Uspekhi Mat. Nauk 36:2 (1981), 11–80 Russian Math. Surveys 36:2 (1981), 11–92

Theta functions and non-linear equations

B.A. Dubrovin

CONTENTS

where

(5.4.23′) $$[B, V] = \Omega, \quad A = I^2, \quad B = I.$$

The systems (5.4.23) have been explicitly integrated by the present author in [42]. They all have a commutative representation of the form

(5.4.24) $$\left[\frac{d}{dt} - [B, V] + zB, \quad zA - [A, V]\right] = 0$$

on matrices depending on a superfluous parameter z; consequently, their solutions can be expressed in terms of θ-functions of Riemann surfaces Γ of the form

(5.4.25) $$\det(zA - [A, V] - w\cdot 1) = 0.$$

The set of these surfaces Γ is the same as that of all plane non-singular algebraic curves (in $\mathbf{C}P^2$) of degree n (their genus is $(n-1)(n-2)/2$) and their degeneracies. Explicit formulae for a general solution of (5.4.23) can be obtained from [42] and have the form $V = (V_{ij})$, where

(5.4.26) $$v_{ij} = \pm \frac{\lambda_i}{\lambda_j} \frac{\theta(A(P_i) - A(P_j) + tU + z_0)}{\theta(tU + z_0)\,\varepsilon(P_i, P_j)} \quad (i \neq j)$$

(5.4.27) $$[\varepsilon(P, Q]^{-1} = \frac{\sqrt{\partial_{U(P)}\theta[\nu](0)\,\partial_{U(Q)}\theta[\nu](0)}}{\theta[\nu](A(P) - A(Q))},$$

(5.4.28) $$\lambda_i = \lambda_i^0 \exp\{t \sum_{k \neq i} c_i^k b_k\},$$

(5.4.28′) $$c_i^k = -\frac{d}{dP} \log \varepsilon(P, P_i)|_{P=P_k}.$$

Here $\lambda_i^0, \ldots, \lambda_n^0$ are arbitrary non-zero constants; the θ-function is constructed from a curve of the form (5.4.25); $P_1, \ldots, P_n$ are the points at infinity on this curve, where $w/z \to a_i$ as $P \to P_i$; the vector U has the form

(5.4.29) $$U = \sum_{j=1}^{n} b_j U(P_j),$$

where $U(P)$ is a period vector of differentials Ω_P with a double pole at P; z is an arbitrary vector; and finally, v is any non-degenerate odd half-period (that is, grad $\theta[v](0) \neq 0$).

Appendix

THE PERIODIC NON-ABELIAN TODA CHAIN AND ITS TWO-DIMENSIONAL GENERALIZATION

I.M. Krichever

The equations of a non-Abelian Toda chain were suggested by Polyakov, who found polynomial integrals for them. These equations, which have the form

(1) $$\partial_t(\partial_t g_n \cdot g_n^{-1}) = g_{n-1} g_n^{-1} - g_n g_{n+1}^{-1}, \quad \partial_t = \frac{\partial}{\partial t},$$

where the g_n are matrices of order l, admit a commutative representation of Lax type $\partial_t L = [P, L]$. Here

(2) $$L\psi_n = g_n g_{n+1}^{-1}\psi_{n+1} - \dot{g}_n g_n^{-1}\psi_n + \psi_{n-1}, \qquad \dot{g}_n = \partial_t g_n,$$

(3) $$P\psi_n = \frac{1}{2}(g_n g_{n+1}^{-1}\psi_{n+1} + \dot{g}_n g_n^{-1}\psi_n - \psi_{n-1}).$$

Using this representation, explicit expressions in terms of Riemann θ-functions have been obtained in the present survey for periodic solutions, $g_{n+N} = g_n$, of the equations (1).

In contrast to the continuous case when the algebraic-geometric constructions give only the so-called finite-zone solutions, in a difference version all the periodic solutions of the Lax equations turn out to be algebraic-geometric. This is connected with the fact that shift by a period, which commutes with l, is a difference operator.

In [46] the present author obtained a classification of commuting difference operators (see also [47]). In the same paper a construction of quasiperiodic solutions of difference operators of Zakharov-Shabat type and Lax type was proposed. Apart from general solutions of similar type, the non-abelian Toda chain has separatrix families of solutions or, in the terminology of [14], finite-zone solutions of rank $l > 1$. Their dimension is more than half the dimension of the phase space.

First we recall the scheme of integration ([15], [46]) of the "ordinary" Toda chain

(4) $$\begin{cases} \dot{v}_n = c_{n+1} - c_n, \\ \dot{c}_n = c_n(v_n - v_{n-1}). \end{cases}$$

Let R be a hyperelliptic Riemann surface of genus g of the form

(5) $$w^2 = \prod_{i=1}^{2g+2}(z - z_i);$$

P^+ and P^- the points of R of the form $P^{\pm} = (\infty, \pm)$. To integrate the system (4) we introduce the Baker-Akhiezer function $\psi(n, t, P)$ which is, meromorphic on R everywhere except for at P^+ and P^-, where it has g poles and as $P \to P^{\pm}$, an asymptotic expansion of the form

(6) $$\psi(n, t, P)\,|_{P\to P^{\pm}} = i^n \lambda_n^{\pm 1} z^{\pm n}(1 + \xi_1^{\pm}(n, t) z^{-1} + \ldots)\exp\left(\mp \frac{tz}{2}\right).$$

For this function there are difference operators $L = (L^{nm})$ and $A = (A^{nm})$ such that

(7) $$\frac{\partial\psi}{\partial t} = A\psi, \quad L\psi = z\psi.$$

These operators have the form

(8) $$L^{nm}=-i\sqrt{c_{n+1}}\,\delta_{n,m-1}+v_n\delta_{n,m}+i\sqrt{c_n}\,\delta_{n,m+1},$$

(9) $$A^{nm}=\frac{i}{2}\sqrt{c_{n+1}}\,\delta_{n,m-1}+w_n\delta_{n,m}+\frac{i}{2}\sqrt{c_n}\,\delta_{n,m+1}.$$

Here $w_n-w_{n-1}=\frac{1}{2}(v_n-v_{n-1})-\frac{1}{2}(\log c_n)$, and

(9′) $$\sqrt{c_n}=\lambda_{n-1}/\lambda_n,$$

(9″) $$v_n=\xi_1^+(n+1,t)-\xi_1^+(n,t).$$

The compatibility condition for (7) coincides with the equations of the Toda chain. Expressing the Baker-Akhiezer function (6) in terms of θ-functions of R and calculating the coefficients λ_n and $\xi_1^+(n,t)$, we obtain an explicit form of the solutions of the Toda chain:

(10) $$v_n=\frac{d}{dt}\log\frac{\theta((n+1)U+tV+z_0)}{\theta(nV+tV+z_0)},$$

(11) $$c_n=\frac{\theta((n+1)U+Vt+z_0)\,\theta((n-1)U+Vt+z_0)}{\theta^2(nU+Vt+z_0)}.$$

Here z_0 is an arbitrary vector; the vectors $U=(U_j)$ and $V=(V_j)$ are determined as follows:

(12) $$U_j=\int_{P^-}^{P^+}\omega_j$$

(ω_1, ..., ω_g is a canonical basis of holomorphic differentials on R),

(13) $$2V_j=\oint_{b_j}\Omega_{P^+}+\oint_{b_j}\Omega_{P^-},$$

where Ω_{P^+} and Ω_{P^-} are normalized differentials of the second kind with a double pole at P^+ and P^-, respectively.

Periodic solutions of the Toda chain with period N are distinguished in our system as follows: R must have the form

(14) $$w^2=(P_N(z)+1)(P_N(z)-1),$$

where $P_N(z)$ is a polynomial. We emphasize that all periodic solutions of the Toda chain are obtained in this way.

1. Thus, we consider periodic solutions of (1). The restriction of L to the space of eigenfunctions of the shift operator by a period, that is, $\psi_{n+N}=w\psi_n$, where ψ_n is an l-dimensional vector, is a finite-dimensional linear operator. Its matrix has the form

(15) $$\tilde{L}=\begin{pmatrix} b_{N-1} & 1 & 0 & \dots & 0 & wa_{N-1}\\ a_{N-2} & b_{N-2} & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & a_1 & b_1 & 1\\ w^{-1} & 0 & \dots & 0 & a_0 & b_0 \end{pmatrix},$$

where the block $(l\times l)$-elements are $b_n=-\dot{g}_ng_n^{-1}$, $a_n=g_ng_{n-1}^{-1}$.

It follows from the Lax representation that the coefficients of the polynomial $Q(w, \lambda) = \det(\widetilde{L} - \lambda\cdot 1)$ are the integrals of (1). However, in contrast to the Abel case they are not independent.

***Lemma* 1.** *The polynomial $Q(w, \lambda)$ has the form*

$$(16)\quad (w-\lambda^N)^l + (w^{-1}-\lambda^N)^l + \sum_{k=1}^{l-1} (r_k^+(\lambda)(w-\lambda^N)^k + r_k^-(\lambda)(w^{-1}-\lambda^N)^k) - \\ - R_0(\lambda) + \sum a_{ij}\lambda^i w^j.$$

The last summation is over the pairs i, j such that $i \geqslant 0$, $i + N \mid j \mid \leqslant (N-1)l$. The polynomials $r_k^\pm$ have only k non-zero coefficients:

$$r_k^\pm(\lambda) = \sum_{i=(N-1)(l-k)-k+1}^{(N-1)(l-k)} b_{ki}^\pm \lambda^i.$$

The coefficients a_{ij} and $b_{ki}^\pm$ are a complete system of integrals in involution with the single relation

$$(17)\qquad R_0(\lambda) + (-\lambda^N)^l = \sum_k r_k^+(\lambda)(-\lambda^N)^k = \sum_k r_k^-(\lambda)(-\lambda^N)^k.$$

The number of independent integrals is $Nl^2 - l + 1$.

The restrictions on the form $Q(w, \lambda)$ are equivalent to the following condition: all the roots w of $Q(w, \lambda) = 0$ for large λ must be expandable in Laurent series in λ^{-1}, one half of them must be of the form $\lambda^N + O(\lambda^{N-1})$, and the other half of the form $\lambda^{-N} + O(\lambda^{-N-1})$.

We consider the algebraic curve $\mathfrak{R}$, given in $\mathbf{C}^2$ by the equation $Q(w, \lambda) = 0$. In general position we may assume that it is non-singular and that $Q(w, \lambda) = 0$ for almost all λ has $2l$ distinct roots w_j. Then to each point P of $\mathfrak{R}$, that is, $P = (w_j, \lambda)$ there corresponds the unique eigenvector $\varphi_n(t) = (\varphi_n^1, \ldots, \varphi_n^l)^t$, normalized by the condition $\varphi_0 \equiv 1$. All remaining coordinates $\varphi_n(t)$ are meromorphic functions on $\mathfrak{R}$. Their poles lie at the points $\gamma_i(t)$, where the left upper principal minor $\widetilde{L} - \lambda\cdot 1$ vanishes and $[\operatorname{rank}(\widetilde{L} - \lambda\cdot 1) = Nl - 1]$.

***Lemma* 2.** *The number of poles $\gamma_i(t)$ is $Nl^2 - l^2 = g + l - 1$, where g is the genus of $\mathfrak{R}$.*

Thus, to every set of initial conditions $g_n(0)$ and $\dot{g}_n g_n^{-1}(0)$ there corresponds a curve $\mathfrak{R}$, that is, a polynomial Q and a set of $Nl^2 - l^2$ points $\gamma_i(0)$ on it. The solutions differing by a transformation $g_n \to Gg_n$, where G is a constant matrix, are the kernel of this mapping.

We consider the problem of recovering L from the indicated data.

Let Q be as in Lemma 1. Then $\mathfrak{R}$ is compactified at infinity in λ by the points $P_j^\pm$ at which w has poles of order N and zeros of multiplicity N, respectively.

***Lemma* 3.** *For any set of $Nl^2 - l$ points γ_i in general position there exists one and only one vector-function $\psi_n(t, P)$ with the following properties:*

1° *it is meromorphic on $\mathcal{R}$ except at $P_j^{\pm}$ with poles at γ_i;*

2° *if we form from $\psi_n(t, \lambda_j^{\mp})$ as columns, the matrices $\psi_n^{\pm}(t, \lambda)$ then they have the form*

$$\psi_n^{\pm}(t, \lambda) = \lambda^{\pm n}\left(\sum \xi_{n,s}^{\mp}(t)\lambda^{-s}\right)e^{\mp\lambda t/2}, \qquad \xi_{n,0}^{-} = 1. \tag{18}$$

Here the $\lambda_j^{\mp}$ are inverse images of λ in a neighbourhood of $P_j^{\pm}$.

***Lemma* 4.** *The function $\psi_n(t)$ satisfies the equations*

$$L\psi_n = \lambda\psi_n, \quad (\partial_t - P)\psi_n = 0,$$

where $g_n = \xi_{n,0}^{+}$.

The functions $\varphi_n(t)$ and $\psi_n(t)$ differ in the normalization $\rho_n(t) = \psi_n(\psi_0^l)^{-1}$.

Corollary. *The matrices g_n satisfy the equation* (1). *By the restrictions to Q, the thus constructed solutions are periodic, $g_{n+N} = g_n$.*

For ψ_n we can construct formulae of Baker-Its type, by analogy with [15]. Calculating $\xi_{n,0}^{\pm}$ from them we obtain the following result.

***Theorem* 1.** *For any polynomial of the form* (16) *and any set of $Nl^2 - l^2$ points γ_i in general position the functions*

$$g_n(t) = (g_n^{-})^{-1} g_n^{+} c^n \tag{19}$$

are periodic solutions (1), *where the matrix elements of $g_n^{\pm}$ are*

$$g_{n,ij}^{\pm} = \theta(\omega_j^{\pm} + \vec{U}n + \vec{V}t + \vec{Z}_i)\,\theta^{-1}(\omega_j^{\pm} + \vec{Z}_i). \tag{19'}$$

The constant vectors $\vec{U}$ and $\vec{V}$ are given by the periods of differentials of the third and second kinds with poles at $P_j^{\pm}$; $\omega_j^{\pm}$ are the images of the points $P_j^{\pm}$ under the Abel transformation, and the $\vec{Z}_i$ are the images of the divisors $\gamma_1, \ldots, \gamma_{g-1}, \gamma_{g+i}$, $1 \leqslant i \leqslant l$. also under the Abel transformation. The constant c is determined from the periodicity condition $g_N = g_0$.

The general solution has the form $G_1 g_n G_2$, where the G_i are fixed matrices.

Remark. The calculation of all of the parameters in the formulae of the theorem from the initial data $g_n(0)$ and $g_n g_n^{-1}(0)$ only uses quadratures and a solution of algebraic equations, and the latter is necessary only to find the $\vec{Z}_i$. All the remaining parameters $\vec{\omega}_j^{\pm}$, U, $\vec{V}$, etc. can be expressed by quadratures in terms of the integrals.

2. Considering special cases of multiple eigenvalues of L and a shift by a period, we restrict ourselves to the case of maximal degeneracy of multiplicity l. Then the polynomial Q has the form $Q(w, \lambda) = Q_1^l(w, \lambda)$, $Q_1 = w + w^{-1} + \sum_{i=0}^{N} a_i\lambda^i$. To each point of the hyperelliptic curve $\mathcal{R}$ given

by $Q_1(w, \lambda) = 0$ there corresponds an l-dimensional subspace of joint eigenfunctions. Let $\psi_n(t, P)$ be the matrix whose columns form a basis in this subspace, normalized by the condition $\psi_0(0, P) = 1$. Then ψ_n is a meromorphic matrix, having lN poles γ_s, and

$$\varphi^{ij}_{n,s} = \alpha^j_s \varphi^{il}_{n,s}; \quad \varphi^{ij}_{n,s} = \operatorname{res}_{\gamma_s} \psi^{ij}_n, \tag{20}$$

where the α^j_s are constants independent of n and t. In a neighbourhood $P^\pm$ of the inverse images of $\lambda = \infty$, ψ_n has the form

$$\Psi^\pm_n(t, \lambda) = \lambda^{\pm n} \Big(\sum_{s=0}^{\infty} \xi^\pm_{n,s}(t) \lambda^{-s} \Big) e^{\mp \lambda t/2}. \tag{21}$$

***Lemma* 5.** *For any set of data* (γ_s, α^j_s) (*which are called, as in* [14], *the Tyurin parameters*) *in general position there exists one and only one matrix function* ψ_n *satisfying* (20) *and* (21) *and normalized by the requirement* $\xi^-_{n,0} \equiv 1$.

Just as above, $\xi_{n,0}$ can be proved to be a periodic solution of (1).

3. In conclusion we give a construction of the periodic solutions of the equations

$$(\partial_t^2 - \partial_x^2)\varphi_n = e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n}, \tag{22}$$

to which, as was found in [48], the two-dimensional version by Zakharov-Shabat of the Lax pair for the Abelian Toda chain reduces. These equations generalize, besides the equations of the chain itself, the sine-Gordon equation corresponding to the periodic solutions $\varphi_{n+2} = \varphi_n$.

We consider a non-singular algebraic curve $\mathfrak{R}$ of genus g with two distinguished points $P^\pm$.

***Lemma* 6.** *For any set of points* $\gamma_1, \ldots, \gamma_g$ *in general position there exist unique functions* $\psi_n(z_+, z_-, P)$ *such that*:

1° *they are meromorphic except at* $P^\pm$ *with poles at* $\gamma_1, \ldots, \gamma_g$;

2° *in a neighbourhood of* $P^\pm$ *they are representable in the form*

$$\psi_n(z_+, z_-, P^\pm) = e^{kz_\pm} \Big(\sum_{s=0}^{\infty} \xi^\pm_{n,s}(z_+, z_-) k^{-s} \Big) k^{\pm n};$$

where $\xi^\pm_{n,0} = 1$ *and* $k^{-1} = k^{-1}(P^\pm)$ *are local parameters in neighbourhoods of* $P^\pm$.

***Lemma* 7.** *The following equalities hold*:

$$\partial_{z_+}\psi_n = \psi_{n+1} + (\partial_{z_+}\varphi_n)\psi_n, \quad \partial_{z_-}\psi_n = e^{\varphi_n - \varphi_{n-1}}\psi_{n-1}; \quad e^{\varphi_n} = \xi^-_{n,0}.$$

The compatibility conditions of these equalities are equivalent to the equations

$$\frac{\partial^2}{\partial z_+ \partial z_-}\varphi_n = e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n},$$

which coincide with (22) *written in conical variables.*

***Theorem* 2.** *For each non-singular complex curve* $\mathfrak{R}$ *with two distinguished points the formula*

$$(23)\qquad \varphi_n=\log\frac{\theta(\omega^++U_1t+U_2x+U_3n+W)}{\theta(\omega^-+U_1t+U_2x+U_3n+W)}+\log\frac{\theta(\omega^-+W)}{\theta(\omega^++W)}+nc$$

gives a solution of the equations (22).

Here $\omega^\pm=(\omega_1^\pm,\ldots,\omega_g^\pm)$ are the images of $P^\pm$ under the Abel transformation; the vectors U_i depend on the points $P^\pm$ and are the period vectors of Abelian differentials of the second and third kinds with appropriately chosen singularities at $P^\pm$ (see, by analogy, [15]).

Let us distinguish the periodic solutions $\varphi_{n+N}=\varphi_n$ among the solutions thus constructed. For this purpose there must be a function $E(P)$ on $\mathfrak{R}$ having a pole of order N and a zero of order N at $P^\pm$.

Suppose $\mathfrak{R}$ is given in $\mathbf{C}^2$ by the equation

$$(24)\qquad w^N-E^m+E\left(\sum a_{ij}E^iw^j\right)=0;$$

$N(i+1)+mj\leqslant Nm-2$; N is prime to m. This is an N-sheeted cover of the E-plane, and over $E=0$ and $E=\infty$ all the sheets are glued, that is, the function $E(P)$ given by the projection of $\mathfrak{R}$ has the required properties.

Corollary. *Suppose that* $\mathfrak{R}$ *is of the form* (24); *then the formulae* (23) *give periodic solutions of* (22).

Remark (Dubrovin). The methods developed in Chapter 4 of the present survey allow us, in particular, to make the formula (23) for the solutions of (22) effective. By substituting (23) in (22) we obtain after simple transformations the following relation:

$$(25)\qquad a\,\frac{\theta(U_3+W)\,\theta(U_3-W)}{\theta^2(W)}=b+\partial_{U(P^+)}\partial_{U(P^-)}\log\theta(W).$$

Here W is an arbitrary g-dimensional vector; $U(P)$ for each $P\in\mathfrak{R}$ is a period vector of a differential with a double pole at P $(2U_{1,2}=U(P^+)\pm U(P^-))$; the constants a and b have the form

$$(25')\qquad a=\varepsilon^{-2}(P^+,P^-),\quad b=\frac{d}{dP}\frac{d}{dQ}\log\varepsilon(P,Q)\big|_{P=P^+\ Q=P^-}$$

($\varepsilon(P,Q)$ is defined by (5.4.27)). This is a standard identity in the theory of Abelian functions (see [8], (39)). Applying the addition theorem to (25), we obtain the following system (in the notation of Chapter IV):

$$(26)\qquad a\hat{\theta}[n](2U_3)=b\hat{\theta}[n](0)+\partial_{U(P^+)}\partial_{U(P^-)}\hat{\theta}[n](0),$$

where

$$n\in\frac{1}{2}(\mathbf{Z}_2)^g$$

Here $U_3 = A(P^+) - A(P^-)$, therefore, the system (26), together with (4.2.4), allows us to recover from the period matrix not only the canonical equations of the curve $\mathfrak{R}$, but also the image of the Abel transformation A: $\mathfrak{R} \to J(\mathfrak{R})$ (although, for this we have to solve the transcendental equation (26) for U_3)).

References

[1] G. Springer, Introduction to Riemann surfaces, Addison-Wesley, Reading, MA, 1957. MR **19**-1169
Translation: *Vvedenie v teoriyu rimanovykh poverkhnostei,* Izdat. Inostr. Lit., Moscow 1960. MR **23** # A318.

[2] N.G. Chebotarev, *Teoriya algebraicheskikh funktsii* (Theory of algebraic functions), OGIZ, Moscow 1948.

[3] I.R. Shafarevich, *Osnovy algebraicheskoi geometrii,* Nauka, Moscow 1972. MR **51** # 3162.
Translation: Basic algebraic geometry, Springer-Verlag, Berlin-Heidelberg-New York 1974. MR **51** # 3163.

[4] P.A. Griffiths and J. Harris, Principles of algebraic geometry, Wiley-Interscience, New York 1978. MR **80b**:14001.

[5] J. Igusa, Theta-functions, Springer-Verlag, Berlin-Heidelberg-New York 1972. MR **48** # 3972.

[6] A. Krazer, Lehrbuch der Thetafunktionen, reprint, Chelsea, New York 1970.

[7] H.F. Baker, Abelian functions, University Press, Cambridge 1897.

[8] J.D. Fay, Theta-functions on Riemann surfaces, Lecture Notes in Math. **352** (1973). MR **49** # 569.

[9] E.I. Zverovich, Boundary-value problems of the theory of analytic functions, Uspekhi Mat. Nauk **26**:1 (1971), 113-181.
= Russian Math. Surveys **26**:1 (1971), 117-192.

[10] V.V. Golubev, *Lektsii po integrirovaniyu uravnenii dvizheniya tyazhelogo tverdogo tela okolo nepodvizhnoi tochki* (Lectures on the integration of equations of motion of a heavy solid about a fixed point), Gostekhizdat, Moscow 1953. MR **15**-904.

[11] A.I. Markushevich, *Vvedenie v klassicheskuyu teoriyu abelevykh funktsii* (Introduction to the classical theory of Abelian functions), Nauka, Moscow 1979. MR **81c**:14024.

[12] H. Bateman and A. Erdélyi, Higher transcendental functions, Vol. 3, McGraw-Hill 1955. MR **16**-586.
Translation: *Vysshie transtsendentnye funktsii,* Tom 3, Nauka, Moscow 1967.

[13] S.P. Novikov, (Editor) *Teoriya solitonov* (Theory of solitons), Nauka, Moscow 1980,

[14] I.M. Krichever and S.P. Novikov, Holomorphic bundles over algebraic curves and nonlinear equations, Uspekhi Mat. Nauk **35**:6 (1980), 47-68.
= Russian Math. Surveys **35**:6 (1980), 53-79.

[15] ———————, Methods of algebraic geometry in the theory of non-linear equations, Uspekhi Mat. Nauk **32**:6 (1977), 183-208. MR **58** # 24353.
= Russian Math. Surveys **32**:6 (1977), 185-213.

[16] ———————, Integration of non-linear equations by methods of algebraic geometry, Funktsional. Anal. i Prilozhen. **11**:1 (1977), 15-32. MR **58** # 13168.
= Functional Anal. Appl. **11**:1 (1977), 12-26.

[17] B.A. Dubrovin, V.B. Matveev and S.P. Novikov, Non-linear equations of Korteweg-de Vries type, finite-zone operators, and Abelian varieties, Uspekhi Mat. Nauk **31**:1 (1976), 55-136. MR **55** # 899.
= Russian Math. Surveys **31**:1 (1976), 59-146.

[18] N.I. Akhiezer, A continuous analogy of orthogonal polynomials on a system of integrals, Dokl. Akad. Nauk SSSR **141** (1961), 263-266.
= Soviet Math. Dokl. 2 (1961), 1409-1412.
[19] H.F. Baker, Note on the foregoing paper "Commutative ordinary differential operators" by J.L. Burchnall and T.W. Chaundy, Proc. Royal Soc. London **A118** (1928), 584-593.
[20] S.P. Novikov, A periodic problem for the Korteweg-de Vries equation, I, Funktsional. Anal. i Prilozhen. 8:3 (1974), 54-66.
= Functional Anal. Appl. 8 (1974), 236-246.
[21] B.A. Dubrovin, I.M. Krichever and S.P. Novikov, The Schrödinger equation in a magnetic field and Riemann surfaces, Dokl. Akad. Nauk SSSR **229** (1976), 15-18.
= Soviet Math. Dokl. **17** (1976), 947-951.
[22] I.M. Krichever, Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles on a straight line, Funktsional. Anal. i Prilozhen. **14**:4 (1980), 45-54.
= Functional Anal. Appl. **14** (1980), 282-290.
[23] B.A. Dubrovin, On a conjecture of Novikov in the theory of theta-functions and non-linear equations of Korteweg-de Vries and Kadomtsev-Petviashvili type, Dokl. Akad. Nauk SSSR **251** (1980), 541-544.
= Soviet Math. Dokl. **21** (1980), 469-472.
[24] R. Hirota, Recent developments of direct methods in soliton theory, Preprint, Hiroshima University 1979.
[25] A.N. Tyurin, The geometry of the Poincaré divisor of a Prym variety, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 1003-1043. MR **51** # 2664.
= Math. USSR-Izv. **9** (1975), 951-986.
[26] F. Schottky, Über die Moduln der Thetafunktionen, Acta Math. **27** (1903), 235-288.
[27] C. Neumann, De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur, J. Reine Angew. Math. **56** (1859), 46-63.
[28] J. Moser, Various aspects of integrable Hamiltonian systems, Preprint Courant Institute, New York 1978.
[29] A.P. Veselov, Finite-zone potentials and integrable systems on a sphere with a quadratic potential, Funktsional. Anal. i Prilozhen. **14**:1 (1980), 48-50.
= Functional Anal. Appl. **14** (1980), 37-39.
[30] G.R. Kirchhoff, Mechanik, Berlin 1876.
Translation: *Mekhanika,* Izdat. Akad. Nauk SSSR, Moscow 1962.
[31] A.M. Perelomov, Some remarks on the integration of equation of motion of a solid in an ideal liquid, Funktsional. Anal. i Prilozhen. **15**:2 (1981), 83-85.
= Functional Anal. Appl. **15** (1981), 144-146.
[32] H. Weber, Anwendung det Thetafunktionen zweier Veränderlicher auf die Theorie der Bewegung eines festen Körpers in einer Flüssigkeit, Math. Ann. **14** (1978), 173-206.
[33] F. Kötter, Über die Bewegung eines festen Körpers in einer Flüssigkeit. I, II, J. Reine Angew. Math. **109** (1892), 51-81, 89-111.
[34] V.A. Steklov, *O dvizhanii tverdogo tela v zhidkosti* (On the motion of a solid in a liquid), Kharkov 1893.
[35] B.A. Dubrovin and S.P. Novikov, A periodic problem for the Korteweg-de Vries and Sturm-Liouville equations. Their connection with algebraic geometry, Dokl. Akad. Nauk SSSR **219** (1974), 19-22. MR **58** # 1761.
= Soviet Math. Dokl. **15** (1974), 1597-1601.
[36] S.Yu. Dobrokhotov and V.P. Maslov, Finite-zone almost periodic solutions in VKB-approximations, *Sovremennye problemy matematiki* **15** (1980), 3-94.

[37] B.A. Dubrovin and S.P. Novikov, Ground states in a periodic field. Magnetic-Bloch functions and vector bundles, Dokl. Akad. Nauk SSSR **253** (1980), 1293-1297.
= Soviet Math. Dokl. **21** (1980), 240-244.

[38] A. Andreotti and A.L. Mayer, On periodic relations for Abelian integrals on algebraic curves, Ann. Scuola Norm. Sup. Pisa (3) **21** (1967), 189-238. MR **36** # 3792.

[39] H.M. Farkas and H.E. Rauch, Period relations of Schottky type on Riemann surfaces, Ann. of Math. (2) **92** (1970), 434-461. MR **44** # 426.

[40] R. Garnier, Sur une classe de système différentielles abéliens déduits de la théorie des équations linéaires, Rend. Circ. Mat. Palermo **43** (1919), 155-191.

[41] S.V. Manakov, A remark on the integration of the Euler equations of the dynamics of an n-dimensional rigid body, Funktsional. Anal. i Prilozhen. **10**:4 (1976), 93-94. MR **56** # 13272.
= Functional Anal. Appl. **10** (1976), 328-329.

[42] B.A. Dubrovin, Completely integrable Hamiltonian systems associated with matrix operators, and Abelian varieties, Funktsional Anal. i Prilozhen. **11**:4 (1977), 28-41. MR **58** # 31219.
= Functional Anal. Appl. **11** (1977), 265-277.

[43] A.M. Perelomov, Coherent states and theta-functions, Funktsional. Anal. i Prilozhen. **6**:4 (1972), 47-57. MR **57** # 18562.
= Functional Anal. Appl. **6** (1972), 292-300.

[44] V.I. Arnold, *Matematicheskie metody klassicheskoi mekhaniki*, Nauka, Moscow 1974. MR **57** # 14032.
Translation: Mathematical methods in classical mechanics, Springer-Verlag, Berlin-Heidelberg-New York 1978.

[45] B.A. Dubrovin, The Kadomtsev-Petviashvili equation and relations between the periods of holomorphic differentials on a Riemann surface, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 1015-1028.
= Math. USSR-Izv. **18** (1981).

[46] I.M. Krichever, Algebraic curves and non-linear difference equations, Uspekhi Mat. Nauk **33**:4 (1978), 215-216. MR **80k**:58055.
= Russian Math. Survey **33**:4 (1978), 255-256.

[47] D. Mumford and P. van Moerbeke, The spectrum of difference operators and algebraic curves, Acta Math. **143** (1979), 93-154. MR **80e**:58028.

[48] A.V. Mikhailov, On the integrability of a two-dimensional generalization of the Toda chain, Zh. Eksper. Teoret. Fiz. letters **30** (1974), 443-448.

[49] I.V. Cherednik, On conditions of the reality in "finite-zone integration", Dokl. Akad. Nauk SSSR **252** (1980), 1104-1108.

[50] S. Lang, Introduction to algebraic and Abelian functions, Addison-Wesley, Reading, MA, 1972. MR **48** # 6122.
Translation: *Vvedenie v algebraicheskie i abelevy funktsii*, Mir, Moscow 1976.

[51] O. Forster, Riemannsche Flächen, Springer-Verlag, Berlin-Heidelberg-New York 1977. MR **56** # 5867.
Translation: *Rimanovy poverkhnosti*, Mir, Moscow 1980.

[52] F. Kötter, Die von Steklow und Liapunow entdeckten integralen Fälle der Bewegung eines Körpers in einer Flüssigkeit, Sitzungsber. Königlich Preussische Akad. d. Wiss. Berlin, **1900**, no. 6, 79-87.

[53] B.A. Dubrovin and S.P. Novikov, Periodic and conditionally periodic analogues of the multi-soliton solutions of the Korteweg-de Vries equation, Zh. Eksper. Teoret. Fiz. **67** (1974), 2131-2143. MR **52** # 3759.
= Soviet Physics JETP **40** (1974), 1058-1063.

[54] B.A. Dubrovin, Periodic problems for the Korteweg-de Vries equation in the class of finite-zone potentials, Funktsional. Anal. i Prilozhen. **9**:3 (1975), 41-51. = Functional Anal. Appl. **9** (1975), 215-223.
[55] P.D. Lax, Periodic solutions of Korteweg-de Vries equation, Comm. Pure Appl. Math. **28** (1975), 141-188. MR **51** # 6192.
[56] H.P. McKean and P. van Moerbeke, The spectrum of Hill's equation, Invent. Math. **30** (1975), 217-274. MR **53** # 936.
[57] V.A. Marchenko, "The theory of Sturm-Liouville operators", Kiev, "Naukova Dumka", 1977 (in Russian).
[58] H.P. McKean and E. Trubowitz, Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points, Comm. Pure Appl. Math. **29** (1967), 143-226. MR **55** # 761.
[59] I.M. Krichever, An algebraic-geometric construction of the Zakharov-Shabat equations and their periodic solutions, Dokl. Akad. Nauk SSSR **227** (1976), 291-294. MR **54** # 1298. = Soviet Math. Dokl. **17** (1976), 394-397.
[60] A.P. Its and V.B. Matveev, Schrödinger operators with a finite-zone spectrum and the N-soliton solutions of the Korteweg-de Vries equation, Teoret. Mat. Fiz. **23** (1975), 51-67. MR **57** # 18570.

Translated by J.E. Jayne

Moscow State University

Received by the Editors 30 September 1980