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MULTI-PHASE SOLUTIONS OF THE BENJAMIN–ONO EQUATION AND THEIR AVERAGING

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The Benjamin–Ono equation [1]

$$u_t + 2uu_x - \mathrm{P.V.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u_{yy}(y)}{y-x}\,dy = 0, \tag{0.1}$$

which appears in a number of problems of mathematical physics is a non-local analogue of the Korteweg–de Vries equation. In particular, as shown in [2-3], it describes the propagation of wave packets in a boundary layer. These papers and our discussions with O. S. Ryzhovyi have prompted us to pursue the investigations described below.

We can apply the general approach of the inverse problem method to Eq. (0.1), i.e., we can write it as a compatibility condition for an overdetermined system of supplementary linear problems [4, 5]. As a consequence, the direct and inverse scattering problems for the supplementary linear system can be used to solve the Cauchy problem with quickly decreasing initial values. In the framework of this approach there naturally appear exact "multisoliton solutions" which are rational functions of their arguments (these questions are studied in a large number of papers; we do not claim completeness here, instead directing the reader to [6-8], where even more detailed bibliographies may be found).

One of the main goals of this article is to obtain a large class of quasiperiodic solutions of Eq. (0.1) by using the ideas and the methods of the theory of finite-zone integration.

Here we find it appropriate to make a few remarks. The general method of finite-zone integration allows us to construct periodic and quasiperiodic solutions of nonlinear equations that admit various types of commutative representation. The corresponding solutions in general may be written in terms of theta functions of supplementary Riemann surfaces of finite genus (algebraic curves). As a limit case corresponding to the degeneration of algebraic curves to rational ones with singularities, the finite-zone construction provides a rather simple and effective method for constructing multisoliton and rational solutions of the original nonlinear equations (see [9, 10]). The construction of such limit solutions can be described in a closed form, using only the simplest elements of linear algebra. The authors of [10] have described a similar construction of integrable potentials of a nonstationary Schrödinger operator

$$(i\partial_t + \partial_x^2 + v(x,\ t))\,\psi(x,\ t,\ k) = 0 \tag{0.2}$$

and solved a number of related nonlinear equations. The latter paper is especially relevant to us since, as it will become apparent later, for a special choice of parameters the

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integrated potentials (0.2) result in solutions of the Benjamin-Ono equation. The resulting solutions from the point of view of the general algebrogeometric approach are completely analogous to multisoliton solutions of the nonlinear Schrödinger equation and a series of other non-linear equations (i.e., they correspond to rational curves with double singularities). Equation (0.1) has a special property in that these solutions are quasi-periodic. Therefore, in the sequel we do not call them multi-soliton, preferring to use the term "multiphase solutions."

These solutions have form

$$u = u_0 (K_x + Wt + \Phi \mid I_1, \ldots, I_N), \tag{0.3}$$

where u_0 (z_1, ..., $z_n|I$) is a function that is 2π-periodic in every one of the variables z_i and depends on a set of parameters $\{I_k\}$ like on parameters, vectors K(I) and W(I) also depend on the parameter I_k, and vector Φ is arbitrary.

Solutions of form (0.3) have been obtained using an analogue of Hirota's method in [8]. As noted before, in this article we use a different approach, which subsequently plays an important part in the averaging procedure. In particular, we use a method for parameterization different than the one used in [8].

Equations having a sufficiently large supply of solutions of form (0.3) we can average with Whitham's averaging method (a non-linear WKB-method) (see [11-13]) by constructing asymptotic solutions of form

$$u(x, t) = u_0 (\varepsilon^{-1} S(X, T) + \Phi(X, T) \mid I(X, T)) + \tilde{u}_1 + \ldots, \tag{0.4}$$

where the parameters I and Φ are now functions of "slow" variables $X = \varepsilon x$, $T = \varepsilon t$. If a vector S(X, T) satisfies

$$\partial_X S = K(I(X, T)); \qquad \partial_T S = W(I(X, T)), \tag{0.5}$$

then u(x, t) satisfies the original non-linear equation with an accuracy of $O(\varepsilon)$ [11-13].

Whitham's equations are equations describing the dependence of I_k (X, T) on slow variables. In the second paragraph of this article we derive Whitham's equations for parameters of multiphase solutions of the Benjamin—Ono equation using an approach described in [14]. They are the necessary conditions for the existence of an asymptotic solution of form (0.4) with a uniformly bounded (with respect to x and t) first term u_1 (x, t).

Remarkably, in the initial parameters of the construction these equations split, becoming Riemann—Hopf equations

$$\partial_T I_k = \partial_X I_k^2. \tag{0.6}$$

Consequently, solutions of Whitham's equations are defined implicitly by relations

$$I_k (X, T) = f_k (X + I_k T), \tag{0.7}$$

where the functions $f_k(x)$ are equal to the initial values for the Cauchy problem, i.e., $f_k(X) = I_k(X, 0)$.

1. Construction of Multiphase Solutions of the Benjamin—Ono Equation. The Benjamin—Ono equation is equivalent to the compatibility condition for a system of linear equations

$$\begin{gathered}(i\partial_t + \partial_x^2 - 2U_{j,x})\psi_j = 0, \qquad j = 1, 2, \\ i\partial_x\psi_1 + u\psi_1 = \lambda\psi_2,\end{gathered} \tag{1.1}$$

assuming that U_1 (x, t) and U_2 (x, t) can be analytically extended to the upper and lower half-planes of the variable x, respectively [4, 5]. Indeed, from (1.1) we obtain

$$iu = U_1 - U_2 + c(t), \tag{1.2}$$

$$u_t + 2uu_x + (U_{1,xx} + U_{2,xx}) = 0. \tag{1.3}$$

Since U_1 and U_2 can be analytically extended to the upper and lower half-planes, from (1.2) we see that the corresponding piecewise analytic function can be written in terms of u(x, t) by using an integral of Cauchy type. Using Plemen'—Sokhotskii equations, we obtain

$$U_1 = \frac{i}{2} u + \frac{1}{2\pi} \int \frac{u-c}{x-y} dy, \quad U_2 = -\frac{i}{2} u + \frac{1}{2\pi} \int \frac{u-c}{x-y} dy. \tag{1.4}$$

Substituting (1.4) into (1.3), we obtain (0.1).

Using the mentioned representation of the Benjamin–Ono equation, we now prove the main result of this paragraph as stated below. Fix a set of numbers a_i, b_i, c_i $(i = 1, \ldots, n)$ and define a matrix

$$M_{jm} = c_m e^{i(a_m - b_m)x - i(a_m^2 - b_m^2)t} \delta_{jm} - \frac{1}{b_j - a_m}. \tag{1.5}$$

THEOREM 1.1. Suppose that C, a_m, b_m are real numbers such that

$$C < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n, \tag{1.6}$$

and let

$$|c_i|^2 = -\frac{(b_i - C) \prod_{j \neq i} (a_i - a_j)(b_i - b_j)}{(a_i - C) \prod_j (b_i - a_j)(a_i - b_j)}. \tag{1.7}$$

Then a formula

$$u(x, t) = C + \sum_m (a_m - b_m) - 2 \operatorname{Im} \partial_x \ln \det M(x, t) \tag{1.8}$$

defines a real non-singular quasi-periodic solution of the Benjamin–Ono equation.

Remark. Solutions (1.8) can be written as

$$u = u_0 (Kx + Wt + \Phi \mid a_i, b_i, C), \tag{1.9}$$

where the n-periodic function u_0 and vectors K and W are determined by the values a_i, b_i, C and the components of the vector Φ are equal to

$$\Phi = \arg c_i. \tag{1.10}$$

Proof: Consider a function $\psi_1 (x, t, k)$

$$\psi_1 = \left(1 + \sum_{m=1}^{n} \frac{r_m(x, t)}{k - a_m}\right) e^{ikx - ik^2 t}, \tag{1.11}$$

such that

$$c_m \operatorname{res}_{k=a_m} \psi_1 = \psi_1 (x, t, b_m). \tag{1.12}$$

Linear conditions (1.12) are equivalent to the following non-homogeneous system of linear equations in quantities r_m:

$$\sum_{m=1}^{n} M_{jm}(x, t) r_m(x, t) = 1. \tag{1.13}$$

LEMMA 1.1. The matrix M is nondegenerate for $\operatorname{Im} x \geq 0$.

Proof: Assume that $M(x_0, t_0)$ is degenerate for some real x_0 and t_0. This means that there exists a fucntion ψ_0 of form

$$\psi_0 (k) = \sum_{m=1}^{n} \frac{r_m^0}{k - a_m} e^{ikx_0 - ik^2 t_0}, \tag{1.14}$$

satisfying relations (1.12). Define a differential

$$d\Omega = \psi_0(k) \overline{\psi}_0(\bar{k}) \, dk \prod_{i=1}^{n} \frac{k - a_i}{k - b_i} \tag{1.15}$$

This differential is meromorphic in k and has a zero residue at infinity

$$\operatorname{res}_\infty d\Omega = 0.$$

On the other hand, from (1.12), (1.6), and (1.7) we see that

$$\operatorname{res}_{k=a_m} d\Omega + \operatorname{res}_{k=b_m} d\Omega = |R_m|^2 \frac{\prod_{i\neq m}(a_m - a_i)}{\prod_i (a_m - b_i)}\left(1 - \frac{b_m - C}{a_m - C}\right) > 0, \tag{1.16}$$

where

$$R_m = r_m^0 \exp(ia_m x_0 - ia_m^2 t_0).$$

Therefore, the sum of all residues of $d\Omega$ is positive, which is impossible. This contradiction proves the nondegeneracy of $M(x, t)$ for real x and t.

Define a function

$$U_1 = i\sum_{m=1}^{n}(a_m - b_m) - \partial_x \ln\det M(x, t). \tag{1.17}$$

The definition of M implies that

$$U_1(x, t) = O(e^{-\alpha \operatorname{Im} x}), \quad \alpha = \min_m (b_m - a_m). \tag{1.18}$$

If all differences $a_m - b_m$ can be written as

$$(a_m - b_m) = \frac{2\pi}{T} s_m, \quad s_m \text{ are integers}, \tag{1.19}$$

then the matrix $M(x, t)$ is a periodic function in the variable x. The number of zeros of the function $\det M$ in a region: $\operatorname{Im} x > 0$, $0 < \operatorname{Re} x \leq T$ is equal to

$$N = \frac{1}{2\pi}\int_0^T U_1(x, t)\,dx.$$

This number does not change if we change the parameters a_i and b_i continuously while preserving relations (1.19). It is easy to see that as $|a_i - a_j| \to \infty$ we have $N = 0$. Therefore, we have proved the lemma for an everywhere dense subset of parameters corresponding to periodic matrices M. The function U_1 is analytic with respect to parameters. Therefore, it is in general regular for $\operatorname{Im} x \geq 0$. Q.E.D.

It is known that the function $\psi_1(x, t, k)$ satisfies an equation

$$(i\partial_t + \partial_x^2 - 2U_{1,x}(x, t))\psi_1(x, t, k) = 0, \tag{1.20}$$

where $U_1 = i\Sigma_m r_m(x, t)$ is equal to (1.17) (for more details see [11]). Furthermore, using (1.5) we obtain estimates (1.18) and (1.21):

$$\psi_1 = e^{ikx - ik^2 t}(1 + O(e^{-\alpha \operatorname{Im} x})). \tag{1.21}$$

Define a function $\psi_2(x, t, k)$ by letting

$$\psi_2 = \left(1 + \sum_{j=1}^{n} \frac{\tilde{r}_j(x, t)}{k - b_j}\right) e^{ikx - ik^2 t}, \tag{1.22}$$

which is normalized by conditions

$$\tilde{c}_j \operatorname{res}_{k=b_j} \psi_2 = \psi_2(x, t, a_j), \tag{1.23}$$

where $\tilde{c}_j$ is an arbitrary set of constants. Proceeding as before, we obtain

$$(i\partial_t + \partial_x^2 - 2U_{2,x}(x, t))\psi_2(x, t, k) = 0, \tag{1.24}$$

$$U_2 = i\sum_{m=1}^{n}(b_m - a_m) - \partial_x \ln\det \tilde{M}(x, t), \tag{1.25}$$

where the matrix $\tilde{M}$ consists of elements

$$\tilde{M}_{jm} = \tilde{c}_m \delta_{jm} \exp[i(b_m - a_m)x - i(b_m^2 - a_m^2)t] - \frac{1}{a_j - b_m}. \tag{1.26}$$

In addition, $U_2(x, t)$ and $\psi_2(x, t, k)$ are analytic on the lower half-plane $\mathrm{Im}\, x \le 0$. Furthermore,

$$U_2(x, t) = O(e^{\alpha\, \mathrm{Im}\, x}), \tag{1.27}$$

$$\psi_2 = e^{ikx - ik^2 t}(1 + O(e^{\alpha\, \mathrm{Im}\, x})). \tag{1.28}$$

Let

$$\lambda(k) = -(k-c)\prod_{i=1}^{n}\frac{(k-b_i)}{(k-a_i)}. \tag{1.29}$$

LEMMA 1.2. If constants c_i and $\tilde{c}_i$ are related by

$$\tilde{c}_i^{-1} = -c_i\frac{b_i - c}{a_i - c}\frac{\prod_{j\neq i}(a_i - a_j)(b_i - b_j)}{\prod_j (b_i - a_j)(b_j - a_i)}, \tag{1.30}$$

then the functions $\psi_1(x, t, k)$ and $\psi_2(x, t, k)$ satisfy

$$i\partial_x\psi_1 + u(x, t)\psi_1 - \lambda(k)\psi_2(x, t, k) = 0. \tag{1.31}$$

where

$$u = \sum_{i=1}^{n}(r_i - \tilde{r}_i) + C - \sum_{i=1}^{n}(a_i - b_i) = C - \sum_{i=1}^{n}(a_i - b_i) + i(U_2(x, t) - U_1(x, t)). \tag{1.32}$$

Proof. The definition of $\lambda(k)$ and relations (1.30) immediately imply that the function $\lambda(k)\psi_2(x, t, k)$ satisfies relations (1.12). Let $\tilde{\psi}(x, t, k)$ be the function defined by the left-hand side of (1.31). From (1.32) we see that it has from

$$\tilde{\psi}(x, t, k) = e^{ikx - ik^2 t}\sum_{j=1}^{n}\frac{\hat{r}_j(x, t)}{k - a_j}. \tag{1.33}$$

Since it satisfies relations (1.12), $\hat{r}_m$ are solutions of a homogeneous system of equations

$$\sum_m M_{jm}\hat{r}_m = 0.$$

The matrix M is nondegenerate. Therefore, all $\hat{r}_m$ are equal to zero. Q.E.D.

LEMMA 1.3. Suppose that $\hat{c}_j$, as defined by Eq. (1.30), are such that

$$\hat{c}_i = -\bar{c}_i, \tag{1.34}$$

Then

$$U_1(x, t) = \bar{U}_2(\bar{x}, t). \tag{1.35}$$

Proof. Define functions

$$\overset{+}{\psi_1} = \overline{\psi_2(\bar{x}, t, \bar{k})}, \qquad \overset{+}{\psi_2} = \overline{\psi_1(\bar{x}, t, \bar{k})}. \tag{1.36}$$

A function $\psi_1(x, t, k)\psi_1^+(x, t, k)$ is a rational function with poles at points a_m, b_m. The defining relations (1.12) and (1.23) and condition (1.34) imply that for every m the sum of residues of this function at points a_m and b_m is equal to zero. Therefore, the residue at infinity is also equal to zero:

$$0 = \mathrm{res}_\infty\,\psi_1\overset{+}{\psi_1} = \sum_m r_m(x, t) + \overline{\tilde{r}_m(\bar{x}, t)} = i(\bar{U}_2(\bar{x}, t) - U_1(x, t)). \tag{1.37}$$

This proves the lemma and therefore the theorem.

Lemma 1.3 implies that functions ψ_1^+ and ψ_2^+ satisfy the following equations, which are conjugate to (1.1):

$$(-i\partial_t + \partial_x^2 - 2U_{j,x})\overset{+}{\psi_j} = 0,$$
$$-i\partial_x\overset{+}{\psi_2} + u\overset{+}{\psi_2} = \lambda\overset{+}{\psi_1}. \tag{1.38}$$

We conclude this paragraph by remarking that functions ψ_j and ψ_j^+ ($j = 1, 2$) can be written as

$$\psi_j = R_j\,(Kx + Wt + \Phi, k)e^{ikx-ik^2t},$$
$$\psi_j^+ = R_j^+\,(Kx + Wt + \Phi, k)e^{-ikx+ik^2t}, \tag{1.39}$$

where the fucntions $R_j(z_1, \ldots, z_n, k)$, $R_j^+(z_1, \ldots, z_n, k)$ are periodic functions of variables z_i.

2. Averaging and Whitham's Equations. We now construct asymptotic solutions of form (0.4) and derive equations that relate phases S_1 (X, T), .., S_n (X, T) and the slowly changing parameters.

We first show that if relations (0.5) are satisfied then a function $\tilde{u} = u_0\,(S(X, T)/\varepsilon + \Phi\,(X, T), I\,(X, T))$ satisfies Eq. (0.1) with accuracy $O(\varepsilon)$. We find it convenient to rewrite the integrodifferential operator in (0.1) as a pseudodifferential operator. After some straightforward calculations (see [6-8]) we obtain

$$\text{P. V.}\int_{-\infty}^{\infty} \frac{u_{yy}(y)}{y-x}\,dy = L\Big(-i\frac{\partial}{\partial x}\Big)u, \quad L(p) = -\pi i p\,|p|.$$

Thus, using variables $X = \varepsilon x$ and $T = \varepsilon t$ we can rewrite Eq. (0.1) as follows:

$$\varepsilon\frac{\partial u}{\partial T} + 2\varepsilon u\frac{\partial u}{\partial X} + L\Big(-i\varepsilon\frac{\partial}{\partial X}\Big)u = 0. \tag{2.1}$$

We substitute the function $\tilde{u}$ into (2.1) and compute the expansion of $L(-i\varepsilon\,\partial/\partial X)\tilde{u}$, using a method described in [15]. Namely, we write $\tilde{u}$ as $\tilde{u} = e^{(iS\cdot\hat{\omega})/\varepsilon}u_0(z + \Phi, I)|_{z=0}$, $\hat{\omega} = i\partial/\partial z$, $z = (z_1, \ldots, z_n)$, $z_j \in [0, 2\pi]$. Using the formula for the commutation of a pseudo-differential operator with the exponential cited in [16], we obtain

$$\begin{aligned} L\Big(-i\varepsilon\frac{\partial}{\partial X}\Big)\tilde{u} &= e^{iS\hat{\omega}/\varepsilon}L\Big(\frac{\partial S}{\partial X}\hat{\omega} - i\varepsilon\frac{\partial}{\partial X}\Big)u_0(z+\Phi, I) = L\Big(\frac{\partial S}{\partial X}\hat{\omega}\Big) - i\varepsilon\Big(\frac{\partial L}{\partial p}\Big(\frac{\partial S}{\partial X}\hat{\omega}\Big)\frac{\partial}{\partial X} \\ &+ \frac{1}{2}\frac{\partial^2 L}{\partial p^2}\Big(\frac{\partial S}{\partial X}\hat{\omega}\Big)\cdot\frac{\partial^2 S}{\partial X^2}\hat{\omega}\Big)u_0(z+\Phi, I)\Big|_{z=S/\varepsilon} + O(\varepsilon^2). \end{aligned} \tag{2.2}$$

Thus, making the substitution into Eq. (2.1) and retaining the summands of zeroth order in ε, we obtain

$$Q = \Big(S_T\cdot\frac{\partial}{\partial z}\Big)\tilde{u} + 2\tilde{u}\Big(S_x\cdot\frac{\partial}{\partial z}\Big)\tilde{u} + L\Big(-iS_x\cdot\frac{\partial}{\partial z}\Big)\tilde{u}\,\Big|_{z=S/\varepsilon},$$

S_T, $_x\cdot\partial/\partial z = S_{1T}$, $_x\partial/\partial z_1 + \ldots + S_{nT}$, $_x\partial/\partial z_n$. We show that $Q = 0$ if $\tilde{u}$ is of the form (1.9) and S and I satisfy relations (0.5). Indeed, we replace coordinates $z = (z_1, \ldots, z_n)$ with coordinates $t, x, y_1, \ldots, y_{n-2}$ using formulas $z = Kx + Wt + U_1y_1 + \ldots + U_{n-2}y_{n-2}$, where the vectors K, W, and U_j are linearly independent. Using (0.5) we then obtain

$$Q = \frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} + L\Big(-i\frac{\partial}{\partial x}\Big)u\Big|_{z=S/\varepsilon}, \tag{2.3}$$

where $u = u_0\,(Kx + Wt + \Phi', I)$, $\Phi' = U_1y_1 + \ldots + U_{n-2}y_{n-2} + \Phi(X, T)$, $I = I(X, T)$. Since Eq. (2.3) does not contain derivatives with respect to variables $X, T, y_1, \ldots, y_{n-2}$ and a function $u_0\,(Kx + Wt + \Phi', I)$ satisfies the Benjamin–Ono equation for all Φ' and K, W, and I that satisfy (0.6), the remainder resulting from a substitution of $\tilde{u}$ into (2.1) is equal to $O(\varepsilon)$. Using Eq. (2.2), we can easily compute this remainder $\tilde{F} = \varepsilon\partial\tilde{u}/\partial T + 2\varepsilon\tilde{u}\partial\tilde{u}/\partial X + L\,(-i\varepsilon\partial/\partial X)\tilde{u}$:

$$\tilde{F} = \varepsilon F + O(\varepsilon^2), \tag{2.4}$$

$$F = \Big(\frac{\partial u_0}{\partial T} + 2u_0\frac{\partial u_0}{\partial X} + \frac{\partial L}{\partial p}\Big(-iK\cdot\frac{\partial}{\partial z}\Big)\frac{\partial u_0}{\partial X} + \frac{1}{2}\frac{\partial^2 L}{\partial p^2}\Big(-iK\cdot\frac{\partial}{\partial z}\Big)\frac{\partial^2 S}{\partial X^2}\cdot\frac{\partial u_0}{\partial z}\Big)\Big|_{z=S/\varepsilon}.$$

We now derive relations supplementing (0.5) to a closed system of equations. These relations are obtained by studying the equations for corrections to the leading term of the asymptotics of $\tilde{u}$. Procedures for calculating corrections differ significantly for one- and multiphase cases. In the first case it is sufficiently well understood (see, for example,

[11]), and there exists an algorithm in the theory of perturbations that gives the asymptotic solution as a series in powers of ε: $u = u_0(S/\varepsilon + \Phi, I(x, t)) + \varepsilon u_1 (S/\varepsilon, x, t) + \varepsilon^2 y_2 (S/\varepsilon, x, t)+ \ldots$. Every term of the series has the same "one-phase" structure. In the case of two or more phases this representation for the asymptotic solution no longer holds, as dictated by the appearance of resonances, i.e., a set of points (X_p, T_p) at which the dimension of the cokernel of the inverted operator changes if we assume that the corrections to $\tilde{u}$ have the same "n-phase" structure (see (2.5')). The appearance of resonance points filling a line that is everywhere dense in $\mathbb{R}_X$ for each fixed T significantly changes and complicates the perturbation theory. We derive results only in the two-phase case for the Korteweg–de Vries equation [17]. In this case even the construction of the first-order correction is already a highly nontrivial problem that requires, in particular, a study of an equation that in general is nonlinear.

We do not engage here in a presentation of the theory of perturbations, and will only demonstrate a method for obtaining the necessary conditions for the smallness of the correction to $\tilde{u}$, assuming that the smallness of this correction requires the smallness of a solution $\tilde{u}_1$ of the following equation, which is a linearization of (nonhomogeneous) Eq. (0.1) in terms of $\tilde{u}$:

$$\varepsilon \frac{\partial \tilde{u}_1}{\partial T} + 2\varepsilon \frac{\partial}{\partial X}(\tilde{u}\tilde{u}_1) + L\left(-i\varepsilon \frac{\partial}{\partial X}\right)\tilde{u}_1 = -\varepsilon F(S/\varepsilon, X, T), \tag{2.5}$$

The linear operator on the left-hand side of (2.5) defines a family of operators $\mathscr{L}$ on the torus $\mathrm{T}^n = (z_1, \ldots, z_n | z_j \in [0, 2\pi])$, that depend on X and T as parameters and are obtained from (2.5) by substituting $\varepsilon\partial/\partial t \to S_t \cdot \partial/\partial z = W \cdot \partial/\partial z$ and $\varepsilon \partial/\partial x \to S_x \cdot \partial/\partial z = K \cdot \partial/\partial z$:

$$\mathscr{L} = W\frac{\partial}{\partial z} + 2K\frac{\partial}{\partial z}((\tilde{u}(z+\Phi, X, T))\cdot) + L\left(-iK\frac{\partial}{\partial z}\right). \tag{2.5'}$$

Assume that a smooth function w(z, X, T) is 2π-periodic in every variable $z_1, z_2, \ldots, z_n$, and belongs to the cokernel of an operator $\mathscr{L}$ for every $(X, T) \in \Omega$, i.e.,

$$-\mathscr{L}^* w \equiv \left(W \cdot \frac{\partial}{\partial z} + 2u_0(z+\Phi, X, T) K \cdot \frac{\partial}{\partial z} - L\left(-iK\cdot\frac{\partial}{\partial z}\right)\right) w = 0.$$

We make the following assumption on the functions $S_j(X, T)$ which generalizes to the case $n > 2$ "condition A" known in the theory of averaging (see, for example, [18, p. 175]. Namely, we assume that for X, T $\in \Omega$ the following Wronskian is not equal to zero:

$$\Delta(X, T) = \begin{vmatrix} K_1 & K_2 & \ldots & K_n \\ K_1' & K_2' & \ldots & K_n' \\ \ldots & \ldots & \ldots & \ldots \\ K_1^{(n-1)} & K_2^{(n-1)} & \ldots & K_n^{(n-1)} \end{vmatrix} \neq 0, \quad K_j^{(l)} = \frac{\partial^l K_j}{\partial X^l}. \tag{2.6}$$

LEMMA 2.1. Suppose that a solution $\tilde{u}_1(x, t, \varepsilon)$ of Eq. (2.5) satisfies $\tilde{u}_1 = o(1)$ for $\varepsilon \to 0$. Then for $(X, T) \in \Omega$ we hve an orthogonality condition

$$\int_{\mathrm{T}^n} w(z, X, T) F(z, X, T)\, dz = 0. \tag{2.7}$$

Suppose that for all X, T $\in \Omega$ the average with respect to the variable x exists and depends smoothly on X:

$$\langle wF \rangle \overset{\text{def}}{=} \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} w(Kx, X, T) F(Kx, X, T)\, dx.$$

Then condition (2.7) is equivalent to

$$\langle wF \rangle = 0. \tag{2.8}$$

In the proof of this lemma and in the sequel we shall need the following useful supplementary result.

LEMMA 2.2. Suppose that condition (2.6) is satisfied and f(z, X, T) is a smooth function that is 2π-periodic in variables $z_1, \ldots, z_2$ and finite in a variable $X \in \Omega_T = \Omega \cap \{t = T\}$. Then

$$\int_{-\infty}^{\infty} f\left(\frac{S(X,T)}{\varepsilon}, X, T\right) \mathrm{d}X = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \int_{\mathrm{T}^n} f(z, X, T)\, \mathrm{d}z\, \mathrm{d}X + O(\varepsilon^{1/n}). \tag{2.9}$$

If in addition an average

$$\langle f \rangle = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(Kx, X, T)\, \mathrm{d}x$$

is a smooth function of X then

$$\langle f \rangle = \frac{1}{(2\pi)^n} \int_{\mathrm{T}^n} f(z, X, T)\, \mathrm{d}z. \tag{2.10}$$

<u>Proof.</u> We estimate the derivatives of phases $S\cdot\nu$. Denote the k-th derivative of $S\cdot\nu$ by $\rho_{\nu k}\,|\nu|$, $|\nu| = \sqrt{\nu_1{}^2 + \ldots + \nu_n{}^2}$. Then condition (2.6) implies that equations $S^{(k)}\cdot\nu = \rho_{\nu k}|\nu|$ (k = 1, ..., n) may be solved with respect to the vector $\nu\nu|\nu|$ as follows:

$$\nu_m/|\nu| = \frac{1}{\Delta(X,T)} \sum_{k=1}^{n} A_{mk}(X, T)\rho_{\nu k}, \tag{2.11}$$

where A_{mk} (X, T) consist of sums of products of arbitrary $S^{(k)}$ (k = 1, ..., n), and therefore are bounded in Ω. From (2.11) we immediately obtain

$$1 = \sum_{m=1}^{n} \nu_m^2/|\nu|^2 \leqslant C(\Omega) \sum_{k=1}^{n} \rho_{\nu k}^2, \qquad C(\Omega) = \mathrm{const}.$$

Therefore, $\sum_{k=1}^{n} \rho_{\rho k}{}^2 \geq \tilde{\delta}^2$, $\tilde{\delta} > 0$ does not depend on ν. Thus, for every fixed T and every point X we have the following estimate for at least one value of l ($\leq l \leq$ n):

$$|S^{(l)}\cdot\nu| \geqslant \delta\,|\nu|, \tag{2.12}$$

where $\delta = \delta(\Omega) > 0$ is a constant. Since the vector $\nu|\nu|$ belongs to the unit sphere, in the sequel without loss of generality we can assume that inequality (2.12) holds for a given k and for all $X \in \Omega_T$. Estimate (2.12) then also implies that a function $K\cdot\nu = S_X\cdot\nu$ has no more than n zeros for every T. Indeed, if we assume the opposite then since K (X, T)·ν is smooth, the derivative K_X (X, T)·ν has at least (n − 1) zeros, K_{XX} (X, T)·ν has at least (n − 2) zeros, and so on, and $S^{(l)}\cdot\nu$ has at least $(n - l + 1)$ zeros, which contradicts (2.12). This proves, in particular, that the set of points in Ω_T for which at least one of the expressions K (X, T)·ν is equal to zero is no more than countable for $\nu \in Z^n$.

We now study an integral $\int_{-\infty}^{\infty}$ f (S(X, T)/ε, X, T)dX. We expand the function f(z, X, T) as a multidimensional Fourier series in the variable z as follows:

$$f(z, X, T) = \sum_{\nu\in\mathbf{Z}^n} \mathrm{e}^{\mathrm{i}\nu\cdot z} f_\nu(X, T). \tag{2.13}$$

Since f is smooth, the Fourier coefficients f_ν satisfy the following estimates for all X, T ∈ Ω, and all natural N:

$$\left|\frac{\partial^m f_\nu}{\partial X^m}\right| < \frac{Q_N^{(m)}(\Omega, f)}{|\nu|^N}, \qquad Q_N^{(m)}(\Omega, f) = \mathrm{const}.$$

Using these estimates, inequality (2.12), and known estimates for integrals of quickly oscillating exponentials cited in [19], we deduce that for all ϰ > 0 we have

$$\left|\int_\Omega \mathrm{d}X\, f_\nu(X, T)\, \mathrm{e}^{\frac{iS\cdot\nu}{\varepsilon}}\right| \leqslant \frac{\sqrt[n]{\varepsilon}\, c_\varkappa}{|\nu|^\varkappa}, \tag{2.14}$$

where c_κ is a constant dependent on Ω and f. Writing the function f on the left-hand side of (2.9) in the form (2.12) and using (2.14), we immediately obtain Eq. (2.9).

To prove Eq. (2.10) it suffices to note that the countability of the set of "resonance" points, i.e., points $X \in \Omega_T$, such that for some $\nu \in \mathbf{Z}^n$ we have K(X, T)·ν = 0, implies the existence of a set of points such that $K(X, T)\cdot\nu \neq 0$ for all $\nu \in \mathbf{Z}^n$ which is also everywhere dense in Ω_T. For these points Eq. (2.10) holds because of known results in the theory of averaging (see, for example, [18]). At other points X it follows from the smoothness of the left- and right-hand sides of (2.10).

Proof of Lemma 2.1. Let $\varphi(X, T)$ be a smooth function that is finite on Ω. We multiply Eq. (2.5) by $\varphi(X, T)\cdot w(S/\varepsilon, X, T)$, integrate the resulting expression over $X, T \in \Omega$, and transfer operators $\partial/\partial T$, $\partial/\partial X$ and $L(-i\varepsilon\partial/\partial X)$ over to φw. Applying these operators to a function w, and using Eqs. (2.2) and $\mathcal{L}^+ w = 0$, we obtain

$$\iint_\Omega R(S/\varepsilon, X, T, \varepsilon)\tilde{u}_1 \, dX\, dT = \iint_\Omega \varphi w F \, dX\, dT,$$

where $R(z, X, T, \varepsilon)$ is a smooth function 2π-periodic in z_j. The limit as $\varepsilon \to 0$ of the left-hand side of this equation by our assumption $\tilde{u}_1 = o(1)$ is equal to zero. Lemma 2.2 implies that the limit of the right-hand side is equal to

$$\iint_\Omega \varphi\left[\frac{1}{(2\pi)^n}\int_{T^n} w(z, X, T)\, R(z, X, T)\, dz\right] dX\, dT.$$

The arbitrariness of φ and the smoothness of the expression inside the square brackets imply that the above expression is equal to zero. Equation (2.8) follows from the second statement of Lemma 2.2.

Let us now consider the function $\psi_1\psi_2^+$. Clearly, for real x and t it has the following structure:

$$\psi_1\psi_2^+(x, t, K) = w(Kx + Wt + \Phi', K, I),$$

where $w(z, K, I)$ is 2π-periodic in every $z_1, \ldots, z_n$, I, which are the parameters of the solution (0.3).

LEMMA 2.3. The function $w(z, K, I(X, T))$ belongs to the cokernel of the operator $\mathcal{L}$, if relations (0.5) are satisfied.

Proof. Let $p(z)$ be a smooth function on the torus $T^n = [0, 2\pi]^n$. Let

$$\mathcal{E}(p) = \frac{1}{(2\pi)^n}\int_0^{2\pi} (\mathcal{L}^+ w)\, p(z)\, dz.$$

We transfer the operator $\mathcal{L}^+$ to the function $p(z)$ and write the operator $\mathcal{L}$ in coordinates $x, t, y_1, \ldots, y_{n-2}$ (see (2.3)), using for the operator $L(-i\partial/\partial x)$ formulas $Lp = p_{1,xx} + p_{2,xx}$, $ip = p_1 - p_2$ (see (1.2)), where $p_1(x, t, I)$ and $p_2(x, t, I)$ are extended analytically to the upper and the lower half-planes of the variable x, respectively. Equations (1.1) and (1.37) directly imply

$$w\mathcal{L}p \equiv \psi_1(p_t + 2u_0 p_x + 2u_{0x}p + p_{1,xx} + p_{2,xx})\psi_2^+$$
$$= \frac{\partial}{\partial t}(\psi_1 p \psi_2^+) - i\frac{\partial}{\partial x}(v(\psi_{1x}\psi_2^+ - \psi_1\psi_{2x}^+)) + \frac{\partial}{\partial x}((p_{1x} + p_{2x})\psi_1\psi_2^+) + 2ip_{1x}\psi_1\psi_1^+ - 2ip_{2x}\psi_2\psi_2^+,$$

where $\frac{\partial}{\partial t} = W_1\frac{\partial}{\partial z_1} + \ldots + W_n\frac{\partial}{\partial z_n}$ и $\frac{\partial}{\partial x} = K_1\frac{\partial}{\partial z_1} + \ldots + K_n\frac{\partial}{\partial z_n}$.

The averages with respect to z of the expressions containing $\partial/\partial t$, $\partial/\partial x$, are clearly equal to zero. We compute the average with respect to x of the last two summands: $2i\langle p_{1x}\psi_1\psi_1^+\rangle - 2i\langle p_{2x}\psi_2\psi_2^+\rangle$. Shifting the contours of integration to the complex plane by shifting it to the upper half-plane for the first expression and to the lower one for the second expression, we see that they are equal to zero, since the expressions inside these integrals are exponentially small. Therefore, Lemma 2.2 implies that $\mathcal{E}(p) = 0$ for all p. The smoothness of the function $\mathcal{L}^+ w$ now implies the lemma.

Remark. Thus, Lemmas 2.1-2.3 imply that we can treat Eq. (2.5) as if its solution can be written in the "n-phase" form, i.e., in the form $\tilde{u}_1 = \varepsilon w(S/\varepsilon, X, T)$. As noted before, such representation of the function u holds only in the one-phase case (see [10, 17]).

THEOREM 2.1. System of Eqs. (2.4) and (0.5) is equivalent to equations

$$\partial_T a_i = -\partial_X a_i^2, \quad \partial_T b_i = -b_i^2, \quad \partial_T C = -\partial_X C^2. \tag{2.15}$$

Proof. Suppose we are given an arbitrary deformation of the parameters $a_i(\tau)$, $b_i(\tau)$, $C(\tau)$, of multiphase solutions of the Benjamin–Ono equation. Then the corresponding solutions $u(x, t, \tau)$ and functions ψ_i, ψ_i^+ are functions of the parameter τ.

A truncated derivative $\hat{\partial}_\tau u$ is a function obtained by differentiating the corresponding formulas of the form (0.3) in which K and W are assumed to be constant. By this definition we have

$$\partial_\tau u = \hat{\partial}_\tau u + \sum_{i=1}^{n} (x\partial_\tau K + t\partial_\tau W_i)\frac{\partial u}{\partial \varphi_i}. \tag{2.16}$$

LEMMA 2.4. We have

$$\langle \psi_1 \hat{\partial}_\tau u \psi_2^+ \rangle = \partial_\tau \lambda - i\partial_\tau K \langle \psi_1 \psi_2^+ \rangle, \tag{2.17}$$

$$\left\langle \psi_1 \frac{\partial u}{\partial \varphi_i} \psi_2^+ \right\rangle = 0. \tag{2.18}$$

Proof. Let $\psi_i(x, t, k \mid \tau_1)$, $\psi_i^+(x, t, k \mid \tau)$ be the Baker–Akhiezer functions and the conjugate functions corresponding to the different values of the parameter τ. Then Eqs. (1.1) and (1.3) imply that

$$i\partial_x(\psi_1\psi_2^+) + \psi_1(u(x, t \mid \tau_1) - u(x, t \mid \tau))\psi_2^+ = (\lambda(k \mid \tau_1) - \lambda(k \mid \tau))\psi_2\psi_2^+. \tag{2.19}$$

Differentiating (2.19) with respect to τ_1 and letting $\tau_1 = \tau$, we obtain

$$i(\partial_\tau K)(\psi_1\psi_2^+) + (\psi_1\hat{\partial}_\tau u\psi_2^+) = \partial_\tau\lambda(\psi_2\psi_2^+) + Q. \tag{2.20}$$

Here the remainder Q has form

$$Q = \sum_s (\alpha_s x + \beta_s t + \gamma_s)\,\partial_x \tilde{w}_s(Kx + Wt + \Phi), \tag{2.21}$$

where α_s, β_s, γ_s are constants and $\tilde{w}_s(z_1, \ldots, z_n)$ are periodic functions of variabes z_i.

The vectors K and W define rectangular winding son the torus T^n. Let $T_1(\Phi)$ be the closure of a winding $Kx + Wt + \Phi$. It is a subtorus of T^n. For every function of form $w(Kx + Wt + \Phi)$ we denote its average over the subtorus T_1 by $\langle w\rangle_{T_1}$. It is equal to the average with respect to x and t, i.e., $\langle w\rangle_{T_1} = \langle w\rangle$.

We average Eq. (2.20) over $T_1(\Phi)$ (note that we cannot take its average over x and t, since some of the terms in (2.21) linearly depend on x and t). Equation (2.21) implies that $\langle Q\rangle_{T_1} = 0$. To obtain (2.17) for Eq. (2.20) averaged over T_1, it remains to note that

$$\langle \psi_1\psi_1^+ \rangle = \langle \psi_2\psi_2^+ \rangle = 1. \tag{2.22}$$

The latter equations are obtained by shifting the contour of integration to the complex region of the variable x. Equations (2.18) follow from (2.17) by considering the variation of u with respect to φ_i for constant a_i, b_i C. Since λ and K do not depend on φ_i, the right-hand side of (2.17) in this case is equal to zero.

LEMMA 2.5. We have

$$2\langle \psi_1(\partial_\tau(U_{1,x} + U_{2,x}) + u\partial_\tau u)\psi_2^+ \rangle = 2k\partial_\tau\lambda + i\partial_\tau W\langle \psi_1\psi_2^+ \rangle. \tag{2.23}$$

Proof. Eqs. (1.1) and (1.37) imply that

$$i\partial_t(\psi_1\psi_2^+) + \partial_x(\psi_1'\psi_2^+ - \psi_1\psi_2^{+\prime}) = 2(\delta U_{1,x}\psi_1\psi_2^+) + 2i(u_x\psi_1\psi_2^+), \tag{2.24}$$

where $\delta U_1 = U_1(x, t, \tau_1) - U_1(x, t, \tau)$ (in the sequel we use similar notations $\delta\lambda$, δu for increments of the corresponding functions).

Furthermore, these equations also imply that

$$u(\psi_1\psi_2^+)_x + \delta u\psi_1\psi_{2x}^+ - \lambda(\psi_2\psi_{2x}^+ + \psi_{1x}\psi_1^+) - \delta\lambda(\psi_2\psi_{2x}^+) = 0. \tag{2.25}$$

Using (2.16) and an equation $i\psi_{2x}{}^{+} = -u\psi_2{}^{+} + \lambda\psi_1{}^{+}$, we obtain from (2.25) the following equation:

$$\begin{aligned}&(2\delta U_{1,x}+2\delta uu)\,\psi_1\psi_2^{+}-2\delta u\lambda\psi_1\psi_1^{+}-2i\delta\lambda\,(\psi_2\psi_{2x}^{+})\\&=i\partial_t\,(\psi_1\psi_2^{+})-i\lambda\,(\psi_2\psi_2^{+}+\psi_1\psi_1^{+})_x-\delta\lambda\,(\psi_2\psi_2^{+})_x-2i\lambda\,(\psi_2\psi_{2x}^{+}+\psi_{1x}\psi_1^{+}).\end{aligned} \tag{2.26}$$

We differentiate this equation with respect to τ_1, let $\tau_1 = \tau$, and then average the resulting equation, obtaining

$$\langle\psi_1\,(2\hat{\partial}_\tau U_{1,x}+2u\hat{\partial}_\tau u)\,\psi_2^{+}\rangle-2\lambda\,\langle\hat{\partial}_\tau u\psi_1\psi_1^{+}\rangle-2i\partial_\tau\,\lambda\,\langle\psi_2\psi_{2x}^{+}\rangle=i\partial_\tau W\,\langle\psi_1\psi_2^{+}\rangle \tag{2.27}$$

(it is easy to see by shifting the contours of integration to the complex region of x that the contributions to the average by the summands on the right-hand side of (2.26) are all equal to zero except the first one).

We now rewrite the second-to-last summand on the left-hand side of (2.27) as follows:

$$-2\lambda\,\langle\partial_\tau u\psi_1\psi_1^{+}\rangle=2i\lambda\,\langle\psi_1\,(\partial_\tau U_1-\partial_\tau U_2)\psi_1^{+}\rangle-2i\lambda A=-2i\lambda\,\langle\partial_\tau U_2\psi_1\psi_1^{+}\rangle-2i\lambda A=2i\lambda\,\langle\partial_\tau U_2\,(\psi_2\psi_2^{+}-\psi_1\psi_1^{+})\rangle$$

$$-2i\lambda A=-2\,\langle\partial_\tau U_2\,(\psi_1\psi_2^{+})_x\rangle-2i\lambda A=2\,\langle\partial_\tau U_{2,x}\psi_1\psi_2^{+}\rangle-2i\lambda A, \tag{2.28}$$

where $A = \partial_\tau(C + \Sigma\,(b_i - a_i))$.

We derived (2.28) using equations

$$\langle\partial_\tau U_1\psi_1\psi_1^{+}\rangle=\langle\partial_\tau U_2\psi_2\psi_2^{+}\rangle=0, \tag{2.29}$$

which can also be obtained by shifting to the complex region of x. Substituting (2.28) into (2.27), we obtain the desired Eq. (2.23) (since $\langle\psi_2\psi_{2x}{}^{+}\rangle = -ik$).

Letting $\tau = T$ and $\tau = X$ in Eqs. (2.17) and (2.23), respectively, we obtain

$$\langle\psi_1 F\,[u_0]\,\psi_2^{+}\rangle=2k\partial_X\lambda-i\,(\partial_T K-\partial_X W)\,\langle\psi_1\psi_2^{+}\rangle+2\lambda\partial_X\left(C+\textstyle\sum_i(b_i-a_i)\right). \tag{2.30}$$

Thus, Eqs. (2.4) and (0.5) imply an equation

$$\partial_T\ln\lambda+2k\partial_X\ln\lambda+2\left(C+\textstyle\sum_i(b_i-a_i)\right)_X=0, \tag{2.31}$$

which holds for all k. Substituting Eq. (1.29), which defines $\lambda(k)$, into (2.31), we obtain (after equating to zero the residues at points a_i, b_i, C) Eqs. (2.6). An application of (2.10) concludes the proof of the theorem.

Remarks. 1. To completely describe the leading term of the asymptotic solution $\tilde{u}$ (0.4) we have to derive the equations for the corrections $\Phi_j(X, T)$ to phases. In the one-phase case the derivation requires a study of equations for $\tilde{u}_1 = O(\varepsilon)$ and a correction $\tilde{u}_2 = O(\varepsilon)$ and is well understood (see [11, 20]). Equations for $\Phi_j(X, T)$ have not been found in the multiphase case. As noted before, this is related to the complicated structure of the spectrum of the operator $\mathscr{L}$ (2.5), since the dimension of its kernel and cokernel depend on slow variables X and T (compare with [21]).

2. Clearly, we can replace "parameters" a_i, b_i, C with others, but the corresponding equations then become linked, even though they are possibly useful in the analysis of concrete physical problems. For example, in the one-phase case we can use "parameters" $K = K(X, T) = \partial S/\partial X$,

$$M=M\,(X,T)=\frac{1}{2\pi}\int_0^{2\pi}\tilde{u}\,dz,$$

$$D=D\,(X,T)=\frac{1}{2\pi}\int_0^{2\pi}(\tilde{u}-M)^2\,dz.$$

The corresponding equations have form

$$K_t+\frac{\partial}{\partial X}(K\,(2M-K)+D)=0;\qquad M_t+\frac{\partial}{\partial X}(M^2+D)=0;$$

$$D_t+2\frac{\partial}{\partial X}(D^2/K+(M-K)\,D)+2D\partial M/\partial X=0.$$

3. Condition (2.6) can be weakened. For example, we can replace (2.6) by a condition that for some $N \geq n$ the matrix $\|\partial^j K_i / \partial X^j\|$ ($i = 1, \ldots, n$; $j = 1, \ldots, N$) has a complete rank at every point $(X, T) \in \Omega$.

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