitrary n-dimensional vector Z. Each set of original data determines a solution to the string equations in $(2+1)$-de Sitter space-time. These solutions are almost periodic functions of σ. The closed strings correspond to the 2π-periodic solutions of string equations. The periodicity condition is satisfied if

$$p_2^+ - p_2^- = 2\pi, \qquad U_k^+ - U_k^- = \pi m_k, \quad m_k \in \mathbb{Z}, \tag{1.21}$$

where $U_k^\pm$, $k = 1, \ldots, n$, are the coordinates of the vector $U^\pm$. Relations (1.21) form a set of $n+1$ equations for $2n+2$ parameters E_s. Therefore, for each n we have a $2n+1$-parametric (the vector Z in (1.13) is arbitrary) family of 2π-periodic exact solutions of string equations in de Sitter space-time. As $\tau \to \pm\infty$, the functions $q_i(\sigma, \tau)$, $i = 1, 2$, tend to $\pm\infty$. But it should be emphasized that, as follows from (1.14), the "internal size" of the string defined by the invariant metric

$$ds^2 = u(x_+, x_-)(d\sigma^2 - d\tau^2)/2H^2 \tag{1.22}$$

is an almost periodic function of τ.

§2. Schrödinger Operators with a Finite-Gap Property on an Energy Level

Consider the two-dimensional Schrödinger operator

$$H = (\partial_x - iA_1(x,y))^2 + (\partial_y - iA_2(x,y))^2 + u(x,y) \tag{2.1}$$

in the magnetic field. The inverse problem based on the spectral data corresponding to a single energy level $E = E_0$ was posed and considered for the operator H in [7]. In this paper a class of operators with a finite-gap property on a single energy level was constructed. From the viewpoint of spectral theory this class is distinguished by the condition that the Riemann surface of Bloch functions corresponding to this energy level (the complex Fermi surface) is of finite genus. Veselov and Novikov [8, 9] found conditions on the algebraic-geometric data of the construction [7] that distinguish real smooth potential ($A_i = 0$) operators $H = H_0$,

$$H_0 = \partial_x^2 + \partial_y^2 + u(x,y). \tag{2.2}$$

Here we present the needed results of the cited papers with slight modifications, allowing for the fact that the operator (1.10) is hyperbolic in contrast to the elliptic operator (2.2). The complex theory is the same in both cases. The only difference is in the reality and smoothness conditions on the potentials.

Let Γ be a smooth algebraic curve of genus g with two marked points $P_\pm$ and fixed local coordinates $k_\pm^{-1}(Q)$ in neighborhoods of these points, $k_\pm^{-1}(P_\pm) = 0$. Assume that there exists a holomorphic involution

$$\sigma \colon \Gamma \mapsto \Gamma \tag{2.3}$$

on Γ such that $P_\pm$ are its only fixed points, i.e.,

$$\sigma(P_\pm) = P_\pm. \tag{2.4}$$

The local parameters are assumed to be "odd," i.e.,

$$k_\pm(\sigma(Q)) = -k_\pm(Q). \tag{2.5}$$

The factor-curve will be denoted by Γ_0. The projection

$$\pi \colon \Gamma \mapsto \Gamma_0 = \Gamma/\sigma \tag{2.6}$$

represents Γ as a double covering over Γ_0 with two branch points $P_\pm$. In this realization σ is the permutation of the sheets. Since there are only two branch points, we have

$$g = 2g_0, \tag{2.7}$$

where g_0 is the genus of Γ_0. Let us consider a meromorphic differential $d\Omega(Q)$ of the third kind on Γ_0 with residues ∓ 1 at the points $P_\pm$ and holomorphic outside $P_\pm$. The differential $d\Omega$ has g zeros, which will be denoted by $\widehat{\gamma}_i$, $i = 1, \ldots, 2g_0 = g$,

$$d\Omega(\widehat{\gamma}_i) = 0. \tag{2.8}$$

For each i let us choose a point γ_i on Γ such that

$$\pi(\gamma_i) = \widehat{\gamma}_i, \qquad i = 1, \ldots, g. \tag{2.9}$$

(In what follows such divisors γ_s are said to be admissible.)

By definition, the Baker–Akhiezer function $\psi(x_+, x_-, Q)$ corresponding to this set of data is the unique function that has the following analytic properties with respect to the variable $Q \in \Gamma$:

1) Outside the points $P_\pm$ the function ψ is meromorphic and has at most simple poles at the points γ_s (if they are all distinct);
2) In some neighborhoods of $P_\pm$ the function ψ has the form

$$\psi(x_+, x_-, Q) = e^{k_\pm x_\pm}\left(1 + \sum_{s=1}^{\infty} \xi_s^\pm(x_+, x_-) k_\pm^{-s}\right), \tag{2.10}$$

$k_\pm = k_\pm(Q)$, $Q \to P_\pm$.

For almost all values of $x_\pm$ (which are external parameters in the definition) the function $\psi(x_+, x_-, Q)$ exists and is unique.

Theorem 2.1 [8]. *The Baker–Akhiezer function* $\psi(x_+, x_-, Q)$ *satisfies the equation*

$$(\partial_+\partial_- + u(x_+, x_-))\psi(x_+, x_-, Q) = 0, \tag{2.11}$$

where

$$u(x_+, x_-) = -\partial_-\xi_1^+ = -\partial_+\xi_1^-, \tag{2.12}$$

and $\xi_1^\pm = \xi_1^\pm(x_+, x_-)$ *are the first coefficients in the expansion* (2.10).

In the general case the function $u(x_+, x_-)$ is a meromorphic function of the variables $x_\pm$. Its expression in terms of the Prim theta function was obtained in [9] (see formula (1.14)). The following conditions on the set of algebraic-geometric data are sufficient for reality and regularity of the corresponding potential $u(x_+, x_-)$.

Assume that there is an antiholomorphic involution

$$\tau : \Gamma \mapsto \Gamma \tag{2.13}$$

on Γ such that

$$\tau(P_\pm) = P_\pm, \qquad k_\pm(\tau(Q)) = \overline{k_\pm(Q)}. \tag{2.14}$$

Lemma 2.1. *Let the divisor of the poles* $\gamma_1, \dots, \gamma_g$ *of the Baker–Akhiezer function be invariant with respect to the anti-involution* τ, *that is,*

$$\tau(D) = D, \qquad D = \gamma_1 + \cdots + \gamma_g. \tag{2.15}$$

Then

$$\psi(x_+, x_-, \tau(Q)) = \overline{\psi(x_+, x_-, Q)}, \tag{2.16}$$

and the corresponding potential $u(x_+, x_-)$ *in* (2.11) *is real,*

$$u(x_+, x_-) = \overline{u(x_+, x_-)}. \tag{2.17}$$

According to the Hurwitz theorem, the number of fixed ovals of an antiholomorphic involution is not greater than $g+1$. Curves with $g+1$ fixed ovals are called M-curves.

Lemma 2.2. *Let* Γ *be an* M*-curve with respect to the anti-involution* τ (*i.e.,* τ *has* $g+1$ *fixed ovals* $a_0, a_1, \dots, a_g$), *and let the marked points* $P_\pm$ *and the points* $\gamma_1, \dots, \gamma_g$ *be chosen in such a way that*

$$P_\pm \in a_0, \quad \gamma_s \in a_s. \tag{2.18}$$

Then the corresponding potential $u(x_+, x_-)$ *is real and smooth for all real* $x_\pm$.

Remark. It is likely that the cited conditions are not only sufficient, but also necessary for the algebraic-geometric potentials to be real and smooth.

Both lemmas can be proved in the same way as the corresponding statements for the elliptic equation (2.2) (see [8, 9]; the proof can also be found in [10]).

§3. Self-Consistency Conditions

Let $E(P)$ be a meromorphic function on Γ_0 with simple poles $Q_i^0 \in \Gamma_0$, $i = 1, \dots, N$. The preimages of these points on Γ will be denoted by Q_i, $Q_i^\sigma = \sigma(Q_i)$.

Lemma 3.1. *Let $\psi(x_+, x_-, Q)$ be the Baker–Akhiezer function; then*

$$\sum_{i=1}^{N} \varphi_i(x_+, x_-)\varphi_i^\sigma(x_+, x_-) = 1, \tag{3.1}$$

where

$$\varphi_i(x_+, x_-) = r_i \psi(x_+, x_-, Q_i), \tag{3.2}$$
$$\varphi_i^\sigma(x_+, x_-) = r_i \psi(x_+, x_-, Q_i^\sigma), \tag{3.3}$$

and the constants r_i are given by the formula

$$r_i^2 = \frac{\operatorname{res}_{Q_i} E(Q)\, d\Omega(Q)}{E_+ - E_-}, \qquad E_\pm = E(P_\pm). \tag{3.4}$$

Proof. Consider the differential

$$\widehat{d\Omega}(x_+, x_-, Q) = \frac{\psi(x_+, x_-, Q)\psi^\sigma(x_+, x_-, Q)E(Q)\, d\Omega(Q)}{E_+ - E_-}, \tag{3.5}$$

where, by definition,

$$\psi^\sigma(x_+, x_-, Q) = \psi(x_+, x_-, \sigma(Q)). \tag{3.6}$$

The functions $\psi(x_+, x_-, Q)$, $\psi^\sigma(x_+, x_-, Q)$ are meromorphic outside the marked points $P_\pm$, where they have essential singularities. It follows from (2.5) and (2.10) that the differential $\widehat{d\Omega}$ is meromorphic everywhere on Γ, including the points $P_\pm$. Condition (2.8) implies that this differential is holomorphic at the points γ_s and γ_s^σ. Therefore, $\widehat{d\Omega}$ can have poles only at the points Q_i, Q_i^σ, and $P_\pm$. The residues of this differential at these points are equal to

$$\operatorname{res}_{Q_i} \widehat{d\Omega} = \operatorname{res}_{Q_i^\sigma} \widehat{d\Omega} = \varphi_i(x_+, x_-)\varphi_i^\sigma(x_+, x_-), \tag{3.7}$$
$$\operatorname{res}_{P_\pm} \widehat{d\Omega} = \mp \frac{E_\pm}{E_+ - E_-}. \tag{3.8}$$

The sum of residues of a meromorphic differential on a compact Riemann surface is equal to zero, i.e., the sum of the right-hand sides in (3.7) and (3.8) is equal to zero, whence follows (3.1).

Lemma 3.2. *If $u(x_+, x_-)$ is the potential of the operator* (2.11) *corresponding to the Baker–Akhiezer function, then*

$$2u(x_+, x_-) = \sum_{i=1}^{N} \partial_+\varphi_i(x_+, x_-)\partial_-\varphi_i^\sigma(x_+, x_-) + \partial_-\varphi_i(x_+, x_-)\partial_+\varphi_i^\sigma(x_+, x_-). \tag{3.9}$$

Proof. Consider the meromorphic differential

$$d\Omega_1(x_+, x_-, Q) = \frac{E(Q)\, d\Omega(Q)}{E_+ - E_-}(\partial_+\psi\partial_-\psi^\sigma + \partial_-\psi\partial_+\psi^\sigma). \tag{3.10}$$

It is holomorphic everywhere except for the points Q_i, Q_i^σ, and $P_\pm$. The sum of its residues at the points Q_i and Q_i^σ is equal to the right-hand side in (3.9). It follows from (2.10) and (2.12) that

$$\operatorname{res}_{P_\pm} d\Omega_1 = \mp 2\frac{E_\pm}{E_+ - E_-}\partial_\mp \xi_1^\pm = \mp 2 \frac{E_\pm}{E_+ - E_-} u. \tag{3.11}$$

Again using the fact that the sum of all residues of a meromorphic differential is equal to zero, we obtain (3.9).

Lemma 3.3. *Suppose that the differential of the function* $E(Q)$ *is zero at the points* $P_\pm$*, i.e.,* $dE(P_\pm) = 0$*. Then*

$$\sum_{i=1}^{N} \partial_\pm \varphi_i(x_+, x_-)\, \partial_\pm \varphi_i^\sigma(x_+, x_-) = 0. \tag{3.12}$$

Proof. The left-hand side of (3.12) is equal to the sum of residues at the points Q_i and Q_i^σ of the differentials

$$d\Omega_\pm = (E(Q) - E_\pm)\, \partial_\pm \psi(x_+, x_-, Q)\, \partial_\pm \psi^\sigma(x_+, x_-, Q)\, d\Omega(Q). \tag{3.13}$$

Hence, to prove (3.12) it suffices to show that the residues of these differentials at the points $P_\pm$ are zero. The condition $dE(P_\pm) = 0$ implies that in some neighborhoods of $P_\pm$ we have

$$E(Q) = E_\pm + O(k_\pm^{-4}). \tag{3.14}$$

It follows from (3.14) and from the expansion (2.10) for the Baker–Akhiezer function that

$$\operatorname{res}_{P_\pm} d\Omega_+ = 0, \qquad \operatorname{res}_{P_\pm} d\Omega_- = 0. \tag{3.15}$$

So far, as well as in the beginning of the preceding section, we have considered the complex theory. Let us now suppose that the conditions of Lemma 3.3 are satisfied. These conditions are sufficient for reality and regularity of the potential $u(x_+, x_-)$ corresponding to the algebraic-geometric data

$$\{\Gamma, \gamma_1, \dots, \gamma_g, P_\pm\}. \tag{3.16}$$

Under these conditions the factor-curve $\Gamma_0 = \Gamma/\sigma$ is an M-curve with respect to the anti-involution $\tau_0 \colon \Gamma_0 \to \Gamma_0$ induced by the anti-involution $\tau \colon \Gamma \mapsto \Gamma$. Over each of the fixed ovals a_i^0, $i = 1, \dots, g_0$, of τ_0 there are two fixed ovals a_i and a_i^σ of the anti-involution τ, $\sigma(a_i) = a_i^\sigma$. The fixed oval a_0 of τ_0 which contains the points $P_\pm$ is divided by these points into two segments a_0^+ and a_0^-. The preimage of one of them, say, a_0^+, is a fixed cycle a_0 of the anti-involution τ, i.e.,

$$\tau(Q) = Q, \qquad Q \in a_0. \tag{3.17}$$

The preimage of a_0^- is a cycle $\tilde{a}_0$ fixed with respect to the anti-involution $\tau\sigma$, that is,

$$\tau\sigma(Q) = Q, \qquad Q \in \tilde{a}_0. \tag{3.18}$$

In addition to the conditions of Lemma 3.3, we assume that the meromorphic function $E(Q)$ is real and that

$$r_i^2 = \frac{1}{E_+ - E_-} \operatorname{res}_{Q_i^0} E(Q)\, d\Omega(Q) > 0. \tag{3.19}$$

Moreover, assume that one of poles Q_1^0 lies on the curve a_0^+ and all the others are on the curve a_0^-. This means that

$$\tau(Q_1) = Q_1, \quad \tau(Q_1^\sigma) = Q_1^\sigma, \tag{3.20}$$

$$\tau(Q_i) = Q_i^\sigma, \qquad i > 1. \tag{3.21}$$

It follows from (2.16) that

$$\varphi_1 = \overline{\varphi_1}, \quad \varphi_1^\sigma = \overline{\varphi_1^\sigma}, \tag{3.22}$$

$$\varphi_i = \overline{\varphi_i^\sigma}, \qquad i > 1. \tag{3.23}$$

Theorem 3.1. *Let* $\psi(x_+, x_-, Q)$ *be the Baker–Akhiezer function determined by the curve* Γ*, the pole divisors* γ_i*, the points* $P_\pm$*, and local parameters in the corresponding neighborhoods, satisfying the conditions of Lemma* 3.2*. Let a meromorphic function* $E(Q)$ *satisfy conditions* (3.19)–(3.21)*. Then the vector function* $q = q_i(x_+, x_-)$ *with coordinates given by* (1.17)*, where* φ_i *and* φ_i^σ *are defined in* (3.2) *and* (3.3)*, is a real nonsingular solution to the system of equations*

$$\begin{aligned} &(\partial_+\partial_- + u)q = 0, \qquad \langle q, q\rangle = 1, \\ &u = \langle \partial_+ q, \partial_- q\rangle, \qquad \langle \partial_\pm q, \partial_\pm q\rangle = 0. \end{aligned} \tag{3.24}$$

§4. Theta Function Formulas

In this section we present exact theta function formulas for solutions of the string equations in (2+1)-de Sitter space-time. We begin with an explicit construction of the double covering over the hyperelliptic curve Γ_0 defined by the equation

$$y^2 = (E - E_-)(E - E_+)\prod_{i=1}^{2n}(E - E_i) = R_{2n+2}(E)\,. \tag{4.1}$$

We assume that the real roots of the polynomial R_{2n+2} are enumerated as follows:

$$E_- < E_1 < E_2 < \ldots < E_{2n-1} < E_{2n} < E_+\,. \tag{4.2}$$

The real ovals of the anti-involution

$$\tau_0\colon (y, E) \mapsto (\overline{y}, \overline{E}) \tag{4.3}$$

are the cycles a_i^0 over the forbidden zones $[E_{2i-1}, E_{2i}]$.

As usual we introduce the basis of holomorphic differentials

$$\omega_j = \frac{1}{\sqrt{R}}\sum_{j=0}^{n-1} r_{ij}E^j\, dE, \tag{4.4}$$

normalized by the conditions

$$\int_{E_{2i-1}}^{E_{2i}} \omega_j = \frac{1}{2}\,\delta_{ij}\,. \tag{4.5}$$

The b-periods of these differentials define the so-called Riemann matrix

$$B_{kj} = 2\int_{E_-}^{E_{2j-1}} \omega_k\,. \tag{4.6}$$

The basis vectors e_k and the vectors B_k that are the columns of the matrix (4.6) generate a lattice $\mathcal{B}$ in $\mathbb{C}^n$. The n-dimensional complex torus

$$J(\Gamma_0) = \mathbb{C}^n/\mathcal{B}, \qquad \mathcal{B} = \sum n_k e_k + m_k B_k, \quad n_k, m_k \in \mathbb{Z}, \tag{4.7}$$

is called the Jacobian variety of Γ_0. The vector with coordinates

$$A_k(P) = \int_{E_-}^{P} \omega_k \tag{4.8}$$

defines the Abel map

$$A\colon \Gamma_0 \to J(\Gamma_0)\,. \tag{4.9}$$

The Riemann matrix is symmetric and has positive-definite imaginary part. The entire function of n variables

$$\theta(z) = \theta(z|B) = \sum_{m\in\mathbb{Z}^n} e^{2\pi i(z,m)+\pi i(Bm,m)}, \tag{4.10}$$

$$z = (z_1, \ldots, z_n), \quad m = (m_1, \ldots, m_n), \quad (z,m) = z_1m_1 + \ldots + z_nm_n,$$

determined by this matrix, is called the Riemann theta function. It has the monodromy properties

$$\theta(z + e_k) = \theta(z), \qquad \theta(z + B_k) = e^{-2\pi i z_k - \pi i B_{kk}}\theta(z)\,. \tag{4.11}$$

Although the function $\theta(A(P) - Z)$ is a many-valued function of P, the zeros of this function are well-defined according to (4.11). For Z in a generic position, the equation

$$\theta(A(\gamma_s) - Z) = 0, \qquad s = 1, \dots, n, \tag{4.12}$$

has n roots. At the same time,

$$Z_k = \sum_{i=1}^{n} \int_{E_{2i}}^{\gamma_i} \omega_k . \tag{4.13}$$

It follows from (4.13) and (4.12) that for the vectors $Z^{\pm}$ with coordinates

$$Z_k^{\pm} = \int_{E_2}^{E_{\pm}} \omega_k , \tag{4.14}$$

the roots of Eq. (4.12) are the points

$$E_{\pm}, \quad E_{2i}, \; i = 2, 3, \dots, n .$$

Therefore, the function

$$\lambda(P) = \frac{\theta(A(P) - Z^{+})\theta(A(P) - Z^{-})}{\theta^2(A(P) - \xi)}, \qquad \xi = \frac{Z^{+} + Z^{-}}{2}, \tag{4.15}$$

has simple zeros at the points $E_{\pm}$ and double zeros at the points E_{2i}, $i = 2, \dots, n$. It also has double poles at the points

$$\widehat{\gamma}_0 \in a_0^0, \; \widehat{\gamma}_2 \in a_2^0, \; \widehat{\gamma}_3 \in a_3^0, \dots, \; \widehat{\gamma}_n \in a_n^0 .$$

(It should be mentioned that it follows from (4.11) that the right-hand side in (4.15) has trivial monodromy with respect to the lattice $\mathcal{B}$. This means that $\lambda(P)$ is a well-defined function on Γ_0.)

The function $\lambda(P)$ is real on real cycles of Γ_0. Since its zeros are double, it follows that it has constant signs on each of the cycles $a_1^0, \dots, a_n^0$. Let

$$s_i = \tfrac{1}{2} \operatorname{sgn} \lambda(P)|_{P \in a_i^0}, \quad \varepsilon_i = \tfrac{1}{2} + s_i, \qquad i = 1, \dots, n . \tag{4.16}$$

Then the function

$$\mu(P) = (-1)^{\varepsilon_1} \lambda(P) \prod_{i=2}^{n} \left(\frac{E - E_i}{E - E_{2i-1}} \right)^{\varepsilon_i} \tag{4.17}$$

has simple zeros at $E_{\pm}$, double zeros at E_{2i} or E_{2i-1} (depending on the sign of s_i), and double poles at the points $\widehat{\gamma}_s$, $s = 0, 2, 3, \dots, n$. This function is positive on the cycles a_i^0, $i = 1, 2, \dots, n$.

We denote by Γ the Riemann surface of the function

$$f = \sqrt{\mu(P)} . \tag{4.18}$$

The surface Γ is a double covering over Γ_0 with two branch points $E_{\pm}$. The holomorphic differentials on Γ_0 are even differentials on Γ with respect to permutations of the sheets. Also, there is an n-dimensional space of odd differentials. As a basis of such differentials we shall choose

$$\widetilde{\omega}_j^{\mathrm{od}} = \frac{dE}{\sqrt{\mu(P)}} \, \frac{\theta(A(P) - \alpha_{+})\theta(A(P) - \alpha_{-})}{\theta(A(P))\theta(A(P) - e_1/2)} Y_j(P), \tag{4.19}$$

where

$$Y_j(P) = \frac{\theta(A(P) - e_j/2)\theta(A(P) - C_j)}{\theta(A(P) - \xi)\theta(A(P) + \hat{e} - \hat{s})} . \tag{4.20}$$

The vectors $\alpha_\pm$ in (4.19) are equal to

$$\alpha_\pm = A(\pm\infty) - A(E_2), \tag{4.21}$$

where $\pm\infty$ are the preimages of the point $E = \infty$ on the two sheets of Γ_0. The vector C_j in (4.19) is

$$C_j = -\alpha_+ - \alpha_- - \tfrac{1}{2}e_j + \tfrac{1}{2}e_1 + \xi + \hat{s} - \hat{e}. \tag{4.22}$$

It is uniquely determined by the condition that the right-hand side of (4.19) is invariant by translation by periods. In (4.19)–(4.22), e_j are basis vectors; $\hat{e} = (0, 1, 1, \dots, 1)$; $\hat{s} = (0, s_2, \dots, s_n)$ (where s_i are given by (4.16)). The basis of normalized differentials $\omega_j^{\mathrm{od}} = \sum \alpha_{ij}\widetilde{\omega}_i^{\mathrm{od}}$ is defined by the conditions

$$\oint_{E_{2i-1}}^{E_{2i}} \omega_j^{\mathrm{od}} = \delta_{ij}. \tag{4.23}$$

(Here and below $\oint_P^Q$ stands for the integral along the cycle lying on one of the sheets of Γ over a cycle that surrounds the points P and Q.) The matrix

$$B_{ij}^{\mathrm{Pr}} = \oint_{E_-}^{E_{2i-1}} \omega_j^{\mathrm{od}} \tag{4.24}$$

of b-periods of normalized odd differentials defines the Prim theta function

$$\theta_{\mathrm{Pr}}(z) = \theta(z|B^{\mathrm{Pr}})$$

via (4.10). Let us introduce some more definitions. First of all, for a point $Q \in \Gamma$ we define

$$A_k^{\mathrm{od}}(Q) = \int_{E_-}^{Q} \omega_k^{\mathrm{od}}, \qquad k = 1, \dots, n. \tag{4.25}$$

This vector is determined up to the lattice of periods of the Prim function. For Z in a generic position the equation

$$\theta_{\mathrm{Pr}}(A^{\mathrm{od}}(\gamma_s) - Z) = 0, \qquad s = 1, \dots, 2n, \tag{4.26}$$

has $2n$ roots. Moreover, for an arbitrary Z they form an admissible divisor. Recall that this means that the projections of these points $\widehat{\gamma}_s = \pi(\gamma_s)$ on the hyperelliptic curve Γ_0 are zeros of a third-kind meromorphic differential, $d\Omega(\widehat{\gamma}_s) = 0$. The differential has the form

$$d\Omega(Q) = \frac{(E_- - E_+)\,dE}{(E - E_+)(E - E_-)} \frac{H(Q)}{H(E_-)}, \tag{4.27}$$

where $E = E(Q)$ is the projection of Q onto the E-plane and

$$H(Q) = \frac{\theta_{\mathrm{Pr}}(A^{\mathrm{od}}(Q) - Z)\,\theta_{\mathrm{Pr}}(A^{\mathrm{od}}(Q) + Z)\,\theta^2(A(P) - e_*/2)}{\theta_{\mathrm{Pr}}^2(A^{\mathrm{od}}(Q))\,\theta(A(P))\,\theta(A(P) - e_*)}. \tag{4.28}$$

(Here θ stands for the Riemann theta function of the hyperelliptic curve Γ_0 and the vector e_* is $(1, 1, \dots, 1)$.)

The converse statement is also true, i.e., any admissible divisor γ_s in a generic position can be obtained as the roots of Eq. (4.26) for the corresponding vector Z. That is why below we consider the components of Z as free parameters instead of admissible divisors.

Important remark. The exact construction of the covering Γ over the initial curve Γ_0 and the explicit formulas for normalized odd differentials and the Prim theta function were presented for the case in which Γ_0 is a hyperelliptic curve. Nevertheless, formula (4.29) below is valid in the general case.

Theorem 4.1 [9]. *The Baker–Akhiezer function defined in §2 can be represented in the form*

$$\psi = \frac{\theta_{\mathrm{Pr}}(A^{\mathrm{od}}(Q) + U^+x_+ + U^-x_- - Z)\theta_{\mathrm{Pr}}(A^{\mathrm{od}}(Z))}{\theta_{\mathrm{Pr}}(A^{\mathrm{od}}(Q) - Z)\theta_{\mathrm{Pr}}(U^+x_+ + U^-x_- - Z)} e^{i(p^+(Q)x_+ + p^-(Q)x_-)}. \tag{4.29}$$

Here $p^\pm(Q)$ are Abelian integrals of the second-kind normalized differential $dp^\pm$ on Γ that have poles of the second order at the points $P_\pm$, respectively; the vectors $2\pi U^\pm$ are the vectors of b-periods of these differentials.

In our case Γ_0 is a hyperelliptic curve, and the differentials $dp_\pm$ with second-order poles at the points $E_\pm$ have the form

$$dp^\pm = \frac{h^\pm\, dE}{(E - E_\pm)\sqrt{\mu(P)}} + \alpha_j^\pm \omega_j^{\mathrm{od}}. \tag{4.30}$$

The constants $\alpha_j^\pm$ are determined by normalization conditions

$$\oint_{E_{2i-1}}^{E_{2i}} dp^\pm = 0\,. \tag{4.31}$$

Hence

$$U_k^\pm = \frac{1}{\pi}\oint_{E_-}^{E_{2k-1}} dp^\pm. \tag{4.32}$$

There are two points on Γ_0 over $E = \infty$. We denote them by $\pm\infty$. At one of them, say $Q_1^0 = +\infty$, the function $\mu(Q_1^0) > 0$. According to the results of the previous sections, the values of the Baker–Akhiezer function at the points Q_1, Q_1^σ and Q_2, Q_2^σ that cover the points Q_1^0 and $Q_2^0 = -\infty$ define solutions of the string equations in $(2+1)$-de Sitter space-time. So let

$$A_j = A^{\mathrm{od}}(Q_j), \quad p_j^\pm = \int_{E_-}^{Q_j} dp^\pm, \qquad j = 1, 2, \tag{4.33}$$

and

$$r_j^2 = H(Q_j)/H(E_-), \tag{4.34}$$

where $H(Q)$ is given by (4.28). (The constants r_j^2 are the residues of the differential (3.5), where $d\Omega$ is given by (4.27).)

Theorem 4.2. *Any set* (4.2) *of real points E_s, an n-dimensional real vector Z, and real constants $h^\pm$ define real smooth solutions of the string equations* (3.24) *in* $(2+1)$*-de Sitter space-time with the help of formulas* (1.13) *and* (1.17) (*where the vectors $U^\pm$, A_j, the constants $p_j^\pm$ and r_j, $j = 1, 2$, are defined by* (4.32), (4.33), *and* (4.34)).

In general, since

$$\operatorname{Im} p_1^\pm \neq 0,$$

the components q_1, q_2 of the corresponding solutions are unbounded or tend to zero as $\tau, \sigma \to \pm\infty$. The constants $p_2^\pm$ are real. Hence, the functions q_3, q_4 are almost periodic functions of the variables τ and σ.

Corollary. *If for given E_s the constants $h^\pm$ are chosen in such a way that*

$$p_1^+ - p_1^- = 0, \tag{4.35}$$

then the corresponding solutions of the string equations are almost periodic functions of the variable σ. The additional conditions (1.21) *specify a $2n+1$-parametric family of solutions which are 2π-periodic in σ.*

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