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ALGEBRAIC CURVES AND NON-LINEAR DIFFERENCE EQUATIONS

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In [1] we have given an account of a scheme for the integration of certain non-linear differential equations by methods of algebraic geometry. After a slight modification, the main ideas and results of the scheme can be carried over to difference equations.

1. Let

$$L_1^{ij}=\sum_{\alpha=-n_1}^{n_2} u_\alpha(s)\,\delta_{i,\,j-\alpha},\qquad L_2^{ij}=\sum_{\beta=-m_1}^{m_2} v_\beta(s)\,\delta_{i,\,j-\beta}$$

be difference operators whose coefficients are $(l\times l)$-matrices. We stipulate that their highest and lowest coefficients are non-singular diagonal matrices with distinct diagonal elements.

We consider equations in the coefficients of these operators that are equivalent to the equality $[L_1, L_2]=0$.

The operator L_2 induces on the solution space of the equation $L_1y=Ey$ a finite-dimensional linear operator $L_2(E)$. Its characteristic polynomial $Q(w, E)$ defines a complex curve $\mathfrak{R}$, and the projection $(w, E)=P\to E$ defines a meromorphic function on it.

THEOREM 1. *For any pair of commuting difference operators we can find a polynomial in two variables such that* $Q(L_2, L_1)=0$.

If all the eigenvalues of $L_2(E)$ are distinct, as in the case of pairwise coprime numbers n_2, m_2 and n_1, m_1, then to each point (w, E) of $\mathfrak{R}$ there corresponds an eigenvector of $L_2(E)$ that is unique up to a proportionality factor.

THEOREM 2. *If* $(n_2, m_2)=1$ *and* $(n_1, m_1)=1$, *then* $E(P)$ *has* l *poles* $(P_1^+,\ldots,P_l^+)$ *of order* n_2 *and* l *poles* $(P_1^-,\ldots,P_l^-)$ *of order* n_1. *The coordinates* $\psi_j(i, P)$ *of the eigenvector-functions of* L_1 *and* L_2 *belong to the space associated with the divisor* $\Delta=D+(i-1)D_\infty+P_j^+-P_j^-$, *where* D *is an effective divisor whose degree* g *is equal to the genus of the curve for almost all solutions of the original equations, and* $D_\infty=(P_1^++\ldots+P_l^+)-(P_1^-+\ldots+P_l^-)$.

We consider the inverse problem of recovering the operators from a curve with distinguished points $P_j^\pm$ and a divisor D of degree g.

Since $\deg\Delta=g$, by the Riemann–Roch theorem the $\psi_j(i, P)$ are uniquely determined by the conditions of Theorem 2 up to a normalization. Having fixed one, we have the following theorem.

THEOREM 3. *For any function* $E(P)$ *with poles on* $\mathfrak{R}$ *only at the points* $P_j^\pm$, *there exists a unique operator* L *such that* $L\psi(i, P)=E(P)\psi(i, P)$.

2. In this section we construct exact solutions for certain non-linear differential-difference equations. Suppose that we are given a set of polynomials $Q_j^\pm(k)$ and $R_j^\pm(k)$.

THEOREM 4. *For every effective divisor* D *on a curve* $\mathfrak{R}$ *of genus* g ($\deg D=g$) *with fixed local coordinates* $k_{j\pm}^{-1}(P)$ *in neighbourhoods of the* $P_j^\pm$, *one and (apart from a proportionality factor) only one there exists function* $\varphi_{j_1}(i, y, t, P)$ *that is meromorphic outside* $P_j^\pm$, *and for which* D *is the divisor of the poles. In a neighbourhood of* $P_j^\pm$ *the function*

$$\varphi_{j_1}(i,\ y,\ t,\ P)\exp\{Q_j^\pm(k_{j\pm}(P))\,y+R_j^\pm(k_{j\pm}(P))\,t\}$$

has a pole (zero) of order i *if* $j=j_1$, *and of order* $i-1$ *if* $j\neq j_1$.

By defining the normalization of $\varphi_j(i, y, t, P)$ arbitrarily we obtain the vector-valued function $\psi(i, y, t, P)$.

THEOREM 5. *There exist unique difference operators whose coefficients depend on* y *and* t, *such that*

$$\left(L_1-\frac{\partial}{\partial y}\right)\psi(s,\ y,\ t,\ P)=0\quad\text{and}\quad\left(L_2-\frac{\partial}{\partial t}\right)\psi(s,\ y,\ t,\ P)=0.$$

COROLLARY. *These operators satisfy the equation*

(1) $$[L_1, L_2] = \frac{\partial L_2}{\partial y} - \frac{\partial L_1}{\partial t}.$$

3. EXAMPLE. We consider the equations of a Toda chain:

$$\dot{v}_n = c_{n+1} - c_n, \qquad \dot{c}_n = c_n (v_n - v_{n-1}).$$

By Theorem 4, there is a unique function $\psi(n, t, P)$ with poles at the points $d_1, \ldots, d_g$ of $\mathfrak{R}$ defined by $w^2 = \prod_{i=1}^{2g+2} (E - E_i)$, and with the following asymptotic expansion at the inverse images of $E = \infty$ $(P^{\pm})$:

$$\psi^{\pm}(n, t, E) = i^n \lambda_n^{\pm 1} E^{\pm n} (1 + \xi_1^{\pm}(n, t) E^{-1} + \ldots) \exp\left(\mp \frac{1}{2} tE\right).$$

By Theorem 5, the operators

$$L^{nm} = i\sqrt{c_n}\, \delta_{n, m+1} + v_n \delta_{n, n} - i\sqrt{c_{n+1}}\, \delta_{n, m-1},$$
$$A^{nm} = \frac{i}{2}\sqrt{c_n}\, \delta_{n, m+1} + w_n \delta_{n, n} + \frac{i}{2}\sqrt{c_{n+1}}\, \delta_{n, m-1}$$

satisfy the equations $L\psi = E\psi$ and $A\psi = \partial\psi/\partial t$. Here $\sqrt{c_n} = \lambda_{n-1}/\lambda_n$, $v_n = \xi_1^+(n+1, t) - \xi_1^+(n, t)$, and $w_n = v_n/2 + \dot{\lambda}_n/\lambda_n$.

The equations (1) are equivalent to the system $\dot{v}_n = c_{n+1} - c_n$,

$$\frac{\dot{c}_n}{c_n} = (v_n - v_{n-1}) - (w_n - w_{n-1}) = \frac{1}{2}(v_n - v_{n-1}) - \frac{1}{2}\frac{\dot{c}_n}{c_n},$$

which is the same as the equations of a Toda chain.

We must remark that this representation of equations is different from the commutation representation, used in earlier work (for a bibliography, see [2]).

By expressing $\psi(n, t, P)$ in terms of Riemann's theta-function as in the formula of Its [3] and also § 3 of [1], we obtain the following formulae in which we have used the notation of [1]:

$$\dot{\log} c_n = \frac{d}{dn} \log \frac{\theta(\omega^+ + W)\, \theta((n-1)U + tV + W + \omega^-)}{\theta(\omega^- + W)\, \theta((n-1)U + tV + W + \omega^+)} + \text{const},$$
$$v_n = \frac{d}{dn}\frac{d}{dt} \log \frac{\theta(nU + tV + \omega^+ + W)}{\theta(tV + \omega^+ + W)} + \text{const},$$

where the vectors ω^+, V, and U and the constants depend only on the curve $\mathfrak{R}$, and d/dn denotes the difference derivative. A formula for the variables v_n analogous to ours was first derived by Novikov [2]. In 1977 the author became aware of a similar paper of Mumford.

References

[1] I. M. Krichever, Integration of non-linear equations by methods of algebraic geometry, Funktsional. Analiz i Prilozhen. **11** (1977), 15–31.
= Functional Anal. Appl. **11** (1977), 12–26.

[2] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, Non-linear equations of Korteweg-de Vries type, finite zone linear operators, and Abelian varieties, Uspekhi Mat. Nauk **31**:1 (1976), 55–136.
MR 55 # 899.
= Russian Math. Surveys **31**:1 (1976), 59–146.

[3] A. R. Its and V. B. Matveev, On a class of solutions of the Korteweg-de Vries equation, in: *Problemy matematicheskoi fiziki* (Problems of mathematical physics), No. 8, Leningrad State Univ., Leningrad 1976.

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