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# VIRASORO-TYPE ALGEBRAS, RIEMANN SURFACES AND STRINGS IN MINKOWSKY SPACE

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## Introduction

According to the ideas of [1] the algebrogeometric model of a string in Minkowsky space (more precisely, of a "diagram," near which the string fluctuates) is the following. Given is a compact Riemann surface $\Gamma$ with two distinguished points $P_\pm \in \Gamma$. There exists a unique differential of the third kind dk with two simple poles at the points $P_\pm$, with residues $\pm 1$ and purely imaginary periods with respect to all contours on $\Gamma$. The real part of the corresponding integral $\tau(z) = \operatorname{Re} k(z)$ is single-valued on $\Gamma$ and represents "time." The level lines $\tau(z) = \text{const}$ represent the positions of the string at the present time. To the collection $m = m_+ + m_-$ of strings corresponds a Riemann surface $\Gamma$ with two collections of points $P_{+,i}$, $P_{-,j}$, $i = 1, \ldots, m_+$, $j = 1, \ldots, m_-$ with a differential dk, with real residues $c_{+i}$, $c_{-j}$ at all points $P_\pm$, $c_+ > 0$, $c_- < 0$, and purely imaginary periods on $\Gamma$. In exactly the same way the function $\tau(z) = \operatorname{Re} k(z)$ is single-valued and plays the role of "time". As $\tau \to \pm\infty$ the contours $\tau = \text{const}$ split into free strings. The connected components concentrated near the points $P_\pm$ play the role of asymptotically free "in" and "out" strings. In [1] a rich collection of algebraic objects connected with this picture for $m = 1$ was constructed, which for genus $g = 0$ reduce to the theory of the Virasoro algebra and its representations. In the present paper we demonstrate that these algebraic forms arise in the process of quantization of strings on such algebrogeometric models, "diagrams". To the asymptotic "in" and "out" states correspond the ordinary Fock spaces of free strings — small contours $\tau = \text{const}$ near the points $P_\pm$. The global algebrogeometric objects on a surface $\Gamma$ with distinguished points $P_\pm$ permit one in principle to trace the whole course of the interaction.

The algebrogeometric objects in the theory of Polyakov, Belavin, Knizhnik, etc., of strings in Euclidean space—time, as is known, lead to problems on the space of moduli of Riemann surfaces. The algebraic forms we introduce do not appear in this theory.

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But to the minds of the authors the approach we develop relates the algebrogeometric theory of strings with traditional ideas of operator quantum theory of strings in Minkowsky space and lets us use the mathematical techniques of the method of finite-zone integration in the theory of solitons.

## 1. Analogs of the Heisenberg and Virasoro Algebras Related to Riemann Surfaces

In this section we recall the definitions and constructions of [1] which are needed later.

Let $\Gamma$ be an arbitrary compact Riemann surface of genus g with two distinguished points $P_\pm$. By $\mathcal{A} = \mathcal{A}(\Gamma, P_\pm)$ we denote the commutative ring of meromorphic functions on $\Gamma$, which are holomorphic away from the points $P_\pm$. In [1], an additive basis for $\mathcal{A}$, formed by functions $A_n(Q)$, $Q \in \Gamma$, which for $|n| > g/2$ are uniquely determined by their behavior in neighborhoods of $P_\pm$ of the form

$$A_n(Q) = a_n^\pm z_\pm^{\pm n - g/2}(1 + O(z_\pm)), \quad a_n^+ = 1, \tag{1.1}$$

was introduced. Here $z_\pm(Q)$ are fixed local coordinates in neighborhoods of $P_\pm$. (For $|n| \leq g/2$ the definition of $A_n$ is changed slightly (cf. the details in [1]). Here and below the indices n run through integral values if g is even and half-integral ones n = ..., −3/2, −1/2, 1/2, ... if g is odd.

Definition. By the generalized Heisenberg algebra connected with the curve $\Gamma$ and pair of points $P_\pm$ is meant the algebra generated by generators $\alpha_n$ and a central element t with relations

$$[\alpha_n, \alpha_m] = \gamma_{nm} t, \quad [\alpha_n, t] = 0, \tag{1.2}$$

where the numbers $\gamma_{nm}$ are equal to

$$\gamma_{nm} = \frac{1}{2\pi i} \oint A_m \, dA_n. \tag{1.3}$$

The integral in (1.3) is taken over any contour separating the points $P_\pm$. Since all such contours are homologous, and the $A_n$ are holomorphic away from $P_\pm$, one has that the $\gamma_{nm}$ do not depend on the choice of contour.

An important property of the cited central extension of the commutative algebra $\mathcal{A}$ is the "locality" of the corresponding cocycle:

$$\gamma_{nm} = 0, \qquad \text{for } |n+m| > g, \quad |n|, |m| > g/2, \tag{1.4}$$

$$\gamma_{nm} = 0, \qquad \text{for } |n+m| > g+1, \quad |n| \text{ or } |m| \leqslant g/2. \tag{1.5}$$

Later we need to consider the space of differentials, holomorphic on $\Gamma$ away from the points $P_\pm$, at which they have poles. A basis in this space is formed of the differentials $d\omega_n(Q)$, which for $|n| > g/2$ are uniquely determined, starting from the following behavior in neighborhoods of the points $P_\pm$:

$$d\omega_n = w_n^\pm z_\pm^{\mp n + g/2 - 1}(1 + O(z_\pm)) dz_\pm. \tag{1.6}$$

For $|n| \leq g/2$, $n \neq -g/2$ we define $d\omega_n$ from the following conditions: in the neighborhoods

$$d\omega_n = z_+^{-n+g/2}(1 + O(z_+)) \, dz_+, \qquad Q \to P_+, \tag{1.7}$$

$$d\omega_n = w_n^- z_-^{n+g/2-1}(1 + O(z_-)) \, dz_-, \quad Q \to P_-. \tag{1.8}$$

Finally, we define $d\omega_{-g/2}$ as the unique differential dk:

$$d\omega_{-g/2} = dk, \tag{1.9}$$

having simple poles at $P_\pm$ with residues $\pm 1$ and purely imaginary periods on all cycles. (This differential plays a distinguished role in the theory; in [1] it was denoted by dp.)

It follows from (1.1) and (1.6)-(1.9) that for $|n| > g/2$

$$\frac{1}{2\pi i} \oint A_n \, d\omega_m = \delta_{nm}. \tag{1.10}$$

In [1] the generators $A_n$ for $|n| \le g/2$ were defined up to a constant, which one can fix uniquely by requiring that (1.10) hold for all n and m.

For $g = 0$, when $\Gamma$ is the ordinary completion of the complex plane, and $P_\pm$ are the points $z = 0$ and $z = \infty$, the functions $A_n$ coincide with $z^n$, the generators of the Laurent basis in the space of smooth functions on the circle $|z| = \text{const}$. It turns out that the functions $A_n$ are the "Laurent" basis for curves of arbitrary genus on a special system of contours $C_\tau$. These contours are the level lines of a single-valued function

$$C_\tau = \{Q, \operatorname{Re} k(Q) = \tau\}. \tag{1.11}$$

As $\tau \to \pm\infty$ the contours $C_\tau$ are small circles enveloping the point $P_\mp$. Upon passing through zeros of the differential dk as $\tau$ changes, the contours $C_\tau$ undergo topological reconstruction: they split into disconnected cycles, which afterwards merge. Aside from the dependence on $\tau$ there is the following theorem, whose proof is completely analogous to the proof of Lemma 1.2 of [1].

<u>THEOREM 1.1.</u> For any continuously differentiable function F(Q) on the contour $C_\tau$, $Q \in C_\tau$ one has

$$F(Q) = \frac{1}{2\pi i}\sum_n A_n(Q)\oint_{C_\tau} F(Q')\, d\omega_n(Q') \tag{1.12}$$

(which generalizes the Laurent expansion to the case of arbitrary curves). The dual is the expansion of any smooth differential df(Q) in a series

$$df(Q) = \frac{1}{2\pi i}\sum_n d\omega_n(Q)\oint_{C_\tau} df(Q')\, A_n(Q'). \tag{1.13}$$

Now we proceed to the description of the analogs of the Virasoro algebra. Let $\mathcal{L}^\Gamma = \mathcal{L}(\Gamma, P_\pm)$ be the algebra of meromorphic vector fields on $\Gamma$, holomorphic away from the points $P_\pm$. For $g > 1$ a basis in this space is formed by the fields $e_n$, which in neighborhoods of $P_\pm$ have the form

$$e_n = \varepsilon_n^\pm z_\pm^{\pm n - g_0 + 1}(1 + O(z_\pm))\frac{\partial}{\partial z_\pm}, \quad g_0 = \frac{3g}{2}, \quad \varepsilon_n^+ = 1. \tag{1.14}$$

(The case $g = 1$ is analyzed in detail in Sec. 5 of [1].) It was proved in [1] that the restrictions of these fields to any contour $C_\tau$ form a Laurent basis in the space of all smooth vector fields on this contour.

We denote by $d^2\Omega_n$ generators in the space of quadratic differentials (tensors of type (2, 0)) on $\Gamma$, holomorphic away from $P_\pm$, where they have poles. For $g > 1$ they can be chosen uniquely so that in neighborhoods of $P_\pm$ we have

$$d^2\Omega_n = (\varepsilon_n^\pm)^{-1} z_\pm^{\mp n + g_0 - 2}(1 + O(z_\pm))(dz_\pm)^\lambda. \tag{1.15}$$

Then

$$\frac{1}{2\pi i}\oint_{C_\tau} e_n\, d^2\Omega_m = \delta_{nm}. \tag{1.16}$$

<u>THEOREM 1.2.</u> For any smooth vector field E(Q) on $C_\tau$ one has

$$E(Q) = \frac{1}{2\pi i}\sum_n e_n(Q)\oint E(Q')\, d^2\Omega_n(Q'). \tag{1.17}$$

For any smooth quadratic differential $d^2f$ on $C_\tau$ one has

$$d^2f(Q) = \frac{1}{2\pi i}\sum_n d^2\Omega_n(Q)\oint e_n(Q')\, d^2f(Q'). \tag{1.18}$$

Definition. By the analog of the Virasoro algebra connected with the curve $\Gamma$ with distinguished points we shall mean the algebra $\hat{\mathscr{L}}^\Gamma$ which is a one-dimensional "local" central extension of the algebra $\mathscr{L}^\Gamma$. It is generated by the elements $E_n$ and t with the commutation relations

$$[E_n, t]=0; \quad [E_n, E_m]=\sum_{k=-g_0}^{g_0} c_{nm}^k E_{n+m-k}+\chi_{nm} t. \tag{1.19}$$

Here $c_{nm}^k$ are the structural constants of the algebra $\mathscr{L}^\Gamma$, in which the following relations hold [1]

$$[e_n, e_m]=\sum_{k=-g_0}^{g_0} c_{nm}^k e_{n+m-k}. \tag{1.20}$$

The numbers $\chi_{nm}$ define a 2-cocycle on the algebra $\mathscr{L}^\Gamma$, which must satisfy the locality condition

$$\chi_{nm}=0, \quad \text{if} \quad |n+m|>3g. \tag{1.21}$$

By virtue of the convergence of Laurent series with respect to the basis $e_n$ on any contour $C_\tau$, any local cocycle on $\mathscr{L}^\Gamma$ extends to a cocycle on the algebra of all smooth vector fields on the circle. Since for the latter algebra there exists a unique nontrivial homology class of two-dimensional cocycles [2], all "local" central extensions of the algebra $\mathscr{L}^\Gamma$ are isomorphic. More precisely, if $\chi_{nm}$ and $\chi'_{nm}$ are two different cocycles, which define extensions of $\mathscr{L}^\Gamma$ by (1.19) and satisfy (1.21), then one can find numbers $s_{-g_0}, \ldots, s_{g_0}, \hat{c}$ such that

$$\chi'_{nm}=\hat{c}\left(\chi_{nm}-\sum_k c_{nm}^k s_{n+m-k}\right). \tag{1.22}$$

Here the correspondence $E_n' = E_n + s_n t$, $|n| < g_0$, $t' = \hat{c}t$ establishes an isomorphism of central extensions, defined by the cycles $\chi_{nm}$ and $\chi'_{nm}$. In what follows we shall be interested not so much in the algebra $\hat{\mathscr{L}}^\Gamma$ itself, which is the unique local central extension of $\mathscr{L}^\Gamma$, as in the fixed basis of operators $E_n$, t in it. Hence in the future we shall be interested in all the local cocycles $\chi_{nm}$ and not just their homology classes.

In [1] "local" cocycles on the algebra $\mathscr{L}^\Gamma$ were defined with the help of "projective" complex structures given on $\Gamma$. These cocycles can be defined more effectively with the help of projective connections given on $\Gamma$.

One says that a holomorphic projective connection R is given on $\Gamma$, if for any local coordinate system $z_\alpha(Q)$, defined in a domain $U_\alpha \subset \Gamma$, there is defined a holomorphic function $R_\alpha(z_\alpha)$. In addition, on the intersection of charts $U_\alpha \subset U_\beta$ the corresponding functions must be related by

$$R_\beta(z_\beta)\left(\frac{\partial z_\beta}{\partial z_\alpha}\right)^2=R_\alpha(z_\alpha)+\mathfrak{S}(f_{\alpha\beta}). \tag{1.23}$$

Here $f_{\alpha\beta}$ are the transition functions, $z_\beta=f_{\alpha\beta}(z_\alpha)$; $\mathfrak{S}(h)$ is the Schwartz derivative,

$$\mathfrak{S}(h)=\frac{h'''}{h'}-\frac{3}{2}\left(\frac{h''}{h'}\right)^2. \tag{1.24}$$

There are several canonical projective connections — of Fuks, Schottky, etc. [3, 4]. It follows from (1.23) that the difference of any two projective connections is a quadratic differential. In what follows we shall consider projective connections R, holomorphic on $\Gamma$ away from the distinguished points $P_\pm$, at which R has a pole of at most the first order. One can represent any such connection uniquely in the form

$$R=R_0+\sum_{n=-g_0}^{g_0} s_n\, d^2\Omega_n, \tag{1.25}$$

if one fixes any holomorphic projective connection $R_0$.

Let f and g be two arbitrary vector fields. Then the formula

$$\tilde{\chi}(f, g) = \left(\frac{1}{2}(f'''g - fg''') - R(f'g - fg')\right)dz, \tag{1.26}$$

which is defined in each local coordinate system in which f and g have the form $f(z)\partial/\partial z$, $g(z)\partial/\partial z$ (the dash denotes the derivative with respect to the local parameter z), determines a well-defined 1-form on $\Gamma$. If the projective connection R has the form (1.25), then the cocycle

$$\chi_{nm} = \frac{\hat{c}}{24\pi i} \oint_{C_\tau} \tilde{\chi}(e_n, e_m) \tag{1.27}$$

defines a "local" central extension of the algebra $\mathscr{L}^\Gamma$.

We shall return later to the question of the explicit calculation of projective connections corresponding to cocycles which will be constructed in the next section in the course of quantization of strings, and the connection of this problem with the problem of "accessory" parameters.

## 2. Analogs of the Heisenberg and Virasoro Algebras in String Theory

The standard phase of a classical boson closed string in D-dimensional Minkowsky space is defined as the space of $2\pi$-periodic functions $x^\mu(\sigma)$ and $2\pi$-periodic differentials $p^\mu(\sigma)$ with Poisson bracket

$$\{p^\mu(\sigma'), x^\nu(\sigma)\} = \eta^{\mu\nu}\Delta(\sigma, \sigma'). \tag{2.1}$$

Here $\eta^{\mu\nu}$ is the Minkowsky metric with signature $(-1, 1, 1, \ldots)$, $\Delta(\sigma, \sigma')$ is the "delta-function on the circle" (more precisely, $\Delta(\sigma, \sigma')$ is a function of the variable $\sigma'$ and the differential with respect to the variable $\sigma'$), where for any function on the circle one has

$$f(\sigma) = \oint f(\sigma')\Delta(\sigma, \sigma'). \tag{2.2}$$

Such a definition of phase space does not permit the naive inclusion in consideration of motion of a string in which topological reconstruction occurs (division and merging) in the course of the motion in time. A possible way of avoiding this difficulty prompts the analysis of the system of contours $C_\tau$, constructed above on any Riemann surface with two distinguished points.

The realization of a two-dimensional Riemann surface $\tilde{\Gamma}$ as "world line" of a string automatically induces a partition of $\tilde{\Gamma}$ into a system of contours corresponding to the position of the string at a specific moment of time. The complex structure on $\tilde{\Gamma}$ arises from the requirement that time is a harmonic function on $\tilde{\Gamma}$. In the role of "diagrams" only those world lines appear for which at no time $\tau$ do new components of the string appear or vanish, and the compactification is algebraic.

We consider the problem of quantization of such diagrams. The ordinary Fourier coefficients must be replaced by the coefficients of the decomopsitions with respect to the functions $A_n$, which form a basis in the space of smooth functions on the contours $C_\tau$ for all $\tau$.

Thus, let $x^\mu(Q)$ and $p^\mu(Q)$ be operator-valued functions and differentials on $\Gamma$, commuting if Q and Q' lie on different contours $C_\tau$ (i.e., at different moments of time) and such that

$$[x^\mu(Q), p^\nu(Q')] = -i\eta^{\mu\nu}\Delta_\tau(Q, Q'),\quad Q, Q' \in C_\tau, \tag{2.3}$$

where $\Delta_\tau$ is the "delta-function" on the contour $C_\tau$. It follows from Theorem 1.1 that

$$\Delta_\tau(Q, Q') = \frac{1}{2\pi i} \sum_n A_n(Q)\, d\omega_n(Q'). \tag{2.4}$$

We denote by $X_n^\mu$ the operator coefficients of the expansion

$$x^\mu(Q) = \sum_n X_n^\mu A_n(Q). \tag{2.5}$$

Since $p^\mu$ is a differential with respect to the variable Q, it should be expanded with respect to the basis of differentials $d\omega_n$

$$p^\mu(Q) = \sum_n P_n^\mu d\omega_n(Q). \tag{2.6}$$

It follows from (2.3) and (2.4) that $P_n^\mu$ and $X_m^\nu$ satisfy the canonical commutation relations

$$[P_n^\mu, X_m^\nu] = \frac{1}{2\pi} \eta^{\mu\nu}\delta_{nm}. \tag{2.7}$$

We define the operators $\alpha_n^\mu$ as the coefficients of the decomposition of the differential

$$\pi p^\mu + (x')^\mu = \sum_n \alpha_n^\mu d\omega_n(Q), \quad Q = (\tau, \sigma), \quad x' = \partial x/\partial\sigma. \tag{2.8}$$

We have

$$\sum_n \alpha_n^\mu d\omega_n(Q) = \sum_n (X_n^\mu dA_n(Q) + \pi P_n^\mu d\omega_n). \tag{2.9}$$

Using (1.13), we get

$$dA_n = \sum_m \gamma_{nm} d\omega_m, \tag{2.10}$$

where the constants $\gamma_{nm}$ are given by (1.3). Finally

$$\alpha_n^\mu = \pi P_n^\mu + \sum_m \gamma_{mn} X_m^\mu. \tag{2.11}$$

These operators satisfy the commutation relations

$$[\alpha_n^\mu, \alpha_m^\nu] = \gamma_{nm}\eta^{\mu\nu} \tag{2.12}$$

and realize a representation of the direct product of D copies of the Γ-analog of the Heisenberg algebra.

The conjugate functions $\bar{A}_n(Q)$ and differentials $\overline{d\omega_n}(Q)$ also form bases in the spaces of smooth functions and differentials on the contours $C_\tau$. Hence one can define operators $\bar{X}_n^\mu$ and $\bar{P}_n^\mu$ from the expansions

$$x^\mu(Q) = \sum_n \bar{X}_n^\mu \bar{A}_n(Q), \tag{2.13}$$

$$p^\mu(Q) = \sum_n \bar{P}_n^\mu \overline{d\omega_n}(Q). \tag{2.14}$$

They satisfy the relations

$$[\bar{X}_n^\mu, \bar{P}_m^\nu] = \frac{1}{2\pi} \eta^{\mu\nu}\delta_{nm}. \tag{2.15}$$

The operators

$$\bar{\alpha}_n^\mu = -\pi \bar{P}_n^\mu + \sum_m \bar{\gamma}_{mn} \bar{X}_m^\mu \tag{2.16}$$

(where the $\bar{\gamma}_{nm}$ are the constants conjugate to $\gamma_{nm}$) are the coefficients of the expansion

$$(x')^\mu - \pi p^\mu = \sum_n \bar{\alpha}_n^\mu \overline{d\omega_n}(Q). \tag{2.17}$$

They satisfy the commutation relations of the conjugate Γ-analog of the Heisenberg algebra

$$[\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu] = \eta^{\mu\nu}\bar{\gamma}_{nm}. \tag{2.18}$$

<u>LEMMA 2.1.</u> One has

$$[\alpha_n^\nu, \bar{\alpha}_m^\mu] = 0. \tag{2.19}$$

<u>Proof.</u> Expanding $\bar{A}_n$ and $\overline{d\omega_n}$ with respect to the bases $A_k$ and $d\omega_k$ respectively, we get

$$X_k = \sum_s \bar{X}_s f_{sk}, \quad P_k = -\sum_s \bar{P}_s \bar{f}_{ks}, \tag{2.20}$$

where

$$f_{sk} = \frac{1}{2\pi i} \oint \bar{A}_s \, d\omega_k. \tag{2.21}$$

From this

$$[\alpha_n^\mu, \bar{\alpha}_m^\nu] = \frac{1}{2} \sum (\bar{f}_{nl}\bar{\gamma}_{lm} - \gamma_{ln} f_{ml}) =$$

$$= -\frac{1}{8\pi^2} \sum_l \left( \oint \bar{A}_l \overline{dA_m} \oint A_n \, d\bar{\omega}_l - \oint A_l dA_n \oint \bar{A}_m \, d\omega_l \right) = \frac{i}{4\pi} \oint (dA_n \bar{A}_m + \overline{dA_m} A_n) = 0. \tag{2.22}$$

In passing to the next to the last equation we used (2.4).

The operators $\alpha_n^\mu$ and $\bar{\alpha}_n^\mu$ with $n > g/2$ will be called death operators, and those with $n \leq g/2$, birth operators of "in"-states. In the standard way one can introduce the Fock space, generated by the birth operators of "in"-states from a "vacuum," which is defined by the relations

$$\alpha_n^\mu \mid \Phi^{in} \rangle = \bar{\alpha}_n^\mu \mid \Phi^{in} \rangle = 0, \quad n > g/2. \tag{2.23}$$

We note that the death operators commute with one another, so the conditions (2.23) are consistent.

The subspaces $\mathcal{A}_\pm$, generated by $A_n$ with $c \pm n \geq g/2$, are dense among functions which are holomorphic in neighborhoods of the points $P_\pm$ respectively. Hence the Fock space of in-states we have defined coincides with the ordinary Fock space, constructed in the standard way from the expansions of $x^\mu$ and $p^\mu$ in Fourier series on a small contour enveloping the point $P_+$.

Analogously, if the Fock space of "out-states" is defined as the space generated by the operators $\alpha_n^\mu$, $\bar{\alpha}_n^\mu$ with $n \geq -g/2$ from an out-vacuum, satisfying (2.24), then they coincide with the standard Fock space of a small contour in a neighborhood of the point $P_-$

$$\langle \Phi^{out} \mid \alpha_n^\mu = \langle \Phi^{out} \mid \bar{\alpha}_n^\mu = 0, \quad n < -g/2. \tag{2.24}$$

It follows form the comments made that all the restrictions connected with the requirements of positivity of the norm of physical states and closure of the Lorentz relations and reducing in the case $g = 0$ to the distinguishing of the critical dimension $D = 26$ [5; 6] remain completely valid in the case $g > 0$ considered by us also. Our subsequent goal is the proof of the fact that for any $g$ the construction of a consistent theory is possible for $D = 26$.

For a classical string the Hamiltonian density and momentum are the half sum and difference of the expressions

$$T = \frac{1}{2}(x' + \pi p)^2, \quad \bar{T} = \frac{1}{2}(x' - \pi p)^2, \tag{2.25}$$

which are quadratic differentials on each contour $C_\tau$. In order to define the quantized analogs of these expressions we need to introduce the concept of normal ordering of birth and death operators. Since the operators $\alpha_n^\mu$, $\alpha_m^\nu$ with $|n|$, $|m| < g/2$ are noncommutative, the possibility of introducing such mutually inequivalent concepts is rather large.

Let $\Sigma^\pm$ be a partition of the set of integral points of the two-dimensional plane into two subsets such that $\Sigma^+$ differs by a finite set of points from the half-plane $\Sigma_0^+$: $(n, m)$, $n \leq m$. For any such admissible partition $\Sigma^\pm$ we define the concept of normal product

$$:\alpha_n\alpha_m: = \alpha_n\alpha_m, \ (n, m) \in \Sigma^+; \ :\alpha_n\alpha_m: = \alpha_m\alpha_n, \quad (n, m) \in \Sigma^-. \tag{2.26}$$

Remark. This definition of normal product is far from the most general one. In the next section we need an extension of it (cf. (3.24)). Since $\alpha_n^\mu$, $\alpha_m^\nu$ commute if (1.4) holds, the concept of normal product depends on the partition into two subsets not of the whole plane, but only of the strip $|n + m| \leq g$. One can give fundamental examples of such partitions with the help of collections of numbers $\sigma_{-g}, \ldots, \sigma_g$. Here we shall put

$$(n, m) \in \Sigma^+, \quad \text{if} \quad n + m = s, n \leqslant \sigma_s, \ s = -g, \ldots, g. \tag{2.27}$$

For any choice of concept of normal product we define the quantum operators

$$\begin{gathered} T(Q) = \frac{1}{2} \sum_{n, m} :\alpha_n \alpha_m: d\omega_n(Q)\, d\omega_m(Q), \\ \bar{T}(Q) = \frac{1}{2} \sum_{n, m} :\bar{\alpha}_n \bar{\alpha}_m: \overline{d\omega_n}(Q)\, \overline{d\omega_m}(Q). \end{gathered} \tag{2.28}$$

Here and later $\alpha_n \alpha_m = \sum \eta^{\mu\nu} \alpha_n^\mu \alpha_m^\nu$.

These operators are quadratic differentials on $C_\tau$. Hence they can be decomposed with respect to the basis of quadratic differentials $d^2\Omega_n$, $\overline{d^2\Omega_n}$

$$T = \sum_k L_k\, d^2\Omega_k; \quad \bar{T} = \sum_k \bar{L}_k\, \overline{d^2\Omega_k}. \tag{2.29}$$

From Theorem 1.2 it follows that

$$L_k = \frac{1}{2} \sum_{n, m} l_{nm}^k :\alpha_n \alpha_m:; \quad \bar{L}_k = \frac{1}{2} \sum_{n, m} \bar{l}_{nm}^k :\bar{\alpha}_n, \bar{\alpha}_m:, \tag{2.30}$$

where the constants $\ell_{nm}^k$ have the form

$$l_{nm}^k = \frac{1}{2\pi i} \oint e_k\, d\omega_n\, d\omega_m. \tag{2.31}$$

Here for $|k| > g_0$ we have

$$l_{nm}^k = 0, \qquad \text{if} \quad |n + m - k| > \frac{g}{2}. \tag{2.32}$$

For $|k| \leq g_0$ the indices n and m for which the constants $\ell_{nm}^k$ can differ from zero satisfy the same relation (2.32) for $|n|$, $|m| > g/2$ and the relation $|n + m - k| \leq g + s$ if one or two of the numbers $|n|$, $|m|$ do not exceed $g/2$. Here s is equal to 1, 2, respectively. In any case it follows from the definition of normal product and (2.32) that the action of the operators $L_k$ and $\bar{L}_k$ is well defined in the Fock spaces of "in"- and "out"-states. By (2.32) and (1.4), the operators $L_k$, $\bar{L}_k$ with $|k| > g_0$ do not depend on the choice of normal order and here

$$L_k \mid \Phi^{\text{in}} \rangle = 0 = \langle \Phi^{\text{out}} \mid L_{-k}, \ k > g_0. \tag{2.33}$$

THEOREM 2.1. The operators $L_k$ satisfy the commutation relations

$$L_i L_j - L_j L_i = \sum_{k=-g_0}^{g_0} c_{ji}^k L_{i+j-k} + D \cdot \chi_{ij}^\Sigma \cdot 1. \tag{2.34}$$

Here D is the dimension of the space and the 2-cocycle $\chi_{ij}^\Sigma$ has the form (1.26), where the projective connection R depends only on the method of normal ordering and is independent of D.

Proof. One verifies directly that

$$[L_i, L_j] = \frac{1}{4} \sum_{n, m, k, s} l_{nm}^i l_{ks}^j (\gamma_{mk} :\alpha_n \alpha_s: + \gamma_{ns} :\alpha_k \alpha_m: + \gamma_{ms} :\alpha_n \alpha_k: + \gamma_{nk} :\alpha_s \alpha_m: + D \cdot F_{nmks}) = \frac{1}{2} \sum_{n, s} f_{ns}^{ij} :\alpha_n \alpha_s: + D\chi_{ij}^\Sigma, \tag{2.35}$$

where

$$f_{ns}^{ij} = \sum_{m, k} (l_{nm}^i l_{ks}^j - l_{ks}^i l_{nm}^j)\, \gamma_{mk}. \tag{2.36}$$

In addition

$$\chi_{ij}^{\Sigma}=\frac{1}{2}\sum_{n,\,m,\,k,\,s} l_{nm}^{i} l_{ks}^{j} F_{nmks}. \qquad (2.37)$$

Here

$$F_{nmks}=0, \qquad (n,s)\in\Sigma^{\pm},\ (k,m)\in\Sigma^{\pm},$$

$$F_{nmks}=\gamma_{ns}\gamma_{km}, \quad (n,s)\in\Sigma^{+},\ (k,m)\in\Sigma^{-}, \qquad (2.38)$$

$$F_{nmks}=\gamma_{ns}\gamma_{mk}, \quad (n,s)\in\Sigma^{-},\ (k,m)\in\Sigma^{+}. \qquad (2.39)$$

It follows from (1.4), (1.5), and (2.32) that the sum (2.37) contains only a finite number of nonzero terms. Hence it is well-defined.

By virtue of (1.3) and (2.31) we have

$$\sum_{m,\,k} l_{nm}^{i} l_{ks}^{j}\gamma_{mk}=\frac{1}{(2\pi i)^3}\sum_{m,\,k}\oint(d\omega_n\, d\omega_m e_i)\oint(d\omega_k\, d\omega_s e_j)\oint A_k\, dA_m=\frac{1}{(2\pi i)^2}\sum_{k}\oint d\omega_k\, d\omega_s e_j\oint d\,(d\omega_n e_i)\, A_k. \qquad (2.40)$$

To get the last equation it suffices, in taking the first integral and the sum over m, to use (2.4). Analogously, from (2.40) we get

$$\sum_{m,\,k} l_{nm}^{i} l_{ks}^{j}\gamma_{mk}=\frac{1}{2\pi i}\oint(d\omega_s e_j)\, d\,(d\omega_n e_i). \qquad (2.41)$$

Consequently

$$f_{ns}^{ij}=\frac{1}{2\pi i}\oint d\omega_n\, d\omega_s\,([e_j,e_i])=\sum_{k=-g_0}^{g_0} c_{ji}^{k} l_{ns}^{i+j-k} \qquad (2.42)$$

and (2.34) is proved. It follows directly from (2.39) that the cocycle $\chi_{ij}^{\Sigma}$ is local. The theorem is proved.

At the present time the authors have still not obtained a complete answer to the important and interesting question of the explicit calculation for all methods of normal ordering of the corresponding projective connections $R_{\Sigma}$. At the end of this section we give an expression in terms of Cauchy type integrals for the cocycle $\chi_{ij}^{\Sigma}$ in the case of a normal ordering defined by a partition of $\Sigma$ of the following form:

$$(n,m)\in\Sigma^{-},\ n>N,\ m<N. \qquad (2.43)$$

For this we define the necessary class of meromorphic analogs of Cauchy kernels, which are special cases of general kernels of similar type [7].

For any integer or half-integer (depending on the parity of g) N, $|N|>g/2$, we denote by $K_N(z_1,z_2)dz_2$ a meromorphic analog of the Cauchy kernel, which is a differential with respect to the variable $z_2$ which in neighborhoods of $P_{\pm}$ has the form

$$K_N(z_1,z_2)dz_2=z_2^{-N+g/2-1}O(1)dz_2,\ z_2\to P_{+}, \qquad (2.44)$$

$$K_N(z_1,z_2)dz_2=z_2^{N+g/2}O(1)dz_2,\ z_2\to P_{-}. \qquad (2.45)$$

With respect to the variable $z_1$ the kernel $K_N$ is a meromorphic function, which, in neighborhoods of $P_{\pm}$, has the form

$$K_N=z_1^{N+1-g/2}O(1),\quad z_1\to P_{+}, \qquad (2.46)$$

$$K_N=z_1^{-N-g/2}O(1),\quad z_2\to P_{-}. \qquad (2.47)$$

Away from $P_{\pm}$ the kernel $K_N$ is holomorphic everywhere except $z_1=z_2$. Here

$$K_N(z_1,z_2)\,dz_2=\frac{dz_2}{z_2-z_1}+\text{regular terms}. \qquad (2.48)$$

The cited properties (by virtue of the same arguments which have been used repeatedly) determine $K_N$ uniquely.

Example. We give an explicit formula for $K_N(z_1, z_2)$ in the case $g = 1$, $P_\pm$: $z = \pm z_0$:

$$K_N(z_1, z_2) = \frac{\sigma^{N+1/2}(z_1 - z_0)\,\sigma^{N+1/2}(z_2 + z_0)\,\sigma(z_1 - z_2 + (2N+1)z_0)}{\sigma((2N+1)z_0)\,\sigma(z_2 - z_1)\,\sigma^{N+1/2}(z_1 + z_0)\,\sigma^{N+1/2}(z_2 - z_0)}. \tag{2.49}$$

(For $g > 1$ one can get analogous expressions in terms of the Riemann theta-functions.)

LEMMA 2.2. If $\tau(z_1) < \tau(z_2)$, then

$$K_N(z_1, z_2)\,dz_2 = \sum_{n=N+1}^{\infty} A_n(z_1)\,d\omega_n(z_2). \tag{2.50}$$

For $\tau(z_1) > \tau(z_2)$ we have

$$K_N(z_1, z_2)\,dz_2 = -\sum_{n=-\infty}^{N} A_n(z_1)\,d\omega_n(z_2). \tag{2.51}$$

We give only a brief sketch of the proof. The convergence of the right sides of (2.50) and (2.51) follows since the terms of the series are majorized by the terms of the geometric progression $Me^{n(\tau(z_1)-\tau(z_2))}$. It follows from (2.4) that (2.50) and (2.51) are analytic continuations of one another and define a global analog of the Cauchy kernel $K_N$. Its behavior in neighborhoods of $P_\pm$ is easy to determine, using the asymptotic form of $A_n$ and $d\omega_n$.

We note that (2.50) and (2.51) define $K_N$ for all N. The asymptotics of $K_N$ for $|N| \leq g/2$ differ slightly from (2.44)-(2.47), similarly to the way the asymptotics of $A_n$ and $d\omega_n$ for $|n| \leq g/2$ differ from the general case.

It follows from (2.37), (1.4), and (2.3) that the cocycle $\chi_{ij}^{\Sigma}$, corresponding to the partition (2.43), is equal to

$$\chi_{ij}^{\Sigma} = \frac{1}{32\pi^4}(\lambda_{ij}^N - \lambda_{ji}^N), \tag{2.52}$$

where

$$\lambda_{ij}^N = \sum \oint d\omega_n\,d\omega_m\,e_i \oint d\omega_k\,d\omega_s e_j \oint dA_n A_s \oint dA_k A_m. \tag{2.53}$$

The summation in (2.53) is over $n \leq N$, $s > -N$, $k > N$, $m \leq N$. We choose the contours of integration in (2.53) so that the corresponding variables $z_1$, $z_2$, $z_3$, $z_4$ satisfy the conditions $\tau(z_1) < \tau(z_3) \leqslant \tau(z_4) < \tau(z_2)$. Then it follows from Lemma 2.2 that

$$\lambda_{ij}^N = \oint\oint\oint\oint dz_1\,dz_2\,dz_3\,dz_4\,[e_i(z_1)\,e_j(z_2)\,K_N(z_4, z_1)\,K_{-N}(z_3, z_2)\,(d_3K_N(z_3, z_1))\,(d_4K_N(z_4, z_2))]. \tag{2.54}$$

Here $d_3$ and $d_4$ denote differentiation with respect to the variables $z_3$ and $z_4$ respectively. Shrinking the contours $C_{\tau_3}$ and $C_{\tau_4}$ to the points $P_\pm$, one can get an expression of the form (1.26), where the corresponding projective connection is a linear combination of the first coefficients of the expansions of $K_{\pm N}$ in a neighborhood of the diagonal $z_1 = z_2$.

## 3. Operator Realization of Multilooped String Diagrams and Conformal Anomalies

As already said above, the Fock spaces of "in"- and "out"-states coincide with Fock spaces corresponding to small contours in neighborhoods of the points $P_\pm$, respectively. Hence the physical states in the case of arbitrary g are defined by the conditions

$$L_i|\Phi_{h_+}^{\text{in}}\rangle = 0, \quad i > g_0; \qquad L_{g_0}|\Phi_{h_+}^{\text{in}}\rangle = h_+|\Phi_{h_+}^{\text{in}}\rangle, \tag{3.1}$$

$$\langle\Phi_{h_-}^{\text{out}}|L_i = 0, \quad i < -g_0; \quad \langle\Phi_{h_-}^{\text{out}}|L_{-g_0} = \langle\Phi_{h_-}^{\text{out}}|h_-, \tag{3.2}$$

where the constants $h_\pm$ are equal to

$$h_+ = 1, \quad h_- = \varepsilon_{-g_0}^-. \tag{3.3}$$

(We recall that in a neighborhood of $P_-$ the field $e_{-g_0}$ has the form $\varepsilon^-_{-g_0} z\ \partial/\partial z$.) The actions of the operators $L_i$ with $i \geq g_0$ on $|\Phi^{in}_{h_+}\rangle$ and $L_i$ with $i \geq -g_0$ on $\langle\Phi^{out}_{h_-}|$ generate subspaces $V^{in}_{h_+}$, $V^{out}_{h_-}$. As follows from (2.34), the correspondences $E_i \to L_i$ and $E_i \to -L_i$ define representations of the analog of the Virasoro algebra on $V^{out}_{h_-}$ and $V^{in}_{h_+}$ respectively. In addition these spaces are Verma modules over $\mathcal{L}^\Gamma$. Our next goal will be the construction of scalar products between the elements of such spaces

$$\langle\Phi \mid \Psi\rangle,\ \Phi \in V^{out}_{h_-},\ \Psi \in V^{in}_{h_+} \tag{3.4}$$

so that with respect to this product the operators $L_i$ will be self-adjoint, i.e.,

$$\langle\Phi \mid L_i \mid \Psi\rangle = \langle\Phi L_i \mid \Psi\rangle = \langle\Phi \mid L_i\Psi\rangle. \tag{3.5}$$

For this we need the realization by the Verma modules of the analog of the Virasoro algebra given in [1] which generalizes the corresponding construction of [8] for the case $g = 0$.

In [1] there were defined bases $f_j$ in the spaces $F_\lambda(x_0)$ of tensor fields of weight $\lambda$, which are holomorphic on $\Gamma$ away from the points $P_\pm$ and the contour $\sigma$ joining these points. Here the limit values on $\sigma$ of any field $f \in F_\lambda(x_0)$ are related by

$$f^+ = f^- e^{2\pi i x_0}. \tag{3.6}$$

It was shown in [1] that the action of $e_i$ on $f_j$ has the form

$$e_i f_j = \sum_{k=-g_0}^{g_0} R^k_{ij} f_{i+j-k}, \tag{3.7}$$

where the $R^k_{ij}$ are constants. In particular,

$$R^{\pm g_0}_{ij} = (\pm j \pm x_0 - S(\lambda) + \lambda(\pm i - g_0 + 1)) \frac{\varphi^\pm_j \varepsilon^\pm_i}{\varphi^\pm_{i+j\mp g_0}}. \tag{3.8}$$

Here $\phi^\pm_j$ are the constants defined by the form of $f_j$ in neighborhoods of $P_\pm$ (cf. [1] for the details), and

$$S(\lambda) = g/2 - \lambda(g - 1) \tag{3.9}$$

(we note that by the authors' oversight the corresponding formula (2.2) in [1] is given with a misprint).

We denote by $H_{\lambda,N}(x_0)$ the space generated by the basis consisting of half-infinite forms — exterior products of the form

$$f_{i_1+N} \wedge f_{i_2+N} \wedge \cdots \wedge f_{i_{m-1}+N} \wedge f_{m+N} \wedge f_{m+1+N} \wedge \cdots, \quad i_1 < i_2 < \cdots < i_{m-1} < m, \tag{3.10}$$

where the indices of the fields f, starting from some index, run through all values in succession to $\infty$.

If one defines the action of $e_i$ on the forms (3.10) by the Leibniz rule, then it follows from (3.7) that this action is well defined for $|i| > g_0$. For any cocycle $\chi_{ij}$, defining a local central extension of $\mathcal{L}^\Gamma$, there exists a unique extension of the actions of $e_i$ with $|i| > g_0$ to a representation of the algebra $\hat{\mathcal{L}}^\Gamma$. Let us agree to normalize the cocycles so that they have the form (1.27) with $\hat{c} = 1$. Then the operator t acts on $H_{\lambda,N}(x_0)$ by multiplication by the number

$$D = -2(6\lambda^2 - 6\lambda + 1), \tag{3.11}$$

called <u>the central charge</u>. The vector

$$|\Psi_0> = f_{N+1} \wedge f_{N+2} \wedge f_{N+3} \wedge \dots \tag{3.12}$$

satisfies the conditions

$$E_i |\Psi_0> = 0,\ i > g_0;\ E_{g_0} |\Psi_0> = -h_+ |\Psi_0>. \tag{3.13}$$

The highest weight $h_+$ depends on $x_0$, $\lambda$, and the choice of the cocycle $\chi_{ij}$. If the corresponding projective connection R has the form

$$R = \rho^{\pm} z_{\pm}^{-2} (1 + O(z_{\pm})) \tag{3.14}$$

in a neighborhood of $P_{\pm}$, then

$$\chi_{i,-i\pm 3g} = \frac{(i \mp g_0)^3 - (i \mp g_0) - 2\rho \pm (i \mp g_0)}{12} \varepsilon_i^{\pm} \varepsilon_{-i\pm 3g}^{\pm}. \tag{3.15}$$

Calculating the action of $[L_i, L_{-i+3g}]$ on $|\Psi_0>$ with the help of (3.8), and using (1.19) and (3.15), we get the formula (cf. the note)

$$h_+ = \frac{1}{2}\left(\frac{D}{6}\rho_+ + (N - S(\lambda) + x_0 + 1)(2\lambda + N + x_0 - S(\lambda))\right). \tag{3.16}$$

The action of the operators $L_i$ on $V_{h_-}^{out}$ determines the Verma module generated by the vector $<\Phi_{h_-}^{out}|$, annihilated by $e_i$ with $i < -g_0$. One can realize such a module by defining the action of $e_i$ on left semiinfinite forms. However we shall need a realization of it of a different form. We denote by $H_{\lambda,N}(x_0)$ the space generated by the forms

$$f_{j_1+N}^+ \wedge f_{j_2+N}^+ \wedge \dots \wedge f_{j_{m-1}+N}^+ \wedge f_{m+N}^+ \wedge f_{m+1+N}^+ \wedge \dots \tag{3.17}$$

Here $f_j^+$ are elements of the space $F_{1-\lambda}(-x_0)$, uniquely defined up to propotionality by their asymptotic form in neighborhoods of $P_{\pm}$

$$f_j^+ = z^{\mp j \mp x_0 - S(\lambda)+1} O(z_{\pm}) (dz)^{1-\lambda}. \tag{3.18}$$

One can choose the corresponding constants uniquely so as to have

$$\frac{1}{2\pi i} \oint f_n f_m^+ = \delta_{nm}. \tag{3.19}$$

It follows from this that the action of $e_i$ on $f_j^+$ is equal to

$$e_i f_j^+ = -\sum_{k=-g_0}^{g_0} R_{i,j-i+k}^k f_{j-i+k}^+. \tag{3.20}$$

Just as in the preceding case one can define the action of $e_i$ on $H_{\lambda,N}^+(x_0)$. Here the vector

$$\langle \Phi_0 | = f_{N+1}^+ \wedge f_{N+2}^+ \wedge \dots \tag{3.21}$$

satisfies the conditions

$$\langle \Phi_0 | E_i = 0,\ i < -g_0,\ \langle \Phi_0 | E_{-g_0} = \langle \Phi_0 | h_-.$$

The corresponding representation of $\hat{\mathscr{L}}^{\Gamma}$ has the same central charge (3.11), and the highest weight $h_-$ is given by

$$h_- = \frac{\varepsilon_{-g_0}^-}{2}\left(\frac{D}{6}\rho_- + (S(\lambda) + N + x_0)(S(\lambda) + N + x_0 + 1 - 2\lambda)\right). \tag{3.22}$$

We define the scalar product between elements of $H_{\lambda,N}(x_0)$ and $H_{\lambda,N}^+(x_0)$, by defining it on the basis elements (3.10) and (3.17)

$$\langle j_1, \dots, j_{m-1} | i_1, \dots, i_{m-1} \rangle = \delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{j_{m-1} i_{m-1}} \tag{3.23}$$

and extending it to the rest by linearity. With respect to this product, the operators $E_i$ are skew-symmetric, as follows from (3.20). The product between $H_{\lambda,N}(x_0)$ and $H_{\lambda,N}^+(x_0)$ defines a product between $V_{h_+}^{in}$ and $V_{h_-}^{out}$, where the operators $L_i$ are self-adjoint with respect to this product. It is important to note that we have defined a nontrivial scalar product between spaces corresponding to the highest weights $h_+$, $h_-$, which are given by (3.16) and (3.22). For each fixed method of normal ordering, the set of such admissible pairs $h_+$, $h_-$ is a one-parameter set. Hence the question arises naturally, can $h_+$, $h_-$ simultaneously assume the values (3.3).

Apparently if one confines oneself to just the normal orderings introduced in Sec. 2, then the answer to this question is negative. We extend this concept by defining

$$:\alpha_n\alpha_m: = \alpha_n\alpha_m + \tilde{\gamma}_{mn}. \tag{3.24}$$

Here $\tilde{\gamma}_{mn}$ are arbitrary constants, equal to zero for all but a finite number of points of the half-plane $n \le m$ and equal to $\gamma_{mn}$ for all but a finite number of points of the half-plane $n > m$. To the case (2.26) corresponds $\tilde{\gamma}_{mn} = 0,\ (n, m) \in \Sigma^+,\ \tilde{\gamma}_{mn} = \gamma_{mn}\ (n, m) \in \Sigma^-$ .

By varying the extended concept of normal ordering one can realize an arbitrary cocycle $\chi_{nm}$. In particular, the parameters $\rho_\pm$ in (3.16) and (3.22) can be considered arbitrary.

The scalar product introduced lets us define, for any operator from the associated ring generated by the operators $L_i$, the concept of its mean

$$\langle L_{i_1} \ldots L_{i_k}\rangle_{x_0} = \langle \Phi^{\text{out}}_{h_-} | L_{i_1} \ldots L_{i_k} | \Phi^{\text{in}}_{h_+}\rangle \tag{3.25}$$

(we use the index $x_0$ to recall that the admissible pairs $h_+$, $h_-$ are parametrized by means of $x_0$).

By definition the mean $\langle L_i\rangle_{x_0}$ is equal to zero for $|i| > g_0$. Now the means

$$S_i = \langle L_i\rangle_{x_0},\quad i = -g_0, \ldots, g_0 \tag{3.26}$$

are a priori nontrivial quantities. Since these means are the coefficients of the expansion of the energy-momentum tensor, these quantities correspond to the so-called conformal anomaly, for which $\langle T(z)\rangle \neq 0$.

Unfortunately the restricted size of the paper does not permit us to give a sufficiently detailed calculation even for the simplest case, after $\langle L_{\pm g_0}\rangle = h_\pm$, of a mean $\langle L_{g_0-1}\rangle_{x_0}$. We shall merely indicate some of the most essential points.

We have

$$\langle L_i L_{-i+3g-1}\rangle = \sum_{n=N+1}^{N+i-g_0+1} R^{g_0-1}_{i,\, n-i+g_0-1} R^{g_0}_{-i+3g-1,\, n} + \sum_{n=N+1}^{N+i-g_0} R^{g_0}_{i,\, n-i+g_0} R^{g_0-1}_{-i+3g-1,\, n}. \tag{3.27}$$

Here $R^{g_0}_{ij}$ is given by (3.8), and to get $R^{g_0-1}_{ij}$ it suffices to equate the coefficients of $z^{i+j-g_0-S+1}$ in (3.7). We denote by $\varphi_{j,\lambda}$ the coefficient of the expansion $(z \to P_+)$

$$f_j = z^{j-S(\lambda)+x_0}\left(1 + \varphi_{j,\lambda} z + O(z^2)\right)(dz)^\lambda. \tag{3.28}$$

Then

$$R^{g_0-1}_{ij} = (j - S(\lambda) + x_0 + \lambda(i - g_0 + 1))(\varphi_{i,-1} + \varphi_{j,\lambda} - \varphi_{i+j-g_0,\lambda}) + \varphi_{j,\lambda} + \lambda\varphi_{i,-1}. \tag{3.29}$$

Further direct calculations lead to the equation $(I = i - g_0)$

$$\begin{aligned}\langle L_i L_{-i+3g-1}\rangle &= \varphi_{i,-1}\left(D\,\frac{I(I+1)(I+2)}{12} + (2I+1)h_+\right) + \\ &+ \varphi_{-i+3g-1}\left(D\,\frac{I^3 - I}{12} + 2Ih_+\right) + (N - S(\lambda) + x_0 + 1)(2I+1)\varphi_{N,\lambda}.\end{aligned} \tag{3.30}$$

It follows from (2.34) that

$$\langle L_i L_{-i+3g-1}\rangle = (2I + 1)\langle L_{g_0-1}\rangle + c^{g_0-1}_{-i+3g-1,\, i} h_+ + D\chi^{\Sigma}_{i,\, -i+3g-1}. \tag{3.31}$$

In calculating the cocycle $\chi_{i,-i+3g-1}$ it is necessary to use, in addition to its definition, the relations

$$l^i_{n,\, i-n-9/2} = 1, \qquad l_{n,\, i-n-9/2-1} = \varphi_{n,\, 1} + \varphi_{i-n-9/2,\, 1} + \varphi_{i,\, -1}, \tag{3.32}$$

$$\gamma_{n,\, g-n} = n - {}^9/_2, \quad \gamma_{n,\, g-n-1} = (n - {}^9/_2)\, \varphi_{g-n-1,\, 0} + (n - {}^9/_2 + 1)\, \varphi_{n,\, 0}, \tag{3.33}$$

which are simply consequences of the definitions of $\ell^i_{nm}$ and $\gamma_{nm}$. It follows from (1.10) for $m = n + 1$ that

$$\varphi_{n,\, 0} + \varphi_{n+1,\, 1} = 0. \tag{3.34}$$

Using this equation, one can get that for $\sigma_g = g/2$ (in this case $\rho_+ = 0$) and $\sigma_{g-1} = g/2 + 1$

$$\chi^{\Sigma}_{i,\, -i+3g-1} = \frac{I(I+1)(I+2)}{12}\, \varphi_{i,\, -1} + \frac{I^3 - I}{12}\, \varphi_{-i+3g-1,\, -1}. \tag{3.35}$$

Substitution of (3.35) into (3.31) gives ($c^{g_0-1}_{i,j}$ is given by (3.29) for $\lambda = -1$)

$$\langle L_{g_0-1}\rangle_{x_0} = h_+ \varphi_{g_0-1,\, -1} + (N - S(\lambda) + x_0 + 1)\varphi_{N,\, \lambda}. \tag{3.36}$$

Example ($g = 1$). In this case $\phi_{g_0-1,-1} = 0$, and it is easy to get $\phi_{N,\lambda}$ from (5.2) of [1],

$$\varphi_{N,\, \lambda} = \zeta\, (((2N + x_0) + 1)z_0) - (N + x_0 + {}^1/_2)\zeta\, (2z_0). \tag{3.37}$$

It is already clear from this example that the direct calculation of the means $\langle L_i\rangle$ is quite complicated. The authors intend to give an invariant method of deriving these means in the framework of the construction of models of the conformal theory of fields on arbitrary Riemann surfaces of genus $g > 0$ in a subsequent publication.

Note. It is important to note that to physical values $D > 1$ correspond complex values of $\lambda$. In particular, for $D = 26$, $h = 1$, $\lambda = 1/2 \pm 5i/2\sqrt{3}$, $\mu = N + x_0 + g/2\ (2\lambda - 1) = -1/2 \pm i/2\sqrt{3}$. It is interesting that for $\lambda = 2$ and $\mu = 0$ we get the nonphysical situation $D = -26$, $h = -1$.

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