The theorem is proved.

THEOREM 2. Suppose $p, q \geqslant 2$ are relatively prime integers and the sequence of natural numbers λ_ν satisies the condition $\lambda_{\nu+1} \geqslant 2\lambda_\nu$ ($\nu = 0, 1, 2, \ldots$). Then the numbers

$$\alpha_k = \sum_{\nu=0}^{\infty} p^{-\lambda_\nu} q^{-kp^{\lambda_\nu}} \quad (k = 1, 2, 3, \ldots)$$

are algebraically independent, normal in base q, and the continued fraction for α_k can be given explicitly.

Proof. The normality of α_k follows from Theorem 1. The continued fraction for α_k can be constructed as in the example given above for the case $k = 1$, $\lambda_\nu = 2^\nu$, $p = 3$, $q = 2$. The algebraic independence of the α_k follows from the general theorem of [6] on the algebraic independence of values of lacunary series.

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GENERALIZED ELLIPTIC GENERA AND BAKER–AKHIEZER FUNCTIONS

I. M. Krichever

0. Introduction. The classical multiplicative genera of manifolds have, as was shown in [1], the wonderful property of what is now called rigidity. These genera (the signature and Euler characteristic for orientable manifolds, the A-genus for spinor manifolds, the Todd genus and the general T_y-genus for unitary manifolds) coincide with the index of the corresponding elliptic operators. If a compact Lie group G acts on a manifold X, then the kernel and cokernel of the corresponding operator are finite-dimensional G-modules, which allows us to naturally define the concept of an equivalent genus as the character-index $h^G(X)\colon G \to Q$. The value of this character at the identity of the group coincides with the h(X)-genus of the manifold. "Rigidity" means that if G is a connected compact Lie group, then the index-character is constant on the group. It follows that for the A-genus of a

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spinor manifold which admits a nontrivial action of such a group, $A(X)=0$ (i.e., the A-genus is an obstruction for the existence of nontrivial actions of connected compact Lie groups on spinor manifolds [1]).

The proof of these assertions in [1] was based on the Atiyah–Singer index theorem. The author proposed a different proof of these assertions in [2, 3]. The main idea involved the direct investigation of the global analytical properties of the Conner–Floyd expressions (these will be presented in Sec. 2). Moreover, it was proved in [3] that if the first Chern class of a unitary manifold is divisible by a whole number k, $c_1(X)\equiv 0 \pmod{k}$, then the A_k-genus is rigid. (The generating series for the genus A_k, A = 2, 3, 4, ..., is of the form $kxe^x/(e^{kx}-1)$.) In particular, this implies that if there exists a nontrivial action of S^1 on such a manifold which preserves the almost complex structure, then $A_k(X)=0$. This result was later rediscovered in [4].

The theory of elliptic genera and elliptic cohomologies, which arose in recent years in the papers of Witten, Ochanine, Landweber, Strong and which continues to attract the attention of numerous researchers, was stimulated by Witten's hypothesis [5] concerning the rigidity of the index-character of "twisted" Dirac operators acting on sections of the bundles $S\otimes T_{R_i}$, where S is the principal spinor bundle and the T_{R_i} are associated spinor bundles corresponding to the series of special spinor representations $R_0=1$, $R_1=T$, $R_2=\Lambda^2T\oplus T$, $R_3=\Lambda^3T\oplus(T\otimes T)\oplus T, \ldots$ Here T is the tangent bundle. (We refer to [6] as a principal reference for this topic.)

It turned out that Witten's hypothesis is equivalent to the rigidity of the equivariant elliptic genus (see [7] for a detailed presentation of the history). An elliptic genus was defined by Ochanine [8] to be a ring homomorphism

$$\varphi\colon \Omega_*^{SO}\to R$$

Its value on the generators is given by the generating series

$$g_\varphi(x)=\sum_{n=0}^{\infty}\frac{\varphi([CP^{2n}])}{2n+1}x^{2n+1}=\int_0^x\frac{dt}{\sqrt{R(t)}}, \tag{0.1}$$

where $R(t)=1-2\delta t^2+\varepsilon t^4$. This series is the logarithm of the formal Euler group

$$F(u,v)=g_\varphi^{-1}(g_\varphi(u)+g_\varphi(v))=\frac{u\sqrt{R(v)}+v\sqrt{R(u)}}{1-\varepsilon u^2v^2}. \tag{0.2}$$

For spinor manifolds the rigidity of the elliptic genus was proven by Ochanine for the case of semifree actions and for actions preserving the almost complex structure [9].

As was proved by Witten [10], the elliptic genus coincides with the index of a Dirac-type operator on the loop space $\mathcal{L}X$, and its rigidity property for general S^1-actions on spinor manifolds follows from natural (but at that time not yet proved) properties of the supersymmetric nonlinear sigma-model. Witten's program of proving the rigidity of the elliptic genus was rigorously realized by Taubes [11].

An important consequence of Witten's approach was the extremely natural (in the framework of quantum field theory) explanation of the modular properties of the "universal elliptic genus." These were first discovered directly in [12, 13].

The universal elliptic genus is the homomorphism

$$\varphi\colon \Omega_*^{SO} \to Z\left[\frac{1}{2}\right][\delta, \varepsilon], \tag{0.3}$$

given by Eq. (0.1), in which δ, ε are viewed as independent variables. As was shown in the above papers, the homomorphism φ may be viewed as a homomorphism

$$\varphi\colon \Omega_*^{SO} \to Q\,[[q]], \quad q = e^{2\pi i\tau},$$

whose image is the modular forms of weight 2. Here δ and ε are the generators in the ring of such forms and have weights 2 and 4, respectively.

In a recent paper of Hirzebruch [14], a further generalization of elliptic genera using the modular forms of weight N was proposed. It was proved that these "elliptic genera of level N" are rigid for the case of unitary actions of S^1 on manifolds whose first Chern class is divisible by N.

In this paper, we will define a "generalized elliptic genus" for almost complex manifolds, which will contain all the above genera as special and limiting cases. It is important to note that the generating series of this genus [cf. Eqs. (1.1), (1.4), (1.6)] is a simplest function of Baker—Akhiezer type, which plays a key role in the theory of nonlinear integrable equations. In particular, exactly this function was used in [15] to integrate the Moser—Calogero elliptic system. As was shown in [15], it satisfies the functional equation (1.9), which generalizes the classical addition formulas for elliptic functions.

In Sec. 2 we prove that the proposed genus is rigid for unitary S^1-action on SU-manifolds, i.e., almost complex manifolds whose first Chern class is zero. It would be extremely interesting to understand to the index of which operator this generalized elliptic genus corresponds.

1. Generalized Elliptic Genus. In the framework of this paper, we always assume, unless stated otherwise, that the categories in question are unitary, i.e., all manifolds are assumed to be almost complex, and their group actions and bundles are assumed to be unitary.

The rational multiplicative Hirzebruch genus, i.e., the homomorphism $h\colon U_* \to Q$, is given by the series $x/h(x)$, $h(x) = x + \sum_{i=2}^{\infty} \lambda_i x^i$, $\lambda_i \in Q$. The value of such a homomorphism on the class of bordisms of an n-dimensional manifold X is equal to

$$h(X) = \left\langle \prod_{i=1}^{n} \frac{x_i}{h(x_i)}, [X] \right\rangle, \tag{1.1}$$

where the x_i are the Wu generators, whose symmetric polynomials give the Chern classes of the tangent bundle

$$\prod_{i=1}^{n} (1 + x_i) = 1 + \sum_{i=1}^{n} c_i(X). \tag{1.2}$$

S. P. Novikov proved in [16] that h(x) coincides with the series $g_h^{-1}(x)$, which is functionally inverse to the logarithm

$$g_h(x) = \sum_{n=0}^{\infty} \frac{h([CP^n])}{n+1} x^{n+1} \tag{1.3}$$

of the formal group $f_h(u, v)$, which is the image of the formal group of "geometric" cobordisms

$$f(u, v) = g^{-1}(g(u) + g(v)), \quad g(x) = \sum_{n=0}^{\infty} \frac{[CP^n]}{n+1} x^{n+1}$$

under the homomorphism h.

Let Γ be an arbitrary elliptic curve with periods $2\omega_1$, $2\omega_2$, $\operatorname{Im} \omega_2/\omega_1 > 0$. Define the function $\Phi(x, z) = \Phi(x, z \mid \omega_1, \omega_2)$ by the formula

$$\Phi(x, z) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)} e^{\zeta(z)x}, \tag{1.4}$$

where $\sigma(z)$, $\zeta(z)$ are the standard Weierstrass functions (cf. [17]).

Denote by $\hat{\varphi} = \hat{\varphi}(z, k_0 \mid \omega_1, \omega_2)$ the complex-valued multiplicative genus

$$\hat{\varphi}: U_* \to C, \tag{1.5}$$

given by Eq. (1.1), where the series 1/h(x) (which depends on all the parameters listed above) is equal to

$$\hat{\Phi}(x, z, k_0 \mid \omega_1, \omega_2) = \Phi(x, z \mid \omega_1, \omega_2) e^{-k_0 x}. \tag{1.6}$$

As was already mentioned in the Introduction, the function $\Phi(x, z)$ is a simplest Baker–Akhiezer function (the general definition of these was given in [18]). It is a solution of the Lamé equation [19]

$$\left(\frac{d^2}{dx^2} - 2\wp(x)\right)\Phi(x, z) = \wp(z)\Phi(x, z). \tag{1.7}$$

As a function of the variable z, $\Phi(z, x)$ is doubly-periodic and has an exponential singularity at the point z = 0. It follows from the translational properties of the Weierstrass σ-function that

$$\begin{gathered} \Phi(x + 2\omega_l, z) = \Phi(x, z) \exp(2\zeta(z)\omega_l - 2\eta_l z), \\ l = 1, 2, 3, \quad \omega_3 = \omega_1 + \omega_2, \quad \eta_l = \zeta(\omega_l). \end{gathered} \tag{1.8}$$

The function $\Phi(x, z)$ in the variable x is holomorphic everywhere except at the lattice points $2n\omega_1 + 2m\omega_2$, where it has simple poles and, moreover, residue equal to 1 at the point x = 0.

In [15] it was proved that the function $\Phi(x, z)$ satisfies the functional equations

$$\Phi(x + y)[\wp(y) - \wp(x)] = \Phi'(x)\Phi(y) - \Phi'(y)\Phi(x), \tag{1.9}$$

$$\Phi(x)\Phi(-x) = \wp(z) - \wp(x). \tag{1.10}$$

We note that these equations were proposed in [20] to determine the Lax representation for the Moser–Calogero system of particles with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - 2\sum_{i \neq j} \wp(x_i - x_j). \tag{1.11}$$

We show that Eqs. (1.9), (1.10) lead to the fact that the functions $\Phi_l(x) = \Phi(x, \omega_l \mid \omega_1, \omega_2)$ generate the Ochanine elliptic genera. The definition of $\Phi(x, z)$ implies that for $z = \omega_l$ these functions are odd as functions of x, i.e., $\Phi_l(x) = -\Phi_l(-x)$. Hence

$$\Phi_l^2(x) = \wp(x) - e_l, \quad e_l = \wp(\omega_l). \tag{1.12}$$

Thus,

$$2\Phi_l'(x)\,\Phi_l(x) = \wp'(x) = \sqrt{\prod_{l=1}^{3}(\wp(x)-e_l)}. \tag{1.13}$$

The corresponding formal group is by definition equal to

$$f_l(u,v) = \frac{1}{\Phi_l(g_l(u)+g_l(v))}, \quad \Phi_l(g_l(u)) = \frac{1}{u}. \tag{1.14}$$

It follows from Eqs. (1.13) and (1.14) that

$$\Phi_l'(g_l(u)) = \frac{\sqrt{R_l(u)}}{u^2}, \tag{1.15}$$

where the coefficients of the polynomial $R_l = 1 - 2\delta_l u^2 + \varepsilon_l u^4$are equal to

$$2\delta_l = \sum_{i\neq l}(e_l - e_i), \quad \varepsilon_l = \prod_{i\neq l}(e_l - e_i). \tag{1.16}$$

Expanding Eq. (1.14) with the help of Eq. (1.9), and using Eq. (1.15) in the process, we obtain that $f_l(u,v)$ coincides with the formal Euler group Eq. (0.2).

Before working out other special cases, we note that the function $\hat{\Phi}(x, z, k_0 \mid \omega_1, \omega_2)$, defined in Eq. (1.6) satisfies the same functional equations (1.9), (1.10).

Fix on Γ a point z_{nm} of order N, i.e.,

$$z_{nm} = \frac{2n}{N}\omega_1 + \frac{2m}{N}\omega_2, \qquad n, m = 0, 1, \ldots, N-1. \tag{1.17}$$

If we define

$$k_{nm} = -\frac{2n}{N}\eta_1 - \frac{2m}{N}\eta_2 + \zeta(z_{nm}), \tag{1.18}$$

then the function

$$\hat{\Phi}_{nm}(x \mid \omega_1, \omega_2) = \hat{\Phi}(x, z_{nm}, k_{nm} \mid \omega_1, \omega_2) \tag{1.19}$$

transforms as follows under shifts of the variable x:

$$\hat{\Phi}_{nm}(x + 2\omega_1) = \hat{\Phi}_{nm}(x)\, e^{2\pi i n/N}, \tag{1.20}$$

$$\hat{\Phi}_{nm}(x + 2\omega_2) = \hat{\Phi}_{nm}(x)\, e^{2\pi i m/N}. \tag{1.21}$$

From explicit formulas for Φ it immediately follows that the function $\hat{\Phi}_{nm} = \hat{\Phi}_{nm}(x \mid \omega_1, \omega_2)$ generates the elliptic genus of level N introduced in [14].

It is well known in the theory of finite-zone integration that the degeneracy of the potential $2\wp(x)$ in the Lamé equation to $2\cosh^{-2}(x + x_0)$ corresponds to the degeneracy of the elliptic curve Γ to a singular rational curve Γ_{sing} with one self-intersection point. the normalization $C \to \Gamma_{\text{sing}}$ of this singular curve takes the function $\Phi(x, z)$ to a function of the form

$$\Phi_{\text{sing}}(x, k) = (-k + a)\, e^{kx}, \tag{1.22}$$

where $k = z^{-1}$ is a point in the complex plane. The coefficient $a = a(x)$ is uniquely determined by the equality

$$\Phi_{\text{sing}}(x, \eta) = \Phi_{\text{sing}}(x, -\eta), \tag{1.23}$$

which reflects the fact that the points η and $-\eta$ are glued together under normalization.

Thus, the function Φ_{sing} depends on the quantity η as on a parameter, which coincides with the "discrete spectrum" point of the degenerate Lamé equation. Finally, after finding from Eq. (1.23) the coefficient a, we obtain

$$\Phi_{\text{sing}}(x, k \mid \eta) = (-k + \eta \operatorname{cth} \eta x)e^{kx}. \tag{1.24}$$

For different values of the parameters k, k_0, η of the generating function

$$\hat{\Phi}_{\text{sing}}(x, k, k_0 \mid \eta) = (-k + \eta \operatorname{cth} \eta x)e^{(k-k_0)x} \tag{1.25}$$

we get all the classical manifold genera:

$$1^\circ.\ k = k_0, \quad T_{a,b} - \text{genus},\ a = \eta - k_0, \quad b = \eta + k_0, \tag{1.26}$$

which for $a = 1$, $b = 0$ coincides with the Todd genus, for $a = 1, b = 1$ with the signature of the manifold, for $a = 1$, $b = -1$ with the Euler characteristic.

$$2^\circ.\ k = \eta, \quad k_0 = \frac{N-2}{N}\eta - A_N - \text{genus of the manifold}, \quad A_{N=2} = A - \text{genus of the manifold.} \tag{1.27}$$

2. Rigidity Theorem. In [2] for each multiplicative genus $h\colon U_* \to 0$ its equivariant analog was defined:

$$h^G\colon U_*^G \to K(BG) \otimes Q, \tag{2.1}$$

where U_*^G is a ring of bordisms of manifolds with an action of a compact Lie group G. For an arbitrary G-manifold X the projection

$$p\colon X_G = (X \times EG)/G \to BG \tag{2.2}$$

onto the universal classifying manifold gives the well-defined cobordism class

$$\chi_0^G([X, G]) = p_!(1) \in U^*(BG), \tag{2.3}$$

where $p_!$ is the Gysin homomorphism (direct image).

By Dold's theorem [21], to each homomorphism h there corresponds a functor homomorphism

$$\tilde{h}\colon U^*(\cdot) \to K(\cdot) \otimes Q. \tag{2.4}$$

The equivariant genus is determined by the composition of homomorphisms

$$h^G = \tilde{h} \circ \chi_0^G\colon U_*^G \to U^*(BG) \to K(BG) \otimes Q. \tag{2.5}$$

The rigidity of the genus h^G on the given class of manifolds means that its value

$$h^G([X, G]) \in Q \subset K(BG) \otimes Q \tag{2.6}$$

for the case of actions of connected compact groups G belongs to the ring of constants. From the functorial properties of h^G it follows that it is sufficient to prove rigidity for the case $G = S^1$, to which we limit ourselves from now on.

For $G = S^1$ the universal classifying spaces BS^1 is CP^∞, and the ring $U^*(CP^\infty)$ is isomorphic

$$U^*(CP^\infty) = U^*[[u]]$$

to the ring of formal series in the variable u of degree 2 with coefficients in U^*.

The expressions of the class $\chi_0^{S^1}([X, S^1])$ in terms of the invariants of fixed points are called Conner–Floyd expressions. In the case of S^1-actions with isolated fixed points, they have the form

$$\chi_0^{S^1}([X, S^1]) = \sum_s \prod_{i=1}^n \frac{1}{[u]_{j_{si}}} \tag{2.7}$$

(cf. [16, 22, 23]; for arbitrary actions they were first derived in [24], whose formulas are made more precise in [25]). Here $[u]_j = g^{-1}(jg(u))$; $g(u) = \sum_{n=0}^{\infty} \frac{[CP^n]}{n+1} u^{n+1}$ is the j-th degree in the formal group of "geometric" cobordisms $f(u, v) = g^{-1}(g(u) + g(v))$. The integers j_{si}, $i = 1, \ldots, n = \dim_C X$, are determined by the decomposition of the representation of S^1 in the fiber of the tangent bundle over a fixed point m_s into a sum of irreducible representations $\sum_i \eta^{j_{si}}$.

Equation (2.7) means in part that the Laurent series in its right side contains only the regular part, whose constant term coincides with the class of bordisms of the manifold [X].

As was proved in [2], the generator $u \in U^2(CP^\infty)$ under the homomorphism $\tilde{h}$ goes to $\tilde{h}(u) = g_h^{-1}(\ln \eta)$, where η, η^{-1} are the generators of the ring $K(CP^\infty)$. We introduce the formal variable $x = \ln \eta$. Then it follows from Eq. (2.7) that the equivariant genus $\varphi^{S^1}([X, S^1])$, corresponding to the generalized elliptic genus $\hat{\Phi}$ of Eqs. (1.5), (1.6) for a manifold with isolated fixed points has the form

$$\varphi^{S^1}([X, S^1]) = \varphi_X(x) = \sum_s \prod_{i=1}^n \hat{\Phi}(j_{si}x, z, k_0 \mid \omega_1, \omega_2) \tag{2.8}$$

(for brevity we do not explicitly specify the obvious fact that in the left side the function $\varphi_X(x)$ depends on the quantities, z, k_0, ω_1, ω_2 as parameters).

By the definition of the equivariant genus, the function $\varphi_X(x)$ is regular at the point x = 0. Our goal will be to prove that it is a constant for SU-manifolds. We will first show the necessity of confining ourselves to the case of SU-manifolds with the example of manifolds with isolated fixed points and only then go to the general case.

From the definition of $\Phi(x, z)$ it follows that the function $\varphi_X(x)$ could have poles at all point of the lattice $\Lambda = 2n\omega_1 + 2m\omega_2$. It follows from Eq. (1.8) that

$$\varphi_X(x + 2\omega_l) = \sum_s e^{r_s Q_l(z, k_0)} \prod_{i=1}^n \hat{\Phi}(j_{si}x), \tag{2.9}$$

where

$$r_s = \sum_{i=1}^n j_{si}, \quad Q_l = 2(\zeta(z)\omega_l - \eta_l z - k_0\omega_l). \tag{2.10}$$

If all the quantities r_s are equal, i.e., $r_s = N$, then

$$\varphi_X(x + 2\omega_l) = \varphi_X(x) \cdot e^{NQ_l(z, k_0)}. \tag{2.11}$$

From this equality and from the fact that φ_X is regular at the point x = 0 follows that it is regular at all points of the lattice Λ.

It turns out that if X is an SU-manifold, then in fact r_s does not depend on s. In [2] the following Lemma was proved.

LEMMA 2.1. Suppose that the representation of the group S^1 in the fiber of the S^1-bundle ζ over X over a point of the fixed manifold F_s is equal to $\sum_i \eta^{j_{si}}$; then if $c_1(\zeta)$ is divisible by k, all the sums r_s are equal mod K:

$$\sum_{i=1}^{n} \eta_{si} = r_s \equiv N \pmod{k}. \tag{2.12}$$

(Subsequently, this statement was proved anew in [14]).

In the case where $c_1(X) = 0$, it follows from the statement of the Lemma that the sums r_s do not depend on s: $r_s = N$. The number N is called the type of the action of the group S^1 on X.

THEOREM 2.1. For any SU-manifold X the value of the equivariant genus

$$\varphi_X(x) = \varphi^{S^1}([X, S^1]) \equiv \varphi([X]) \subset K(CP^\infty) \otimes C \tag{2.13}$$

is constant. If the action of S^1 on X has type $N \neq 0$, then

$$\varphi_X(x) \equiv \varphi([X]) = 0. \tag{2.14}$$

Remark. For SU-manifolds the dependence of generalized elliptic genera on the parameter k_0 is trivial: if $c_1(X) = 0$, then for an arbitrary series h(x) we have

$$\prod_{i=1}^{n} \frac{x_i}{h(x_i)} e^{-k_0 x_i} = \prod_{i=1}^{n} \frac{x_i}{h(x_i)}, \tag{2.15}$$

since $\sum x_i = c_1 = 0$. Thus, for SU-manifolds the generalized elliptic genera depend on three parameters:

The proof of the theorem repeats (practically without changes) the proofs found in [1, 2] and hence will be presented schematically, with the reader being referred to the above papers for details.

First of all, we mention the Conner–Floyd expressions for the general S^1-action. Let F_s be a connected component of the set of fixed points of the action of S^1 on X. The normal bundle ν_s (as any complex S^1-bundle over a trivial S^1-manifold) is representable in the form $\nu_s = \sum_j \nu_{sj} \otimes \eta^j$, where η^j is the j-th tensor power of the standard representation of S^1, and the action of S^1 on the bundles ν_{sj} is trivial. The set of complex bundles ν_{sj}, of which only a finite number is different from zero, determines the class of bordisms belonging to the group R_n:

$$\beta(F_s) \in R_n = \sum U_s\left(\prod_{j\neq 0} BU(n_j)\right). \tag{2.16}$$

The summation is taken over the sets of nonnegative integers s, n_j such that $n = s + 2\sum n_j$. The sum over all connected components gives the image of the bordism class $[X, S^1] \in U_*^{S^1}$ and induces the homomorphism $\beta\colon U_*^{S^1} \to R_*$,

$$\beta([X, S^1]) = \sum_s \beta(F_s).$$

Choose the generators of the U_*-module $U_*(CP^\infty) = U_*(BU(1))$ to be the bordism classes corresponding to the imbedding $(CP^n) \in U_{2n}(CP^\infty)$. The standard multiplicative structure in R_* lets us choose the U_*-module generators to be monomials

$$(CP^{l_1}_{j_1}) \times \ldots \times (CP^{l_r}_{j_r}). \tag{2.17}$$

Denote by $G_n(u)$ the Laurent series

$$G_n(u) = \frac{1}{f(u, v)} \cap [CP^n], \tag{2.18}$$

where $\cap [CP^n]$ denotes in this case the substitution of $[CP^{n-k}]$ for v^k. Since $v^k \cap [CP^n] = 0$, $k > n$, the series G_n is well-defined and has the form

$$G_n(u) = \frac{\alpha_{n+1}}{u^{n+1}} + \frac{\alpha_n}{u^n} + \ldots$$

Proposition ([25], see also [24]). The composition of $\chi_0^{S^1}$ and the imbedding $U^*(CP^\infty) \subset U^*[[u]] \otimes C[u^{-1}]$ coincides with the composition $\Psi \circ \beta$, where the homomorphism of U_*-modules $\Psi: R_* \to U^*[[u]] \otimes C[u^{-1}]$ is given by the formula

$$\Psi\left(\prod_{m=1}^{r} (CP^{l_m}_{j_m})\right] = \prod_{m=1}^{r} G_{l_m}([u]_{j_m}). \tag{2.19}$$

Suppose that the generalized elliptic genus φ is given by the function $\hat{\Phi}(x)$ (depending on z, k_0, ω_1, ω_2 as parameters). Denote by $\hat{\Phi}_n(x)$ the function obtained from $G_n(u)$ by applying the homomorphism φ to the coefficients of this series and substituting $1/\hat{\Phi}(x)$ for the variable u. Then it follows from Eqs. (2.18) and (1.9) that

$$\hat{\Phi}_n(x) = -\frac{\hat{\Phi}(x)\dfrac{\bar{v}}{v g'_\Phi(v)} + \hat{\Phi}'(x)\bar{v}}{1 - \hat{\Phi}(x)\hat{\Phi}(-x)v\bar{v}} \cap [CP^n], \tag{2.20}$$

where

$$\hat{\Phi}(g_\Phi(v)) = 1/v, \quad \bar{v} = 1/\hat{\Phi}(-g_\Phi(v)) \tag{2.21}$$

and the symbol $\cap [CP^n]$ signifies the substitution of $\varphi([CP^{n-k}])$ for v^k. This implies that $\hat{\Phi}_n(x)$ is holomorphic outside the points of the lattice Λ $2n\omega_1 + 2m\omega_2$, on which it has poles of order n + 1. Moreover, the function $\hat{\Phi}_n(x)$ has the same translational properties as the generating function $\hat{\Phi}(x) = \hat{\Phi}_0(x)$, i.e.,

$$\hat{\Phi}_n(x + 2\omega_l) = \hat{\Phi}_n(x)\, e^{Q_l(z, k_0)}. \tag{2.22}$$

It follows from Eq. (2.19) that for any S^1-manifold X the function $\varphi_X(x) = \varphi^{S^1}([X, S^1])$ has the form

$$\varphi_X(x) = \sum_s a_s \prod_{m=1}^{r} \hat{\Phi}_{l_{ms}}(j_{ms}x). \tag{2.23}$$

Equation (2.23) generalizes the Conner–Floyd expression (2.8) to the case of general S^1-actions.

By definition the function $\varphi_X(x)$ is regular in a neighborhood of x = 0. From Lemma 2.1 and Eq. (2.22) it follows that for SU-manifolds φ_X satisfies the relation (2.11). Thus, $\varphi_X(x)$ is regular at all lattice points $2n\omega_1 + 2m\omega_2$. In principle, the function φ_X could have had poles at points of order j_{ms}, i.e., at the points x, for which $j_{ms}x = 2n\omega_1 + 2m\omega_2$.

As in [2, 3], in order to prove the absence of poles of φ_X at points of order n, it suffices to make use of the expressions for φ_X in terms of the invariants of fixed submanifolds with respect to the action of the subgroup $Z_n \subset S^1$ consisting of n-th-order roots of unity.

Let F_s be the connected components of the set of fixed points of the action of the subgroup Z_n. Then ([2], Theorem 1.1)

$$\varphi_X(x) = \sum_s p_{s!}\left(\frac{1}{e((\nu_s)_{S^1})}\right), \tag{2.24}$$

where $e((\nu_s)_{S^1})$ is the Euler characteristic of the bundle $(\nu_s \times ES^1)/S^1 \to (F_s \times ES^1)/S^1 = (F_s)_{S^1}$ over the manifold $(F_s)_{S^1}$, and $p_s: (F_s)_{S^1} \to CP^\infty$ is the projection map. The normal bundle ν_s to F_s in X is representable in the form

$$\nu_s = \sum_{j=1}^{n-1} \nu_{sj} \otimes \eta^j, \tag{2.25}$$

where the ν_{sj} are bundles on which the subgroup Z_n acts trivially. Since

$$\frac{1}{e((\nu_s)_{S^1})} = \prod_j \prod_{s,k} \frac{1}{f([u]_j, \lambda_{js}^k)}, \tag{2.26}$$

where the λ_{sj}^k are the Wu generators of the bundle ν_{sj}, it follows from Eq. (1.9) that a formula of the following form holds:

$$\tilde{\varphi} \circ p_{s!}\left(\frac{1}{e((\nu_s)_{S^1})}\right) = \prod_{j=1}^{n-1} \sum_\omega H_\omega(jx) D_\omega(nx), \tag{2.27}$$

where the ω are multi-indices, $H_\omega(x)$ is a holomorphic function outside the lattice points $2n\omega_1 + 2m\omega_2$, and $D_\omega(nx)$ is the value of the equivariant characteristic class on the set of bundles ν_{sj} (cf. [1, 2] for the definition) corresponding to the multi-index ω. The explicit form of these functions is not essential. Only the following is important. From the functorial properties of the equivariant characteristic classes ([2], Theorem 1.2) it follows that the function $D_\omega(x)$ is an equivariant characteristic class of the set of S^1-bundles $\tilde{\nu}_{sj}$, where the new action of S^1 on the bundles is defined to be the action of the quotient-group $S^1/Z_n \approx S^1$ (note that the action of Z_n on ν_{sj} is trivial). By definition $D_\omega(x)$ is regular at the point x = 0. Its explicit form shows that $D_\omega(x)$ behaves like Eq. (2.11) under shifts by $2\omega_i$ Hence $D_\omega(x)$ is holomorphic at all the lattice points. It follows from Eq. (2.27) that $\varphi_X(x)$ does not have poles at points of order n. Thus, $\varphi_X(x)$ is holomorphic everywhere and satisfies the relations (2.11). In this way we obtain that the function $\varphi_X(x)$ is a constant, and, moreover, for $N \neq 0$, this constant is equal to zero. The theorem is proved.

In this proof the theorem, the property $c_1(X) = 0$ was used to prove the translational properties of Eq. (2.11). By Eqs. (1.20), (1.21) we have that for the case of genera $\hat{\varphi}_{nm}$, which are given by the series $\hat{\Phi}_{nm}(x)$, the equality $c_1(X) \equiv 0 \pmod N$ suffices. In this case we can repeat in full the proof of Theorem 1 and obtain the following assertion.

THEOREM 2 [14]. Suppose that the first Chern class of the S^1-manifold X is divisible by N. Then the equivariant genus corresponding to the generating function $\hat{\Phi}_{nm}(x)$, is a constant:

$$\varphi_{n,m}^{S^1}([X, S^1]) \equiv \varphi_{nm}([X]). \tag{2.13}$$

If the action of S^1 on X has type $M \neq 0 \pmod{N}$, then $\varphi_{nm}([X]) = 0$.

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