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THE PEIERLS MODEL

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INTRODUCTION

The recently developed theory of "finite-zone" integration of nonlinear equations harmoniously combines ideas from the theory of nonlinear equations, variational principles for Kruskal-type functionals, the spectral theory of linear operators with periodic coefficients, and methods from algebraic geometry (see the review papers [1-4]). This relationship was particularly close in the primary stage, when the algebrogeometric approach to the spectral theory of the periodic Sturm—Liouville operator was used to construct periodic and quasiperiodic solutions to the Korteweg—deVries equation [1]. (It is interesting to remark that the variational principles used initially to define finite-zone solutions [5] are, in a certain sense, kindred to the ones considered below.) Subsequently, the successes of the algebrogeometric language (especially with two-dimensional equations of the Kadomtsev—Petviashvili type) left both the spectral theory and the variational principles in shade. Roughly one and a half to two years ago it was discovered that the algebrogeometric methods can be applied in problems from the theory of quasi-one-dimensional conductors [6-10]. The typical features of the quasi-one-dimensional conductors (the presence of periodic superstructures and of charge density waves, and the appearance of slits on the Fermi surface) are customarily explained in the framework of the theory based on the Peierls model [11].

This model describes the self-consistent behavior of electrons and atoms of an ionic frame. The state of atoms is characterized by their coordinates on the line x_n, $x_n < x_{n+1}$, and by the quantities v_n, which represent the "internal" degrees of freedom. The model neglects the direct interaction between electrons, but takes into account the deformation energy of the lattice. The interaction of electrons with the lattice reduces to the fact that the state of the latter determines the energy spectrum of the electrons. The energy levels of the electrons are points of the spectrum $E_1 < \ldots < E_N$ (which can merge, though not more than in pairs) of the Schrödinger difference operator

$$L\psi_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1} = E\psi_n \tag{1}$$

with periodic boundary conditions

$$\psi_n(E_i) = \psi_{n+N}(E_i).$$

Here $v_n = v_{n+N}$ and $c_n = c_{n+N} = \exp(x_n - x_{n+1})$.

If there are m electrons in the system (we do not deal with spin degeneration), then, by virtue of Pauli's principle, they occupy, at zero temperature, the m lowest energy levels. The total energy of the system is the sum of the energy of electrons and of the deformation energy of the lattice. Its value, fitted to one node, is equal to

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$$\mathcal{H} = \frac{1}{N}\Big(\sum_{i=1}^{m} E_i + \sum_{n=0}^{N-1} \tilde{\Phi}_1(v_n) + \tilde{\Phi}_2(c_n^2)\Big), \tag{2}$$

where $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are the deformation energy.

The problem is to minimize the functional $\mathcal{H}$ with respect to the variables v_n and c_n and for a fixed value of the electron density $\rho = m/N$. One is particularly interested in this problem in the thermodynamic limit $N \to \infty$, $m \to \infty$, $m/N \to \rho = \mathrm{const}$.

If $\rho = 0$, the minimum of $\mathcal{H}$ is trivially seen to be $v_n = v^\circ$, $c_n = c^\circ$, where v° and c° correspond to the minima of $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$, respectively. Moreover, one has $x_n = an + b$, and the lattice is uniform. The spectrum of the operator L with constant coefficients on the entire line reduces to a segment. When $\rho > 0$, v_n and c_n are no longer constants, and slits appear in the spectrum. Under the assumption that the slits on the Fermi surface are small, the model admits a continual approximation. The latter has been investigated in [6-11] for a special choice of the deformation potential. There the electrons's contribution to the energy was defined by the spectrum of either the Dirac equation for $|\rho - \frac{1}{2}| \ll 1$, or the Sturm–Liouville operator for $\rho \ll 1$.

The discrete Peierls model was integrated in [12] for the particular choice of potentials $\tilde{\Phi}_1^0 = \varkappa_2 v^2/2$ and $\tilde{\Phi}_2^0 = \varkappa_2 c^2 - \varkappa_0 \ln c$. It was shown that in the thermodynamic limit to the minimum of $\mathcal{H}$ there corresponds an operator L having two permitted zones of spectrum. Moreover, all of the energy levels lying in the lower zone $e_1 \leqslant E \leqslant e_2$ are occupied, while those lying in the upper zone $e_3 \leqslant E \leqslant e_4$, $e_2 < e_3$, are empty. The energy of the ground state and the corresponding values of v_n and c_n were also found.

The proof that the ground state is one zone used, in an essential way, the fact that for the choice of potentials indicated above the deformation energy is a linear combination of first integrals of the equations of the Toda lattice: $I_0 = \frac{1}{N}\sum_{n=0}^{N-1} \ln c_n$, $I_2 = \frac{1}{N}\sum_{n=0}^{N-1}\left(c_n^2 + \frac{v_n^2}{2}\right)$.

(An operator L is said to be a q-zone operator if it has q bounded gaps in its spectrum or a q + 1 permitted zone). Since the E_i are first integrals of the Toda lattice too, the ground state, as well as all the levels of the functional

$$\mathcal{H}_0 = \frac{1}{N}\sum_{i=0}^{m} E_i + \varkappa_2 I_2 - \varkappa_0 I_0$$

are degenerate. A similar degeneracy is responsible for the so-called Fröhlich conductivity.

In the present paper we consider the general Peierls functional

$$\mathcal{H} = \mathcal{H}_0 + g\mathcal{H}_1, \quad \mathcal{H}_1 = \frac{1}{N}\sum_{n=0}^{N-1} (\Phi_1(v_n) + \Phi_2(c_n^2)). \tag{3}$$

We attach particular importance to the characteristics of the dependence of the energy of the ground state upon the density ρ. It turns out that this energy, $\mathcal{H}^*(\rho)$, which is a smooth function of ρ when $g = 0$, becomes discontinuous at the rational points $\rho = p/q$ when $g \neq 0$. The corresponding jump is of order $\sim g e^{-q}$. Both a number of physical considerations and the results of computer calculations [13-15] indicate that the degeneracy of the ground state is related to ρ being rational or irrational. We shall prove below that for ρ irrational and $g < g_\rho$ the ground state degenerates.

The particularities of the behavior of the system indicated above are related to the fact that the dependence of q-zone operators upon their n coefficients is a discrete analogue of the time dynamics of completely integrable systems. The Peierls functional has the form of a discrete "time" average. The singularities of such quantities are caused not only by resonances among frequencies, as in the continuous case, but also by resonances with the number one.

1. Periodic Shrödinger Difference Operator

In this section we shall briefly discuss the main elements of the spectral theory of the Schrödinger difference operator. This operator was actively studied by many authors in connection with the integration of the Toda lattice. The preference given to this direction led, as we already mentioned, to the situation that, aside from the paper [1] (Ch. III, Sec. 1), which provided the foundation for the algebrogeometric approach to difference problems, the other works essentially did not touch upon the aspects of spectral theory and its relation to the variational principles.

We shall mainly follow the definitions and notation of [12], where the gap mentioned above was filled up. We shall pause to discuss in more detail the explicit expressions for v_n and c_n in terms of theta-functions of Riemann surfaces. Such formulas for v were obtained, with some inaccuracies, in [1] (Ch. III, Sec. 1). Complete formulas for all the coefficients of operator L were obtained in [16] (see also the author's appendix to the review paper [4]).

The shift-by-a-period operator $\hat{T}$: $y_n \to y_{n+N}$ takes the solutions of equation (1) into solutions of the same equation. Denote by $\hat{T}(E)$ the corresponding two-dimensional linear operator. The eigenvalues of $\hat{T}(E)$ are defined by the characteristic equation

$$w^2 - 2Q(E)w + 1 = 0, \tag{4}$$

where $2Q(E) = \mathrm{Sp}\,\hat{T}(E)$ is a polynomial of degree N.

To each pair (w, E) of complex numbers satisfying equation (4) (or, which is the same, to each point of the curve defined by this equation) there corresponds a unique solution to equation (1) satisfying $\psi_{n+N} = w\psi_n$ and normalized by the condition $\psi_0 = 1$. We shall call ψ_n the Bloch solution.

The spectrum of the operator L on the entire line is the union of those segments of the real axis (throughout this paper we are interested only in operators having real coefficients) for which $|Q(E)| \leqslant 1$. These segments are called permitted zones. Their endpoints e_1, e_2, ..., e_{2q+2} are simple roots of the equation $Q^2(E) = 1$. It follows from (4) that these are simple points of the spectra of the periodic and antiperiodic problems for the operator L. An operator L having the (q + 1)-th permitted zone is called a q-zone operator. Notice that $q \leqslant N - 1$.

For each value of E there are two Bloch functions $\psi_n^{\pm}(E)$, corresponding to the two roots of equation (4). This double-walled function of E becomes single-valued on the Riemann surface Γ of function $\sqrt{R(E)}$,

$$R(E) = \prod_{i=1}^{2q+2} (E - e_i). \tag{5}$$

We shall imagine that Γ is glued up from two sheets of the E plane, with cuts along the permitted zones. The sheet where, at infinity, the branch $\sqrt{R} = E^{q+1} + O(E^q)$ of the function $\sqrt{R}$ was selected will be called upper. The Bloch function $\psi_n(P)$ is meromorphic as a function of the point P of surface Γ. It has the following form in the vicinity of $E = \infty$:

$$\psi_n^{\pm}(E) = e^{\pm x_n} E^{\pm n} \left(1 + \sum_{s=1}^{\infty} \xi_s^{\pm}(n) E^{-s}\right) \tag{6}$$

(here the sign ± denotes the upper and lower lists of Γ, respectively). In the complement of infinity $\psi_n(P)$ has q poles γ_i, i = 1, ..., q, and the fact that $c_n > 0$ implies that one of these poles lies on each of the circles over the forbidden zones.

Remark. The points γ_i, or, more precisely, their projections on the E plane, which we shall denote with the same letters, have a natural spectral interpretation; namely, they are points of the spectrum of the operator L in the problem with null boundary conditions $\psi_0 = \psi_N = 0$. The remaining N − q points of the spectrum of such a problem coincide with the degenerate points of the spectra of the periodic and antiperiodic problems for the operator L, i.e., with roots of multiplicity two of the equation $Q^2(E) = 1$.

Consider the converse statement. Let there be given an arbitrary collection of points $e_1 < \ldots < e_{2q+2}$ on the real axis. Then for any collection of points $\gamma_1, \ldots, \gamma_q$ of the corresponding Riemann surface, disposed one over each bounded forbidden zone, there exists a function $\psi_n(P)$, $P \in \Gamma$, unique up to its sign, that behaves as in (5) in the vicinity of infinity and, outside infinity, has q poles at the points $\gamma_1, \ldots, \gamma_q$.

Remark. Condition (5) simply means that $\psi_n(P)$ has a pole of order n on the upper list, and a zero of order n on the lower list. The coefficients of the leading terms are normalized in such a way that their product equals 1. We denote these coefficients by $e^{\pm x_n}$.

The proof of this assertion is a straightforward consequence of the Riemann–Roch theorem. A little further we shall write down explicit expressions for $\psi_n(P)$ in terms of Riemann's theta-function.

It was proved in [16] that the function $\psi_n(P)$ thus constructed satisfies the equation $L\psi_n(P) = E(P)\psi_n(P)$, where the Schrödinger operator L has coefficients

$$c_n = e^{x_n - x_{n+1}}, \tag{7a}$$

$$v_n = \xi_1^+(n) - \xi_1^+(n+1). \tag{7b}$$

(Here E(P) stands for the projection of the point $P \in \Gamma$ on the E plane).

Remark. Our assertions remain valid for arbitrary, including complex, values of e_i and γ_j. The constraints imposed to the disposition of γ_j are necessary and sufficient for the coefficients c_n and v_n to be real.

For an arbitrary collection of points e_i, the operators L constructed above are quasi-periodic. To isolate the truly periodic operators and obtain explicit formulas for v_n and c_n, let us introduce the important notion of the quasimomentum differential. By definition, this is a differential of the form

$$i\,dp = \frac{E^q + \sum_{k=0}^{q-1} a_k E^{q-k}}{\sqrt{R(E)}}\,dE, \tag{8}$$

normalized by the condition

$$\int_{e_{2i}}^{e_{2i+1}} dp = 0. \tag{9}$$

System (9) is equivalent to a system of linear nonhomogeneous equations in the a_k, which can thus be expressed in quadratures.

Let the Ω_j be holomorphic differentials on Γ

$$\Omega_j = \sum_{j=0}^{q-1} d_{ij} \frac{E^j\,dE}{\sqrt{R(E)}}$$

satisfying

$$2\int_{e_{2i}}^{e_{2i+1}} \Omega_j = -\delta_{ij}.$$

(Here and in the sequel the integrals between ramification points are taken on the lower sheet. If the integration contour encircles a cut, then it passes on the upper edge of the cut on the lower sheet.)

The matrix of b periods is defined by the equality

$$B_{ik} = \frac{1}{2}\int_{e_1}^{e_{2i}} \Omega_k.$$

It is known that B is symmetric and has a positive definite imaginary part.

The integer combinations of the vectors of C^q having coordinates δ_{ik} and B_{ik} form a lattice, which in turn defines a complex torus $J(\Gamma)$, called the Jacobi variety (the Jacobian) of the curve Γ. The Abel map $A: \Gamma \to J(\Gamma)$ is defined as follows: The coordinates $A_k(P)$ of the point A(P) are equal to

$$A_k(P) = \int_{e_1}^{P} \Omega_k.$$

Given the matrix of b periods, as well as any other matrix with a positive imaginary part, one can construct an entire function of q complex variables

$$\theta(u_1, \ldots, u_q) = \sum_{k \in Z^q} \exp\{\pi i((Bk, k) + 2(k, u)),$$

where $(k, u) = k_1 u_1 + \ldots + k_q u_q$. The latter enjoys the following easily verifiable properties:

$$\theta(u_1, \ldots, u_j + 1, \ldots, u_q) = \theta(u_1, \ldots, u_j, \ldots, u_q), \tag{10a}$$

$$\theta(u_1 + B_{1k}, \ldots, u_q + B_{qk}) = \exp(-\pi i(B_{kk} + 2u_k))\,\theta(u_1, \ldots, u_q). \tag{10b}$$

Moreover, given any collection of points $\gamma_1, \ldots, \gamma_q$ in general position, one can find a vector Z such that the function $\theta(A(P) - Z)$ has precisely q zeros on Γ, which are just the γ_i's. For a standard choice of the cycles, the vector Z becomes

$$Z_k = \sum_{j=1}^{q} \int_{e_{2j}}^{\gamma_j} \Omega_k + \frac{k}{2}.$$

The Bloch function $\psi_n(P)$ has the form

$$\psi_n(P) = r_n \exp\left(ni \int_{e_1}^{P} dp\right) \frac{\theta(A(P) + nU - Z)}{\theta(A(P) - Z)}, \tag{11}$$

where r_n is a constant and the vector U has the coordinates

$$U_k = \frac{1}{\pi} \int_{e_1}^{e_{2k}} dp, \quad 0 < \ldots < U_k < U_{k+1} < \ldots < 1. \tag{12}$$

From (10) and (12) it follows easily that the right-hand side of (11) is correctly defined on Γ, i.e., it does not depend upon the choice of the path of integration between e_1 and P. Moreover, $\psi_n(P)$ has the required singularities at infinity and at the points γ_i. (A formula of this type was first proposed by Its [17] for the Bloch function of the Sturm–Liouville operator. The general expression for the Baker–Akhiezer-type functions was obtained in [18] (see also [2, 4]).) In the neighborhood of infinity and on the upper sheet one has

$$\exp\left(i \int_{e_1}^{E} dp\right) = E e^{-I_0}(1 - I_1 E^{-1} + O(E^{-2})),$$

where $I_1 = \alpha_1 + \frac{s_1}{2}$, $s_1 = \sum e_i$.

It follows from (5) that $e^{2x_n + 2nI_0}$ equals the ratio of the values of the factors that multiply exp in (11), taken at the images of the points at infinity $\pm z_0 = \pm(z_{01}, \ldots, z_{0q})$, $z_{0k} = \int_{e_1}^{\infty} \Omega_k$.

Thus, by virtue of (7a),

$$e^{2I_0} c_n^2 = \frac{\theta(z_0 + nU - Z)\,\theta(-z_0 + (n+1)U - Z)}{\theta(-z_0 + nU - Z)\,\theta(z_0 + (n+1)U - Z)}. \tag{13a}$$

Riemann's bilinear relations imply [18] that

$$2z_0 = -U. \tag{14}$$

One has, in the vicinity of the infinity on the upper sheet, that

$$A(E) = z_0 + VE^{-1} + O(E^{-2}),$$

where the coordinates V_k are defined by the equality

$$\Omega_k(E) = dE^{-1}(V_k + O(E^{-1})).$$

The definition of Ω_k shows that $V_k = -d_{k,q-1}$.

Expanding (11) in powers of E^{-1}, we get

$$v_n = \frac{d}{dt} \ln \frac{\theta(z_0 + nU - Z + Vt)}{\theta(z_0 + (n+1)U - Z + Vt)} \Big|_{t=0} + a_1 + \frac{s_1}{2}. \tag{13b}$$

As formula (13) makes clear, the operator L has period N if and only if $U_k = m_k/N$, where the m_k are integers, and $0 < \ldots < m_k < m_{k+1} < \ldots < N$.

As formula (13) shows, the coefficients v_n and c_n of the operator L depend on the point Z of the unit torus in addition to the parameters of the θ function and the vectors U, V, defined by the curve Γ. As in the periodic case, the "generating function" for the integrals of the Toda lattice, i.e., for quantities which do not depend on Z, is, for any finite-zone operator, the quasimomentum

$$p(E) = \int_{e_1}^{E} dp. \tag{15}$$

If one expands p(E) in a neighborhood of infinity on the upper sheet:

$$ip(E) = \ln E - \sum_{s=0}^{\infty} I_s E^{-s}, \tag{16}$$

then

$$I_0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln c_n; \quad I_1 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n, \qquad I_2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(\frac{v_n^2}{2} + c_n^2\right); \quad \text{etc.} \tag{17}$$

The proof of these formulas (which are well-known in the periodic case) is based on the relation $i\,dp = \lim_{N\to\infty} \frac{1}{N} \frac{d\psi_N}{\psi_N}$. It is rather elementary, and we omit it.

2. "Integrable Case"

The Peierls functional (3) was initially defined for periodic operators. With the future aim of passing to the limit as $N \to \infty$, we modify the setting of the problem to a certain extent and define the Peierls functional for an arbitrary q zone operator L by the formula

$$\mathscr{H} = \frac{1}{2\pi} \oint_{e_1}^{\mu} E\,dp + \varkappa_2 I_2 - \varkappa_0 I_0 + g\mathscr{H}_1, \tag{18}$$

$$\mathscr{H}_1 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (\Phi_1(v_n) + \Phi_2(c_n^2)), \tag{19}$$

where μ, the so-called chemopotential, is found from the condition

$$\frac{1}{2\pi} \oint_{e_1}^{\mu} dp = \rho. \tag{20}$$

Here $\varkappa_2$, $\varkappa_0$, and ρ are real positive constants. The integral sign $\oint_{e_1}^{\mu}$ means that integration is taken on the lower sheet of Γ along a path joining μ^- and μ^+ and encircling the segment $[e_1, \mu]$ of the real line. If μ lies in a forbidden zone, then $\mu^- = \mu^+ = \mu$, while if μ lies in a permitted zone, $\mu^\pm$ are the preimages of μ on the upper and lower edges of the cut.

For operators of period N and having bounded coefficients $|v_n| < k_1$ and $|c_n| < k_2$, the functionals (3) and (18) differ by a quantity of order $O[(k_1 + k_2)/N]$.

As we already mentioned in the introduction, the minimum of the functional $\mathcal{H}_0$, equal to (18) for $g = 0$, was found in [12]. Let us pause and discuss the results of this paper in more detail.

The functional $\mathcal{H}_0$ depends smoothly on the endpoints of the zones as long as μ lies in a permitted zone. If μ lies in a forbidden zone, $e_{2k} \leqslant \mu \leqslant e_{2k+1}$, then $\mathcal{H}_0$ is a smooth function for those variations satisfying $\int_{e_1}^{e_{2k}} dp = \text{const}$. In both cases it was proven that the equations of the extremals $\delta\mathcal{H}_0 = 0$ have no solutions among the q-zone operators with $q > 1$.

The zone endpoints e_1, e_2, e_3, e_4, corresponding to a minimum of $\mathcal{H}_0$, are determined from the equations

$$\varkappa_2 = \frac{i}{2\pi} \oint_{e_1}^{\mu} \frac{dE}{\sqrt{R(E)}}; \tag{21}$$

$$0 = \oint_{e_1}^{\mu} (2E - s_1) \frac{dE}{\sqrt{R(E)}}; \tag{22}$$

$$\varkappa_0 = \frac{1}{2\pi i} \oint_{e_1}^{\mu} \left(E^2 - \frac{s_1}{2} E + \frac{s_2}{2} - \frac{s_1^2}{8}\right) \frac{dE}{\sqrt{R(E)}}; \tag{23}$$

$$\rho = \frac{1}{2\pi i} \oint_{e_1}^{\mu} (E + a) \frac{dE}{\sqrt{R(E)}}, \tag{24}$$

where $e_2 \leqslant \mu \leqslant e_3$ and the constant a is derived from the normalization condition (9).

Here $R(E) = \prod_{i=1}^{4} (E - e_i)$, $s_1 = \sum e_i$, $s_2 = \sum_{i<j} e_i e_j$.

Let us point once more to the fact that at the corresponding point the functional $\mathcal{H}_0$ is not a smooth function of all of the zone endpoints (because $e_2 \leqslant \mu \leqslant e_3$). If $\mathcal{H}_0(U, \rho)$ denotes the minimum of $\mathcal{H}_0$ among the one-zone operators for fixed ρ

$$\frac{1}{2\pi} \oint_{e_1}^{\mu} dp = U, \tag{25}$$

then the zone endpoints corresponding to this minimum are determined from the equations (21)-(23) and (25). At the same time, if $U > \rho$ then $e_1 \leqslant \mu < e_2$, while when $U < \rho$, $e_3 < \mu \leqslant e_4$.

The derivative $\partial\mathcal{H}_0(U, \rho)/\partial U$ has a discontinuity at the point $U = \rho$, but its right and left limits exist and are equal to $h(e_3) > 0$ and $h(e_2) < 0$, respectively. Here $h(e)$ is the function

$$h(e) = \frac{1}{2\pi i} \oint_{e_1}^{e} \frac{dE}{\sqrt{R(E)}} \left(\oint_{e_2}^{e_3} \frac{\sqrt{R(t)}\, dt}{E - t}\right). \tag{26}$$

Consequently, in a neighborhood of $U = \rho$, one has

$$\mathcal{H}_0(U, \rho) = \mathcal{H}_0(\rho) + h_1 |U - \rho| + h_2 (U - \rho) + O((U - \rho)^2), \quad h_1 + h_2 = h(e_3), \quad h_2 - h_1 = h(e_2). \tag{27}$$

We shall consider the functional $\mathcal{H}$ given by (18) for small g as a perturbation of $\mathcal{H}_0$. In order to apply the considerations usual in these situations, we need that the second variation of the functional $\delta^2 \mathcal{H}_0$ be strictly positive for those variations of the zone endpoints that satisfy the condition

$$\delta^2 \oint_{e_1}^{e_2} dp = 0. \tag{28}$$

(If this condition is not fulfilled, then the leading term of the increment is defined from (27) and is strictly positive.)

The functional $\mathcal{H}_0$ is defined on the stratified manifold $\hat{\mathscr{E}}_N = \bigcup_{q \leqslant N} \mathscr{E}_q$, where $\mathscr{E}_q$ is the set of collections $\hat{e}_q = (e_1, \ldots, e_{2q+2})$ of distinct points of the real line

$$\mathcal{H}_0 = \mathcal{H}_0(\hat{e}_q).$$

A point $\hat{e}_q \in \mathscr{E}_q \subset \hat{\mathscr{E}}_N$ is called an n-zone state of the system. Its neighborhood in $\hat{\mathscr{E}}_N$ consists not only of collections $\hat{e}_q$, which differ from $\hat{e}_q$ by small displacements of the endpoints e_i of the old zones of e_q, but also differ from collections $\hat{e}_q$, $q \leqslant q' \leqslant N$, which arise from $\hat{e}_q$ by the appearance of new slits in the old permitted zones (i.e., to the collection $(e_1, \ldots, e_{2q+1})$ one adds pairs $e_{2k-1} < e_- < e_+ < e_{2k}$).

Let $\hat{e}^*$ be the ground state of the system, defined by (21)-(24) for $e_2 \leqslant \mu \leqslant e_3$. Consider an arbitrary variation of this state (i.e., any collection $(e_1', \ldots, e_4', e_-^j, e_+^j)$, $0 \leqslant j \leqslant N-1$, where the new slits (e_-^j, e_+^j) where opened at the points e^j which lie in the old permitted zones $[e_1, e_2]$ or $[e_3, e_4]$), and suppose that it satisfies condition (28).

<u>THEOREM 1.</u> The second variation of the functional $\mathcal{H}_0$ is equal to

$$\delta^2\mathcal{H}_0(\hat{e}^*) = \sum_{k=1}^{4} \lambda_k (\delta e_k)^2 + \sum_{j=1}^{N-1} \lambda(e^j)(\delta e^j)^2, \tag{29}$$

where

$$\delta l_k = e_k' - e_k, \quad \delta e^j = e_+^j - e_-^j,$$

$$\lambda(e) = \frac{1}{8\pi i} \oint_{e_1}^{\mu} \frac{(e+a)\,dE}{\sqrt{R}\,(E-e)} \geqslant \Lambda > 0, \tag{30}$$

$$\lambda_k = 2\lambda(e_k), \tag{31}$$

and a is determined by the normalization condition (9):

$$\oint_{e_2}^{e_3} \frac{E+a}{\sqrt{R}}\, dE = 0. \tag{32}$$

What the theorem asserts is that the second variation is "diagonal" and positive at the point of minimum.

<u>Proof.</u> By the definition, the differential dp corresponding to the collection $\hat{e}_1 = (e_1', \ldots, e_4')$ and e_-^j, e_+^j has the form

$$i\,dp(\hat{e}_1', e_\pm^j) = \frac{P_N(E)\,dE}{\sqrt{R(E)} \prod_{j=1}^{N-1} \sqrt{(E-e_-^j)(E-e_+^j)}}. \tag{33}$$

Here $P_N(E)$ is a polynomial of degree N with leading coefficient 1, which is uniquely determined by the conditions

$$\oint_{e_2}^{e_3} dp = 0, \tag{34}$$

$$\oint_{e_-^j}^{e_+^j} dp = 0. \tag{35}$$

It follows from (35) that

$$i\,dp(\hat{e}_1, e_\pm^j = e^j) = \frac{E+a}{\sqrt{R(E)}}\, dE,$$

where a is determined from (32).

Differentiating (33), we see that $\delta^2 dp$ has the form

$$i\delta^2 dp\,|_{\hat{e}'_1=\hat{e}^*,\, e^j_\pm = e^j} = \frac{\tilde{P}_{2N+6}(E)\, dE}{\sqrt{R}\, R_1^{3'} \prod_{j=1}^{N-1} (E-e^j)^2}, \tag{36}$$

where $\tilde{P}_{2N+6}$ is a polynomial of degree $2N + 6$.

Condition (35) implies that $\delta^2 dp$ has no residues at the points $E = e^j$. Combined with (28), this shows that the integrals of $\delta^2 dp$ along all cycles of Γ vanish. Thus, the meromorphic function

$$\delta^2 p = \int_{e_1}^{E} \delta^2\, dp,$$

is correctly defined on Γ, and has singularities only at the points e_i and e^j, namely, poles of order three and one, respectively. Consequently, it can be uniquely represented in the form

$$i\delta^2 p = \frac{\tilde{P}_2}{\sqrt{R}} + \frac{1}{\sqrt{R}}\Big(\sum_{i=1}^{4} \frac{\tilde{A}_i}{(E-e_i)} + \sum_{j=1}^{N-1} \frac{\tilde{B}_j}{(E-e^j)}\Big). \tag{37}$$

Differentiating (37) with respect to dE and comparing with the coefficients of the leading singular terms in e_i and e^j in (36), we get

$$\tilde{A}_i = -\frac{1}{2}(e_i + a)(\delta e_i)^2, \tag{38}$$

$$\tilde{B}_i = -\frac{1}{4}(e^j + a)(\delta e^j)^2. \tag{39}$$

The mixed derivatives, i.e., the expressions containing $\delta e_i \delta e^j$ and so on, are grouped in the coefficients of the second degree polynomial $\tilde{P}_2$. Their explicit form is not needed because their contribution to $\delta^2 \mathcal{H}_0$ is zero. Indeed,

$$\delta^2 \mathcal{H}_0 = -\frac{1}{2\pi} \oint_{e_1}^{\mu} \delta^2 p\, dE + \varkappa_2 \delta^2 I_2 - \varkappa_0 \delta^2 I_0. \tag{40}$$

Expanding (37) in a neighborhood of infinity and comparing the result with the expansion

$$i\delta^2 p = -\sum_{k=0}^{\infty} \delta^2 I_k E^{-k},$$

that follows from the definition (16) of I_k, we see that the coefficients of the polynomial $\tilde{P}_2$ figuring in (37) are

$$\begin{gathered} P_2 = \tilde{l}_0 E^2 + \tilde{l}_1 E + \tilde{l}_2, \\ \tilde{l}_0 = -\delta^2 I_0, \quad \tilde{l}_1 = -\delta^2 I_1 + \frac{s_1}{2}\delta^2 I_0, \\ \tilde{l}_2 = -\delta^2 I_2 + \frac{s_1}{2}\delta^2 I_1 + \delta^2 I_0\Big(\frac{s_1}{8} - \frac{s_2}{2}\Big). \end{gathered} \tag{41}$$

Substituting (41) in $\tilde{P}_2$, the "self-consistency" equations (21)-(23) yield

$$-\frac{1}{2\pi i} \oint_{e_1}^{\mu} \frac{\tilde{P}_2}{\sqrt{R}}\, dE + \varkappa_2 \delta^2 I_2 - \varkappa_0 \delta^2 I_0 = 0. \tag{42}$$

(This relation is by no means surprising, because equations (21)-(23) were obtained from similar relations for δp). Therefore,

$$\delta^2 \mathcal{H}_0(\hat{e}_1^*) = -\frac{1}{2\pi i} \oint_{e_1}^{\mu} \frac{dE}{\sqrt{R}} \Big(\sum_{i=1}^{4} \frac{\tilde{A}_i}{(E-e_i)} + \sum_{j=1}^{N-1} \frac{\tilde{B}_j}{E-e^j}\Big),$$

which, in turn, with the aid of (38) and (39), yields (29).

In order to find the coefficients λ_k and $\lambda(e)$ in a more explicit form and establish their positiveness, we use the elliptic parametrization of Γ (the details of which can be found in [12]).

The function

$$z=\int\limits_{e_1}^{E}\frac{dE'}{\sqrt{R(E')}} \tag{43}$$

maps Γ onto a torus having periods 2ω and $2\omega'$:

$$\omega=\int\limits_{e_1}^{e_2}\frac{dE}{\sqrt{R}};\qquad \omega'=\int\limits_{e_2}^{e_3}\frac{dE}{\sqrt{R}}.$$

(Here the integrals are taken on the lower sheet of Γ on paths over the real axis.)

The formula

$$E(z)=\zeta(z+z_0)-\zeta(z-z_0)+h, \tag{44}$$

provides the inversion of the elliptic integral (43); here $\zeta(z)=\zeta(z\mid\omega,\omega')$ is Weierstrass' ζ function. (The necessary information concerning elliptic functions can be found in [19].)

Parameters ω, ω', z_0, and h replace $e_1, \ldots, e_4$, and relative to them equations (21)-(24) become significantly simpler; namely (see [12]),

$$z_0=(\rho-1)\,\omega', \tag{45}$$

$$\varkappa=\frac{i}{\pi}\,\omega, \tag{46}$$

$$\pi i+2\eta z_0=(2\zeta(2z_0)+h)\,\omega, \tag{47}$$

$$\varkappa_0=\frac{2i}{\pi}(\eta+\wp(2z_0)\,\omega). \tag{48}$$

Relative to the new parameters, the following integral becomes

$$\begin{aligned}\oint\limits_{e_1}^{e_2}\frac{dE}{\sqrt{R}\,(E-e)}&=\int\limits_{-\omega}^{\omega}dz\,\frac{\zeta(z+\gamma)-\zeta(z-\gamma)-\zeta(z_0+\gamma)+\zeta(z_0-\gamma)}{\wp(\gamma+z_0)-\wp(\gamma-z_0)}=\\&=\frac{1}{\wp(\gamma+z_0)-\wp(\gamma-z_0)}\left[4\eta\gamma-2\omega(\zeta(z_0+\gamma)-\zeta(z_0-\gamma))\right].\end{aligned} \tag{49}$$

where γ denotes the image of e under the map (43), i.e., $e=\zeta(\gamma+z_0)-\zeta(\gamma-z_0)+h$.

If $e_1 \leqslant e \leqslant e_2$, then γ is purely imaginary and lies on the segment $[0, \omega]$. When $e_3 \leqslant e \leqslant e_4$, $\mathrm{Re}\,\gamma=\omega'$ and γ belongs to the segment $[\omega', \omega'+\omega]$.

One can easily see that the numerator of (49) does not vanish inside the intervals $(0, \omega)$ and $(\omega', \omega+\omega')$, because at their endpoints it takes the values 0 and $4\pi i$, respectively. Thus, had such a zero existed, the number of zeros of the derivative of this numerator would be greater than 4. However, this derivative is just $4\eta+2\omega(\wp(z_0+\gamma)+\wp(z_0-\gamma))$ and has four zeros.

Let us calculate the expression $e+a=\zeta(\gamma+z_0)-\zeta(\gamma-z_0)+h+a$.

The normalizing condition (32) yields

$$\int\limits_{\omega}^{\omega+\omega'}[\zeta(z+z_0)-\zeta(z-z_0)+h+a]\,dz=(h+a)\,\omega'+2\eta' z_0=0.$$

whence, taking advantage of (45), we get

$$e+a=\zeta(\gamma+z_0)-\zeta(\gamma-z_0)+2\eta'(1-\rho). \tag{50}$$

This shows that the "eigenvalue" $\lambda(\gamma)$ of the second variation $\delta^2\mathcal{H}_0$, corresponding to the zone newly open at the point $e=E(\gamma)$, is equal to

$$\lambda(\gamma)=\frac{1}{8\pi i}[\zeta(\gamma+z_0)-\zeta(\gamma-z_0)+2\eta'(1-\rho)]\;\frac{4\eta\gamma-2\omega(\zeta(z_0+\gamma)-\zeta(z_0-\gamma))}{\wp(\gamma+z_0)-\wp(\gamma-z_0)}. \tag{51}$$

From (30) it follows that

$$\lambda_k=2\lim_{\gamma\to\omega_k}\lambda(\gamma), \tag{52}$$

where $\omega_1 = 0$, $\omega_2 = \omega$, $\omega_3 = \omega + \omega'$, and $\omega_4 = \omega'$.

We have

$$\lambda_1=\frac{1}{2\pi i}(\eta'(1-\rho)+\zeta(z_0))\frac{2\eta+\omega\wp(z_0)}{\wp'(z_0)}, \tag{53}$$

$$\lambda_2=\frac{1}{2\pi i}(\eta+\eta'(1-\rho))\frac{2\eta+\omega\wp(z_0+\omega)}{\wp'(z_0+\omega)}, \tag{54}$$

$$\lambda_3=\frac{1}{2\pi i}(\eta+\eta'(2-\rho))-\frac{2\eta+\omega\wp(z_0+\omega+\omega')}{\wp'(z_0+\omega+\omega')}, \tag{55}$$

$$\lambda_4=\frac{1}{2\pi i}\eta'(2-\rho)\frac{2\eta+\omega\wp(z_0+\omega')}{\wp'(z_0+\omega)_l}. \tag{56}$$

The positivity of λ_3, λ_4, $\lambda(\gamma)$, and $\operatorname{Re}\gamma = \omega'$ is plain from their original expressions (30) and (31). In fact, in this case the integration contour can be contracted to the cut between e_1 and e_2, where, for our choice of direction, $1/i\sqrt{R} < 0$. Now (32) implies that $e + a > 0$ for $e > e_3$. Finally, we see that the expression under the integral sign is positive and $\lambda_3 > 0$, $\lambda_4 > 0$, and $\lambda(e) > 0$ (or, which is the same, $\lambda(\gamma) > 0$ and $\operatorname{Re}\gamma = \omega' > 0$).

We proved earlier that $\lambda(\gamma)$ does not vanish in the interval $(0, \omega)$. As (52) shows, its sign can be determined if we find the sign of λ_1.

The expression $2\eta'(1-\rho) + 2\zeta(z_0) = e_1 + a < 0$ is negative. Since $z_0 = (\rho-1)\omega'$ lies on the segment $(-\omega', 0)$, $\wp'(z_0) > 0$. When one goes around the boundary of the rectangle with vertices $\{0, \omega, \omega'+\omega, \omega'\}$, $\wp(z)$ monotonically increases from $-\infty$ to $+\infty$; moreover, since

$$\int_{\omega'-\omega}^{\omega'+\omega}\wp(z)\,dz=-2\eta$$

$\wp(z)$ takes the value $-\eta/\omega$ on the wedge $[\omega+\omega', \omega']$. Consequently,

$$\frac{\eta}{\omega}+\wp(z_0)>0.$$

As $\operatorname{Im}\omega < 0$, we see that λ_1, λ_2, and $\lambda(\gamma)$ (for $\operatorname{Re}\gamma = 0$) are strictly positive, too. To complete the proof of the theorem, we need only remark that $\lambda(\gamma)$ is bounded from above by a positive constant Λ, because its limits at the boundaries of the intervals $(0, \omega)$ and $(\omega + \omega', \omega')$ exist and are positive.

3. Ground State of the General Peierls Model

As we showed in the first section, the set of q-zone operators is isomorphic to the product of $\mathscr{E}_q$ and a q-dimensional torus T^q. For general functions Φ_1 and Φ_2 (which in the sequel will be assumed to be analytic), the functional

$$\mathscr{H}_1=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(\Phi_1(v_n)+\Phi_2(c_n^2)) \tag{57}$$

is a function $\mathscr{H}_1=\mathscr{H}_1(\hat{e}_q, Z)$ of the collections $\hat{e}_q = (e_1, \ldots, e_{2q+2})$, $Z = (z_1, \ldots, z_q)$.

Let us investigate the character of the dependence of $\mathscr{H}_1$ upon Z.

Formula (12) associates to each collection $\hat{e}_q$ the q-dimensional vector $U(\hat{e}_q)$ with coordinates $0 < U_1 < \ldots < U_q < 1$. We call the collection $\hat{e}_q$ nonresonant if there is no integer vector $r = (r_1, \ldots, r_q)$ such that

$$\langle r, U(\hat{e}_q)\rangle = r_0, \tag{58}$$

where r_0 is an integer. Given a nonresonant collection $\hat{e}_q$, we shall denote by $R(\hat{e}_q)$ the group of all $r \in R(\hat{e}_q)$ such that (58) holds for some value of r_0.

THEOREM 2. The functional $\mathcal{H}_1$ is equal to

$$\mathcal{H}_1(\hat{e}_q, Z) = \sum_{r \in R(e_q)} \mathcal{F}_r \exp(2\pi i \langle r, Z\rangle + \pi i r_0), \tag{59}$$

where

$$\mathcal{F}_m = \int_0^1 \dots \int_0^1 \mathcal{F}(z_1, \dots, z_q) \exp\langle -2\pi i m, Z\rangle\, dz_1 \dots dz_q \tag{60}$$

are the Fourier coefficients of the function

$$\mathcal{F}(z) = \Phi_1(v(z)) + \Phi_2(c^2(z)),$$
$$c^2(z) = \frac{\theta(z - U(\hat{e}_q))\,\theta(z + U(\hat{e}_q))}{\theta^2(z)} e^{-2I_0},$$
$$v(z) = \frac{d}{dt} \ln \frac{\theta(z + Vt)}{\theta(z + U(\hat{e}_q) + Vt)}\Big|_{t=0} + a_1 + \frac{s_1}{2}.$$

Formulas (12) and (13) imply that

$$\Phi_1(v_n) + \Phi_2(c_n^2) = \mathcal{F}\left(nU - \frac{1}{2}U + Z\right).$$

The limit of

$$\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{F}\left(nU - \frac{1}{2}U + Z\right)$$

as $N \to \infty$ can be easily found using the Fourier expansion of $\mathcal{F}(z)$.

COROLLARY 1. For nonresonant collections $\hat{e}_q$ the functional $\mathcal{H}_1(\hat{e}_q, Z)$ does not depend upon Z and equals

$$\widetilde{\mathcal{H}}_1(\hat{e}_q) = \int \dots \int \mathcal{F}(z)\, dz. \tag{61}$$

COROLLARY 2. If the frequencies U_k are not all rational, then the corresponding level of the functional $\mathcal{H}_1$ is degenerated.

Proof. Let at least one of the U_k be irrational. Then all the $r \in R(\hat{e}_q)$ are dependent. That is to say, one can find a vector Z_* such that $\langle r, Z_*\rangle = 0$. But then (59) implies that $\mathcal{H}_1(\hat{e}_q, Z + tZ_*)$ does not depend upon t.

Formula (61) determines the "continuous part" of the functional $\mathcal{H}_1$, which, given general Φ_1 and Φ_2, is discontinuous at all resonant collections. If $|r|$ is the minimal order of resonance, $|r| = |r_1| + \dots + |r_q|$, $r \in R(\hat{e}_q)$, then the $|\mathcal{H}_1(\hat{e}_q, Z) - \widetilde{\mathcal{H}}_1(\hat{e}_q)|$ has order $|\mathcal{F}_r|$ and decreases when $|r|$ grows as $e^{-|r|A}$.

Denote by $\mathcal{H}_1^*(\hat{e}_q)$ the function on $\mathcal{E}_q$ given by

$$\mathcal{H}_1^*(\hat{e}_q) = \min_{Z \in T^q} \mathcal{H}_1(\hat{e}_q, Z).$$

By Corollary 1, the latter equals $\widetilde{\mathcal{H}}_1(\hat{e}_q)$ almost everywhere. By virtue of (59), $\int_{T^q} \mathcal{H}_1(\hat{e}_q, Z)\,dZ = \widetilde{\mathcal{H}}(\hat{e}_q)$, and hence $\mathcal{H}_1^*(\hat{e}_q) \leqslant \widetilde{\mathcal{H}}(\hat{e}_q)$.

Consider an arbitrary variation in $\mathcal{E}_N$, $(e_1', \dots, e_4', e_-^j, e_+^j)$, of the one-zone state $\hat{e}_1 = (e_1, \dots, e_4)$.

THEOREM 3. The variation of $\mathcal{H}_1$ is equal to

$$|\delta\mathcal{H}_1| \leqslant |\mathcal{H}_1^*(\hat{e}_1') - \mathcal{H}_1(\hat{e}_1)| + \sum_{j=1}^{N-1} \varkappa(e^j)\,\delta e^j, \tag{62}$$

Here $\varkappa(e) = 0$ whenever

$$\frac{1}{2\pi}\oint_{e_1}^{e^j} dp \neq r_1 U + r_0 = \frac{r_1}{2\pi}\oint_{e_1}^{e_2} dp + r_0, \tag{63}$$

where r_1 and r_0 are integers; in the contrary case, $0 \leqslant \kappa(e) \leqslant Ce^{-r_1 A_1}$.

Proof. Let Ω be the normalized holomorphic differential corresponding to the surface Γ_1 of the function $\sqrt{R(E)}$

$$\Omega = \alpha \frac{dE}{\sqrt{R}}, \quad -\oint_{e_2}^{e_3} \Omega = 1, \quad B_{11} = \oint_{e_1}^{e_2} \Omega.$$

The normalized holomorphic differentials Ω_1 and Ω_2 on the surface Γ_2 corresponding to the collection $(e_1, \ldots, e_4; e_-, e_+)$, have the form

$$\Omega_1 = \alpha_1 \frac{(E-s)\,dE}{\sqrt{R(E)}\sqrt{(E-e_-)(E-e_+)}},$$
$$\alpha_1 = \alpha + O((\delta e)^2), \quad s = e + O((\delta e)^2), \; \delta e = e_+ - e_-,$$
$$2e = e_+ + e_-, \quad \Omega_2 = \frac{(\alpha_2(E-e) + \alpha_3)\,dE}{\sqrt{R}\sqrt{(E-e_-)(E-e_+)}},$$

where

$$\alpha_3 = -\frac{1}{2\pi}\sqrt{R(e)} + O((\delta e)^2),$$
$$\frac{\alpha_2}{\alpha} = \frac{1}{2\pi}\oint_{e_2}^{e_3} \frac{\sqrt{R(e)}}{\sqrt{R(E)}(E-e)}\,dE + O((\delta e)^2). \tag{64}$$

This shows that the matrix of b periods of the curve Γ_2 is

$$B_{11}' = B_{11} + O((\delta e)^2), \quad B_{12} = \frac{\alpha_2}{\alpha}B_{11} - \frac{\sqrt{R(e)}}{2\pi}\oint_{e_1}^{e_2} \frac{dE}{\sqrt{R(E)}(E-e)} + O((\delta e)^2),$$
$$\exp(\pi i B_{22}) = \delta e O(1). \tag{65}$$

Notice that B_{12} (which may be written down explicitly using the formulas in the previous section) is bounded for all e by a constant: $|B_{12}| < B_0$.

For the corresponding θ function we obtain

$$\theta(z_1, z_2) = \theta(z_1) + \sum_{m=-\infty}^{\infty} \exp(2\pi i(mz_1 + z_2) + \pi i(2B_{12}m + B_{11}m^2)) +$$
$$+ \exp(2\pi i(mz_1 - z_2) + \pi i(B_{11}m^2 - 2B_{12}m)) + O((\delta e)^2).$$

whence

$$\mathcal{F}(z_1, z_2) = \mathcal{F}(z_1) + \delta e\,(F_+(z_1)e^{2\pi i z_2} + F_-(z_1)e^{-2\pi i z_2}) + O((\delta e)^2). \tag{66}$$

Here $F_\pm(z_1)$ are periodic functions of z_1 which depend analytically upon $B_{12}(e)$. Inserting (66) in (59), we find that

$$|\delta\mathcal{H}_1^*| \leqslant |\mathcal{H}_1^*(\hat{e}_1') - \mathcal{H}_1^*(\hat{e}_1)| + \sum_{j=1}^{N-1} \delta e^j \left(\sum_{k=1}^{\infty} |F_{kr(e)}^+| + |F_{kr(e)}^-|\right) + O((\delta e)^2), \tag{67}$$

where $r(e)$ is an integer such that

$$\frac{1}{2\pi}\oint_{e_1}^{e} dp \equiv r(e)\,U \pmod{1},$$

and $F_m^\pm$ are the Fourier coefficients of $F_\pm(z)$. The theorem is proved.

Summing up the results obtained, we are led to the following basic statement:

THEOREM 4. Let Φ_1 and Φ_2 be positive on the real axis and let ρ satisfy the condition

$$\left|\rho - \frac{m}{n}\right| > \frac{\alpha}{n^2}$$

for $n > n_0$, where n_0 and α are constants. Then there exists a $g_\rho > 0$ such that for $g < g_\rho$ the energy of the ground state $\mathcal{H}^*(\rho) = \min(\mathcal{H}_0 + g\mathcal{H}_1)$ satisfies the inequality

$$|\mathcal{H}^*(\rho) - \mathcal{H}^*(\hat{e}_1^*)| < \frac{2g^2C}{\Lambda},$$

where $\hat{e}_1^*$ is the ground state of the unperturbed functional $\mathcal{H}_0$. Moreover:

1. The spectrum of the operator L corresponding to the ground state of the system has slits at the points e^r determined from the conditions

$$\frac{1}{2\pi}\int_{e_1}^{e^r} dp \equiv r\rho \pmod 1.$$

2. The width of the slit has order

$$|\delta e^r| \leqslant \frac{gCe^{-rA_1}}{2\Lambda} + O(g^2).$$

3. The ground state is given by formulas (13), where all the frequencies are of the form $U_k = r_k\rho + r_k'$, with r_k and r_k' integers.

4. If ρ is irrational, then the ground state is degenerate.

Proof. Denote $W_h \subset \hat{\mathscr{E}}_N$ the neighborhood of $\hat{e}_1^*$ consisting of those collections $(e_1', \ldots, e_4'; e_-^j, e_+^j)$ that satisfy

$$\sum_{i=1}^{t} |\delta e_i| + \sum_{j=1}^{N-1} |\delta e^j| = \varepsilon \leqslant h, \tag{68}$$

and let $\overline{W}_h$ be the complement of this neighborhood.

Since $\mathcal{H}_0$ has no other extremals aside from $\hat{e}_1^*$, and since Φ_1 and Φ_2 are positive, for sufficiently small h we have

$$\min_{\overline{W}_h} \mathcal{H}^* \geqslant \min_{\overline{W}_h} \mathcal{H}_0 \geqslant \mathcal{H}_0(\rho) + \frac{\Lambda}{2}h^2.$$

If $g\mathcal{H}_1^*(\hat{e}_1^*) \leqslant \frac{\Lambda}{2}h^2$, then the minimum of $\mathcal{H}$ $\mathcal{H}^*(\rho) \leqslant \mathcal{H}_0(\rho) + g\mathcal{H}_1(\hat{e}_1^*)$ is attained in W_h.

Let ρ satisfy the conditions of the theorem. Then $\mathcal{H}_1^*$ is differentiable at $\hat{e}_1^*$ with respect to all the variations, including the variations that modify the period $\frac{1}{2\pi}\int_{e_1}^{e_2} dp = U$.

Indeed, if $|U - \rho| < \varepsilon$, $U = m/n$, then $n > \sqrt{\frac{\alpha}{\varepsilon}}$, whence

$$|\mathcal{H}_1^*(\acute{e}_1) - \tilde{\mathcal{H}}_1(\acute{e}_1)| < C_1 e^{-\sqrt{\alpha/\varepsilon}A_1}$$

and $\mathcal{H}_1^*(\acute{e}_1)$ has a derivative with respect to U, equal to the derivative of $\tilde{\mathcal{H}}_1(\hat{e}_1)$. Suppose that g satisfies the condition

$$g\frac{\partial\mathcal{H}_1}{\partial U}(\hat{e}_1^*) \leqslant \min(h_1 \pm h_2),$$

where h_1 and h_2 are defined by (27). Then (27) implies that min $\mathcal{H}$ is attained for $U = \rho$.

Let $(\hat{e}_1; e_\pm^j)$ be the ground state of the system. Since it belongs to W_h, the results of Theorems 1 and 3 apply, and we have

$$\mathscr{H}_0 + g\mathscr{H}_1 \geqslant \mathscr{H}_0(\rho) + \frac{\Lambda}{2}\varepsilon^2 + g\mathscr{H}_1^*(\hat{e}_1^*) - gC\varepsilon \geqslant \mathscr{H}_0(\rho) + g\mathscr{H}_1^*(\hat{e}_1^*) - \frac{2g^2C}{\Lambda},$$

where ε is defined in (68). The fact that $\delta^2\mathscr{H}_0$ is "diagonal" allows us to obtain a sharper estimate of the width of each new slit:

$$\delta e^j \sim g\frac{\varkappa(e^j)}{2\lambda(e^j)} \leqslant g\frac{\varkappa(e^j)}{\Lambda}.$$

At the same time, all the assertions of the theorem, except for the last one, are proven. The degeneracy of the ground state for irrational ρ is stated in Corollary 2 to Theorem 2.

We should mention that the zone structure of the Schrödinger difference operator corresponding to the ground state of the Peierls model (which, as we have shown, is quasiperiodic with two periods ρ and 1) is entirely similar to the structure of the spectrum of the Sturm–Liouville operator, with an almost periodic potential that was obtained in [20] (for the applications of the results of [20] to the continual approximations to the Peierls problem, see [15]).

In conclusion, the author considers it his duty to express his gratitude to I. E. Dzyaloshinskii and S. A. Brazovskii for fruitful discussions and help in formulating the problem.

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SOME ALGEBRAIC STRUCTURES CONNECTED WITH THE YANG—BAXTER EQUATION

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One of the strongest methods of investigating the exactly solvable models of quantum and statistical physics is the quantum inverse problem method (QIPM; see the review papers [1-3]). The problem of enumerating the discrete quantum systems that can be solved by the QIPM reduces to the problem of enumerating the operator-valued functions L(u) that satisfy the relation

$$R(u-v)\, L'(u)\, L''(v) = L''(v)\, L'(u)\, R(u-v) \tag{1}$$

for a fixed solution R(u) of the so-called quantum Yang—Baxter equation

$$R_{12}(u-v)\, R_{13}(u) R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v). \tag{2}$$

Here we use the notation $L' = L\otimes 1$, $L'' = 1\otimes L$ (see [1, 3]). More detailed information concerning equations (1) and (2) and the notation used here can be found in the review papers to which we have already referred.

In the classical case, (1) is replaced by the equation

$$\{L'(u), L''(v)\} = [r(u-v), L'(u)\, L''(v)]_-, \tag{3}$$

while (2) becomes the classical Yang—Baxter equation

$$[r_{12}(u-v),\ r_{13}(u)]_- + [r_{12}(u-v),\ r_{23}(v)]_- + [r_{13}(u),\ r_{23}(v)]_- = 0. \tag{4}$$

Here we use {,} to denote the Poisson bracket, and $[A, B]_- = AB - BA$ stands for the commutator of the matrices A and B. We shall also make use of the notation $[A, B]_+ = AB + BA$ for the anticommutator.

The problem of enumerating the solutions to equations (1) and (3) has received little attention. This contrasts with the intense study of both the quantum and classical Yang—Baxter equations, which has led to a number of successes. These have revealed, in particular, the deep relationship between the Yang—Baxter equation, the theory of Lie groups [4, 5], and algebraic geometry [6, 7]. However, important results were obtained in [8, 9], where solutions to (1) and (3) corresponding to lattice versions of the nonlinear Schrödinger and sine—Gordon equations were found.

The present paper is devoted to a study of equations (1) and (3) in the case when R(u) and r(u) are, respectively, the simplest solution to equation (1), found by R. Baxter [10], and its classical analog [11]. During our investigation it turned out that it is necessary to bring into the picture new algebraic structures, namely, the quadratic algebras of Poisson brackets and the quadratic generalization of the universal enveloping algebra of a Lie algebra. The theory of these mathematical objects is surprisingly reminiscent of the theory of

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