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INTEGRATION OF NONLINEAR EQUATIONS BY THE METHODS OF ALGEBRAIC GEOMETRY

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UDC 513.015.7+517.944

A method discovered in the late 1960s (see [1]) for integrating nonlinear partial differential equations in the Zakharov—Shabat form is based on the possibility of representing a number of such equations in operator form [2]

$$\left[L_1 - \frac{\partial}{\partial y}, L_2 - \frac{\partial}{\partial t}\right] = 0, \tag{0.1}$$

where L_1 and L_2 are linear differential operators in the variable x whose coefficients are matrix functions of x, y, and t. Originally this method was associated with the inverse-problem method of the scattering theory. A general scheme for using it was described in [2]. The use of the scattering theory restricted the possibilities of integration to the class of rapidly decreasing solutions.

An investigation of the periodic and almost-periodic solutions of the Korteweg—de Vries equation, the first equation for which a representation of the form (0.1) was found, revealed its deep algebrogeometric nature. (A detailed description of the results obtained along this line and a complete bibliography are given in [3].)

In the present paper we propose a general scheme for constructing periodic and almost-periodic solutions of Eqs. (0.1) by using the methods of algebraic geometry. (A brief description is given in [4, 5].)

These methods enable us to find and express in explicit form, in terms of the Riemann θ function, all stationary solutions of Eqs. (0.1), i.e., solutions that are independent of the variables y, t, and, consequently, to give a classification of commutative rings of differential operators in one variable.

The construction is based on the concept of the algebraicity of a differential operator, which means that it has a family of eigenfunctions, parametrized points of the nonsingular algebraic curve $\mathfrak{R}$, which has "good" analytic properties on $\mathfrak{R}$. The inverse problem of reconstructing an operator from such a family is solvable in the case of operators in several variables as well (see also [6]).

I take this opportunity to express my deep gratitude to S. P. Novikov for his constant interest in the work and his valuable advice.

G. M. Krzhizhanovskii Energy Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 11, No. 1, pp. 15-31, January-March, 1977. Original article submitted August 20, 1976.

1. THE NOVIKOV EQUATION

We consider a system of nonlinear equations in the coefficients of the operators

$$L_1 = \sum_{\alpha=0}^{n} u_\alpha(x) \frac{d^\alpha}{dx^\alpha}, \quad L_2 = \sum_{\beta=0}^{m} v_\beta(x) \frac{d^\beta}{dx^\beta},$$

which is equivalent to the condition that they commute, i.e., to the condition*

$$[L_1, L_2] = 0. \tag{1.1}$$

We assume a priori that this is a set of equations in the class of germs of matrix functions $u_\alpha^{ij}(x)$, $v_\beta^{ij}(x)$, $1 \leqslant i, j \leqslant l$, of a real variable. Anticipating some later results, we may point out that, as will be shown in Sec. 4, all of their solutions admit of a meromorphic continuation to the entire complex region and, in addition, almost all the solutions are conditionally periodic functions.

First of all, we stipulate that in all operators, unless otherwise specified, the leading coefficients are constant, nonsingular diagonal matrices $u_n^{ij}(x) = c_i\delta_{ij}$, $v_m(x) = b_i\delta_{ij}$. In addition, for those i, j for which $c_i = c_j$ (the set of such pairs will be denoted by Δ) we set $u_{n-1}^{ij}(x) \equiv 0$.

Since commutativity of the operators is "equivalent" to the existence of a "sufficiently large" number of joint eigenfunctions, we introduce the formal solutions of the equation

$$L_1\Psi(x, k) = k^n\Psi(x, k)\, u_n, \tag{1.2}$$

which have the following form:

$$\Psi(x, k) = \left(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\right) e^{k(x-x_0)}, \tag{1.3}$$

where k is a formal variable and the $\xi_s(x)$ are matrix functions.

LEMMA 1.1. There exists a unique formal solution of Eq. (1.2), denoted by $\Psi(x, k; x_0)$, which satisfies the "normalization" conditions $\xi_0^{ij} = \delta_{ij}$, $\xi_s^{ij}(x_0) = 0$, $s \geqslant 1$, $(i, j) \in \Delta$.

Proof. The matrices $\xi_s(x)$ are determined successively from the equations obtained by equating the coefficients of k^{-s}, $s = -n + 1, \ldots, 0, 1, \ldots$, on the two sides of Eq. (1.2).

These equations

$$\sum_{\alpha=0}^{n} u_\alpha \sum_{l=0}^{\alpha} C_\alpha^l \frac{\partial^{\alpha-l}}{\partial x^{\alpha-l}} \xi_{s+l} = \xi_{s+n} u_n \tag{1.4}$$

can be converted to the form

$$0 = [u_n, \xi_{n+s}] + n u_n \frac{\partial}{\partial x} \xi_{n+s-1} + \text{(terms containing only } \xi_j,\ j < n + s - 1).$$

Consequently, from the s-th equation we find those elements $\xi_{s+n}^{ij}(x)$, for which $(i, j) \notin \Delta$, and the elements $\frac{\partial}{\partial x}\xi_{s+n-1}^{ij}$, if $(i, j) \in \Delta$. This assertion not only completes the proof of the lemma but also enables us easily to derive the following

COROLLARY. The series (1.3) is a solution of Eq. (1.2) if and only if it is representable in the form

$$\Psi(x, k) = \Psi(x, k; x_0)\, A(k, x_0), \tag{1.5}$$

*These equations were integrated completely by S. P. Novikov and B. A. Dubrovin for the case in which L_1 is a second-order operator with scalar coefficient (see [3]). Equations (1.1) in the case when one of the operators with matrix coefficients is of first order were considered by B. A. Dubrovin (see [3]).

where the series $A(k, x_0) = \sum_{s=0}^{\infty} a_s(x_0) k^{-s}$ has nonzero matrix elements of the coefficients $a_s^{ij}(x_0)$ only for the indices $(i, j) \in \Delta$.

Proof. Let us verify that the series given by the right-hand side of Eq. (1.5) satisfies Eq. (1.2). We have

$$L_1\Psi(x, k) = k^n\Psi(x, k; x_0) u_n A(k, x_0) = k^n\Psi(x, k) u_n,$$

since $[A(k, x_0), u_n] = 0$.

For any solution $\Psi(x, k)$ we take as our series $A(k, x_0)$ the series $\Psi^{-1}(x_0, k; x_0)\Psi(x_0, k)$. We can verify at once that this satisfies the requirements of the corollary.

Hereafter we shall assume that all $c_i \neq c_j$, if $i \neq j$; i.e., the set Δ consists of the pairs (i, i).

THEOREM 1.2. The operators L_1 and L_2 are commutative if and only if in the coefficients of the series

$$\Psi^{-1}(x_0, k; x_0) L_2\Psi(x_0, k; x_0) = A(k, x_0) = A(k)$$

the only nonzero elements are diagonal ones and these are independent of x_0.

Proof. If the operators are commutative, then the series $L_2\Psi(x, k; x_0)$ satisfies Eq. (1.2). To see this, we note that $L_1L_2\Psi(x, k; x_0) = L_2L_1\Psi(x, k; x_0) = k^nL_2\Psi(x, k; x_0) u_n$.

As a corollary to Lemma 1.1, $L_2\Psi(x, k; x_0) = \Psi(x, k; x_0) A(k; x_0)$. Now we make use of the fact that $\Psi(x, k; x_1) e^{k(x_1-x_0)}$ is of the form (1.3) and satisfies (1.2). This means that $\Psi(x, k; x_1) e^{k(x_1-x_0)} = \Psi(x, k; x_0) B(k, x_0)$. Then $A(k, x_1) = B^{-1}(k, x_0) A(k, x_0) B(k, x_0) = A(k, x_0)$. Here we make use of the fact that both series have diagonal matrices as their coefficients.

Now let us prove the sufficiency of the conditions of the theorem. Since $L_1L_2\Psi(x, k; x_0) = k^n\Psi(x, k; x_0) A(k)u_n$, it follows that $[L_1, L_2]\Psi(x, k; x_0) = 0$, which is sufficient to make the operator $[L_1, L_2]$ vanish.

COROLLARY 1. The ring of operators which commute with the given operator is commutative.

Proof. Let $[L_1, L_2] = 0$, $[L_1, L_3] = 0$; then $L_2\Psi(x, k; x_0) = \Psi(x, k; x_0) A_1(k)$, $L_3\Psi(x, k; x_0) = \Psi(x, k; x_0) A_2(k)$, and $[L_2, L_3]\Psi(x, k; x_0) = 0$. This means that $[L_2, L_3] = 0$.

COROLLARY 2. Equations (1.1) have an infinite set of first integrals polynomially dependent on the matrix elements $u_\alpha^{ij}(x)$ and $v_\beta^{ij}(x)$ and their derivatives.

Proof. As stated in Theorem 1.2, the first integrals of the system are matrix elements of the coefficients of the series

$$\Psi^{-1}(x, k; x) L_2\Psi(x, k; x) = \sum_{s=-m}^{\infty} A_s k^{-s}.$$

From Eqs. (1.4) it follows that the matrix elements of the series $\frac{d^r}{dx^r}\Psi(x, k; x)$ are polynomially dependent on the matrix elements u_α^i and their derivatives.

COROLLARY 3. The matrix elements $v_\beta^{ij}(x)$ are polynomially expressible in terms of the matrix elements $u_\alpha^{ij}(x)$, their derivatives, and the first integrals of the equations A_s^{ii}, $-m \leqslant s \leqslant 0$.

Thus, the system of nonlinear equations in the coefficients of the operators L_1 and L_2 in (1.1) turns out to be equivalent to a family of systems of equations in the coefficients of the operator L_1 alone, parametrized by the sets of arbitrary complex constants A_s^{ii}, $-m \leqslant s \leqslant 0$. These last systems will be called Novikov equations, since in the case of the Schrödinger operator $-(d^2/dx^2) + u(x)$ they coincide with the equations which describe the stationary solutions of higher analogs of the Korteweg–de Vries equation, the importance of which was first noted in [7].

As will be shown in the proof of Theorem 1.3, only a finite number of the integrals A_s^{ii} are independent.

It should be noted that the proposed scheme is different from the schemes for constructing the polynomial integrals of the higher analogs of the Korteweg–de Vries equation in [7, 8, 23]. Formulas relating sets of Novikov integrals and Gel'fand–Dikii–Lax integrals are given in [9].*

THEOREM 1.3. Suppose that the coefficients of the operator L satisfy Novikov equations for which the constants A_s^{ii}, $-m \leqslant s \leqslant 0$, determining them include at least one nonzero constant with index s which is relatively prime to n. Then the operator L has a family of characteristic vector functions $\psi(x, P)$, i.e., $L\psi(x, P) = E(P)\psi(x, P)$, parametrized by points of the nonsingular algebraic curve $\mathfrak{R}$, $P \in \mathfrak{R}$. The function E(P) is meromorphic on $\mathfrak{R}$ and has l poles, $P_1, \ldots, P_l$, of multiplicity n. Furthermore, $\psi(x, P)$ satisfies the following conditions:

1) for all x it is meromorphic on $\mathfrak{R}$ outside of $P_1, \ldots, P_l$, and its poles $D_1, \ldots, D_N$ are independent of x;

2) in a neighborhood of the point P_j the vector function $\psi(x, P)\, e^{-k(P)(x-x_0)}$ is analytic, $k(P) = \sqrt[n]{\frac{E(P)}{c_j}}$, and its value in P_j is equal to a vector with a single nonzero j-th coordinate, which is equal to 1.

For almost all solutions of the original equation the divisor $D_1 + \ldots + D_N$ is nonspecial, and its degree is equal to $g + l - 1$, where g is the genus of the curve $\mathfrak{R}$.

Proof. For the formal variable E we consider the nl-dimensional linear complex space $\mathscr{L}(E)$, whose basis is constituted by the j-th columns of the matrices $\Psi(x, k_{jr}; x_0)$, defined in accordance with Lemma 1.1, $1 \leqslant j \leqslant l$, $0 \leqslant r \leqslant n-1$, where $c_j k_{jr}^n = E$ $(c_j = u_n^{jj})$. By Corollary 3, the coefficients of the operator L and the constants A_s^{ii} determine the operator L_2, which commutes with L. Consequently, L_2 induces on $\mathscr{L}(E)$ a finite-dimensional linear operator $L_2(E)$, for which the selected basis is characteristic. The characteristic polynomial of the operator is equal to

$$\prod_{j=1}^{l} \prod_{r=1}^{n-1} (y - A^{jj}(k_{jr})).$$

The coefficients of this polynomial are symmetric functions of the variables k_{jr}, and, consequently, they are Laurent series in the variable E^{-1}. We shall prove that they are polynomials in E.

Since these polynomials are expressed in terms of A_s^{jj}, $s \leqslant m(nl-1) = N$, all the other integrals of the Novikov equations are consequences of these. The vanishing of the coefficients of the powers of E^{-1} enables us to write the linear combinations $c_r A_r^{jj}$, where $r = h_1 m + h_2 n \leqslant N$ [we denote the number of such pairs (h_1, h_2) by $N(m, n, l)$], in terms of integrals with lower indices.

COROLLARY. Every Novikov equation has a set of $ml(nl - 1) - N(m, n, l)$ independent polynomial integrals.

We introduce a new basis in $\mathscr{L}(E)$, formed by $c_{js}(x, E)$ columns of the matrices $C_j(x, E)$, determined by the normalization $\frac{d^r}{dx^r} C_j(x_0, E) = \delta_{rj} I$, where I is a unit matrix and $0 \leqslant r \leqslant n-1$.

LEMMA 1.4. The matrix elements of the matrices $\frac{d^N}{dx^N} C_j(x_0, E)$ are polynomially expressible in terms of the matrix elements $u_\alpha^{ij}(x_0)$, $\frac{d}{dx} u_\alpha^{ij}(x_0), \ldots,$ and the variable E.

Proof of this lemma can be obtained by repeatedly using, for reducing the order of the derivative, the equation

$$u_n \frac{d^n}{dx^n} C_j(x, E) = E C_j(x, E) - \sum_{\alpha=0}^{n-1} u_\alpha(x) \frac{d^\alpha}{dx^\alpha} C_j(x, E),$$

which is satisfied by virtue of the definition of $\mathscr{L}(E)$.

COROLLARY. In the new basis the matrix elements of the operator $L_2^{ij}(E)$ are polynomially dependent on the variable E.

*In the present paper we shall not discuss the question of Hamiltonian mechanics related to Eqs. (1.1). I. M. Gel'fand and L. A. Dikii have informed the author that their Hamiltonian structure is investigated in [24].

This means that the characteristic polynomial of $L_2(E)$ has the form $Q(y, E)$, where $Q(\ ,\ )$ is a polynomial in two variables.

The following lemma reflects a well-known fact concerning self-adjoint operators: Operators which commute with each other are functionally dependent.

LEMMA 1.5. The operators L and L_2 are algebraically related by the equation $Q(L_2, L) = 0$.

Proof. The eigenvalues of the operator $L_2(E)$ are given by the equation

$$Q(y, E) = 0, \tag{1.6}$$

Therefore, $Q(L_2, L)\Psi(x, k; x_0) = Q(y, E)\Psi(x, k; x_0) = 0$.

As has already been noted more than once, a linear differential operator vanishes if and only if the one-parameter family of functions $\Psi(x, k; x_0)$ belongs to its kernel.

The statement of the lemma for the case of the operator $L = -(d^2/dx^2) + u(x)$ was first obtained by A. B. Shabat for a reformulation of the method of proof of the fundamental theorem of [7].

Now we shall consider E a complex number. Equation (1.6) defines the algebraic curve $\mathfrak{R}$, and the correspondence $(y, E) = P \in \mathfrak{R} \to E$ specifies a function $E(P)$ on it.

For large values of E the expansion of the eigenvalues of the operator $L_2(E)$ in Laurent series in the variables $(E/c_j)^{-1/n}$ coincides with the series $A^{jj}(k_{jr})$. By the hypothesis of the theorem, these eigenvalues are distinct. Consequently, they are distinct for almost all values of E. Moreover, it follows from the foregoing that the preimage of the "point at infinity" of the completed complex plane $\bar{C} = C \bigcup \infty$ for the mapping $E: \mathfrak{R} \to \bar{C}$ consists of the l points $P_1, \ldots, P_l$, where the local coordinates in a neighborhood of these points are constituted by the functions $k_j^{-1}(P) = (E(P)/c_j)^{-1/n}$.

To every eigenvalue of the operator $L_2(E)$, i.e., to a point P of the curve $\mathfrak{R}$, there corresponds an eigenvector which is unique to within a proportionality factor.

Selecting the normalization for which its first coordinate in the basis $c_{js}(x, E)$ is constant, we can easily verify that its other coordinates are meromorphic functions on the curve $\mathfrak{R}$, which depend, in general, on the choice of the initial point x_0. We denote this characteristic vector function by $\Phi(x, P; x_0)$:

$$\Phi(x, P; x_0) = \sum_{j,s} \lambda_{js}(x_0, P)\, c_{js}(x, P; x_0), \quad \lambda_{11}(x_0, P) \equiv 1.$$

In order to obtain the required eigenfunction $\psi(x, P)$, we proceed as follows. The coordinates of the vector function $\tilde{\Phi}(x_0, P)$, equal to the values of the logarithmic derivatives of the coordinates $\Phi(x, P; x_0)$ at the point x_0, are independent of the choice of normalization of the eigenvector of $L_2(E)$. They are equal to the ratio of the coordinates $\lambda_{2s}(x_0, P)/\lambda_{1s}(x_0, P)$, which means that they are meromorphic on $\mathfrak{R}$.

We can verify at once that the function $\psi(x, P) = \int_{x_0}^{x} \tilde{\Phi}(z, P)\, dz$ satisfies all the requirements of the theorem.

For almost all solutions of the original Novikov equation, Eq. (1.6) specifies an algebraic curve with no singularities, and in order to complete the proof of the theorem in this case, we need only find the number N of poles of the function $\psi(x, P)$. To do this, we construct the matrix $F(x, E)$, whose columns consist of the coordinates of the vectors $\psi(x, P_i), \psi'(x, P_i), \ldots, \psi^{(n-1)}(x, P_i)$, where the P_i are the points at which $E(P_i) = E$. The function $g(x, E) = (\det \|F(x, E)\|)^2$ is independent of the order in which the points P_i are numbered and is a rational function of the complex variable E. Its zeros are those points E for which the functions $\psi(x, P_i)$ are linearly dependent, i.e., the points for which the eigenvalues of the operator $L_2(E)$ merge into one another. The multiplicity of a zero of $g(x, E)$ is equal to the multiplicity of the branch point ν of the curve $\mathfrak{R}$. (The multiplicity of a branch point is one less than the number of sheets of $\mathfrak{R}$ that merge at that point.) The poles of $g(x, E)$ coincide with the images of the poles of $\psi(x, P)$ and with the "point at infinity" $E = \infty$. We have

$$\sum \nu = 2N + N_\infty.$$

Let us find the multiplicity of the pole at infinity. It is equal to twice the multiplicity of the product of the diagonal elements of the matrix $F(x, E)$. In the local parameter $E(P)^{-1/n}$ the multiplicity of a pole of the corresponding coordinate $\psi^{(r)}(x, P)$ is equal to r. This means that the multiplicity of a pole of $g(x, E)$ in the parameter $E(P)^{-1/n}$ is equal to $2(1 + \ldots + (n-1))l = n(n-1)l$. Consequently, $N_\infty = (n-1)l$. As is known (see [10]), the genus of a curve is equal to half the sum of the multiplicities of all branch points minus the number of sheets plus 1. The multiplicity of branching at the points P_j is equal to $n - 1$, and therefore

$$2g = \sum \nu + (n-1)l - 2nl + 2 = \sum \nu - nl - l + 2.$$

Then

$$2N = \sum \nu - nl + l = 2g + 2l - 2.$$

If the curve $\mathfrak{R}$ has singularities, then there exists a birational nonsingular curve $\hat{\mathfrak{R}}$, which is isomorphic to it, i.e., a mapping $\pi\hat{\mathfrak{R}} \to \mathfrak{R}$, which is an isomorphism almost everywhere. The functions π^*E and $\pi^*\psi$ satisfy the requirements of the theorem. If $P \in \hat{\mathfrak{R}}$, then $\pi^*E(P) = E(\pi(P)))$. This completes the proof of the theorem.

Solutions of Novikov equations for which the curve specified by Eq. (1.6) has singularities may be regarded as the limit of solutions of general type, for which the points of the corresponding curves merge with one another. However, we shall give a closed algebro-geometric description for them, analogous to the description given in [11] for a well-known class of nonreflective potentials of a Sturm–Liouville operator (see also Appendix II to [3]).

COROLLARY. On the hypothesis of Theorem 1.3, the degree N of a nonspecial divisor of the poles of the function $\psi(x, P)$ is equal to $g + l - 1 + d$, where d is the number of points $E_1, \ldots, E_d$ for which

$$\det \| F(x, E_s) \| \equiv 0 \tag{1.7}$$

Proof. The points $E_1, \ldots, E_d$ are images under the mapping $E: \mathfrak{R} \to \bar{C}$ of the singularities of the curve $\mathfrak{R}$. When we remove these singularities, i.e., when we pass to the curve $\hat{\mathfrak{R}}$, we will have identical functions $\psi(x, P)$ corresponding to all the preimages of one singularity. Consequently, Eqs. (1.7) are satisfied. As in the proof of the theorem, making use of the fact that $g(x, E)$ vanishes not only at the images of the branch points but also at the points E_i, we find that $N = g + l + d - 1$.

2. COMMUTATIVE RINGS OF DIFFERENTIAL OPERATORS

In Sec. 1, to every operator whose coefficients satisfy Novikov equations we assigned a set of data: a nonsingular complex curve $\mathfrak{R}$, which, in accordance with the ideology of [3] we call the spectrum of the operator, a meromorphic function $E(P)$, which has poles of n-th order at the points $P_1, \ldots, P_l$ and is called the spectral parameter, an effective divisor $D = \Sigma k_s D_s$ (i.e., a set of points with multiplicities $k_s \geqslant 0$), and also the points $E_1, \ldots, E_d$, where $N = \Sigma k_s = g + l - 1 + d$.

Our purpose in the present section will be to solve the inverse problem and reconstruct the operator L from the set $(\mathfrak{R}, E, D, E_1, \ldots, E_d)$.

First we construct the vector function $\psi(x, P)$, which will be an eigenfunction of the operator L. We shall state the necessary theorem in the form in which it can be used in our next section for solving the inverse problem for algebraic operators of several variables.

In a neighborhood of the points $P_1, \ldots, P_l$ of the nonsingular curve $\mathfrak{R}$ we fix the local parameters $z_j(P)$, $z_j(P_j) = 0$. By analogy with the space $\mathcal{L}(D)$ of meromorphic functions on $\mathfrak{R}$, associated with the divisor D ($f(P) \in \mathcal{L}(D)$, if $D + D_f \geqslant 0$, where D_f is the principal divisor of f), we introduce the space $\Lambda(q, D)$, where q is the set of polynomials $q_j(k)$.

The function $\Phi(q, P)$ belongs to $\Lambda(q, D)$ if:

1) outside of the points P_j it is meromorphic, and for a divisor of its poles D_Φ (the multiplicity with which the point D_s occurs in D_Φ is equal but opposite in sign to the multiplicity of the pole of the function in it) we have $D_\Phi + D \geqslant 0$;

2) in a neighborhood of P_j the function $\Phi(q, P) \exp\{-q_j(k_j(P))\}$ is analytic, $k_j(P) = z_j^{-1}(P)$.

THEOREM 2.1 (Akhiezer). For a nonspecial divisor $D \geqslant 0$ of degree $N \geqslant g$, $\dim \Lambda(q, D) = N - g + 1$.

It should be recalled that by nonspecial divisors, which form an open set among all divisors, we mean those divisors for which $\dim \mathscr{L}(D) = N - g + 1$.

A logarithmic differential $d\Phi(q, P)/\Phi(q, P)$ is an Abelian differential on $\mathfrak{R}$, and, therefore, the proof of the theorem is in large measure a repetition of the proof of Abel's theorem and the solution of Jacobi's inversion problem (see [12]). We shall omit it not only because it has been given repeatedly in many studies (see [3, 13-15]), even though in somewhat different form, but also because it can easily be obtained from the explicit formulas for $\Phi(q, P)$ given in Sec. 4.

In the present section we shall confine our attention to the case in which all the polynomials $q_j(k)$ are equal to $k(x - x_0)$. For simplicity, instead of $\Phi(q, P)$ we shall write $\Phi(x, P)$.

If on the curve $\mathfrak{R}$ there exists a meromorphic function $E(P)$ with poles of multiplicity n at the points $P_1, \ldots, P_l$, then as the local parameters $z_j(P)$ we take $\sqrt[n]{c_j/E(P)}$.

COROLLARY. For a nonspecial divisor $D \geqslant 0$ there exists a unique vector function $\psi(x, P)$ whose coordinates $\psi_i(x, P) \in \Lambda(x, P)$, a matrix with columns $\psi(x, P)\exp(-k(P)(x - x_0))|_{P_j}$ is unique, and at the points $E_1, \ldots, E_d$, $d = N - g - l - 1$, Eqs. (1.7) are satisfied.

Proof. We select an arbitrary basis in the space $\Lambda(x, D)$. The conditions on the coordinates of $\psi(x, P)$ become a system of linear equations. Their number is equal to the dimension of $\Lambda(x, D)$.

THEOREM 2.2. There exists a unique operator $L = \sum_{\alpha=0}^{n} u_\alpha(x) \frac{d^\alpha}{dx^\alpha}$ such that

$$L\psi(x, P) = E(P)\psi(x, P), \quad u_n^{ij} = c_j\delta_{ij}.$$

Its coefficients satisfy Novikov equations.

Proof. We construct for the function $\psi(x, P)$ a series of the form (1.3). As the columns of the matrices $\xi_s(x)$ we take the coefficients of the z_j^s in the expansion of a neighborhood of P_j of the analytic function

$$\psi(x, P)\exp(-k_j(P)(x - x_0)).$$

LEMMA 2.3. For any series of the form (1.3) there exists a unique operator L such that

$$L\Psi(x, k) \equiv k^n\Psi(x, k)\, u_n \pmod{O(k^{-1})e^{k(x-x_0)}}.$$

Proof. The coefficients of L can be found successively from Eqs. (1.4) for $s = -n + 1, \ldots, 0$. If these are satisfied, this is equivalent to the required congruence.

We shall now prove that for the constructed operator $L\Psi(x, k) = k^n\Psi(x, k)u_n$. To do this, we consider the function $L\psi(x, P) - E(P)\psi(x, P)$. This satisfies all the requirements defining $\psi(x, P)$ except one. Its values at all the points P_j are equal to zero. From the uniqueness of $\psi(x, P)$ it follows that $L\psi(x, P) = E(P)\psi(x, P)$.

In order to complete the proof of the theorem, it is sufficient to show that there exists a differential operator which commutes with L.

Let $A(\mathfrak{R}, P_1, \ldots, P_l)$ be the ring of functions which are meromorphic on $\mathfrak{R}$ and have poles at the points $P_1, \ldots, P_l$. In the case when $\mathfrak{R} = \bar{C}$ is the completed complex plane and $P = \infty$ is the "point at infinity," we find that $A(\bar{C}, \infty)$ is the ring of polynomials.

LEMMA 2.4. The function $\psi(x, P)$ gives a homomorphism λ from $A(\mathfrak{R}, P_1, \ldots, P_l)$ into the ring of linear differential operators, where to the function $H(P) \in A(\mathfrak{R}, P_1, \ldots, P_l)$ there corresponds an operator $\lambda(H)$ such that

$$\lambda(H)\psi(x, P) = H(P)\psi(x, P)$$

The construction of the operator $\lambda(H)$ is completely analogous to the construction of L. Its coefficients can be found from the congruence

$$\lambda(H)\Psi(x,k)\equiv\Psi(x,k)\left(\sum_{s=0}^{m}h_s k^s\right)(\operatorname{mod}O(k^{-1})e^{k(x-x_0)}), \tag{2.1}$$

where the elements h_s^j of the diagonal matrices $h_s^{ij}=h_s^j\delta_{ij}$ are the coefficients of the k_j^s in the Laurent series expansion of the function H(P) in a neighborhood of P_j.

Since $A(\mathfrak{R}, P_1, \ldots, P_l)$ is a commutative ring, its image λ is also commutative.

In the definition of the function $\psi(x, P)$, and therefore, of the homomorphism λ as well, we had a function E(P). In order to define λ for any curve $\mathfrak{R}$ with the indicated points and for a divisor of degree $N = g + l - 1$, we stipulate that as E(P) we will always select a function from $A(\mathfrak{R}, P_1, \ldots, P_l)$ with poles of identical minimal multiplicity at the points P_j. Moreover, λ depends only on the equivalence class of D. Two divisors are called equivalent, $D \sim D'$, if $D - D'$ is a divisor of some meromorphic function f(P) on $\mathfrak{R}$. The vector functions $\psi(x, p)$ and $\psi'(x, P)$, constructed for D and D', are related by the equation $\psi(x, P) = B(P)\psi'(x, P)$, where $B^{ij}(P) = f^{-1}(P_j)f(P)\delta_{ij}$; therefore, $\lambda = \lambda'$.

Combining the results obtained in Secs. 1 and 2, we obtain the following.

COROLLARY. For any commutative subring of differential operators A in which there exists a pair of operators of relatively prime orders, there exist a nonsingular curve $\mathfrak{R}$ with the indicated points and a class of divisors (D) such that the homomorphism λ defined by them establishes an isomorphism $\lambda\colon A(\mathfrak{R}, P_1, \ldots, P_l) \to A$.

The space of solutions of Eqs. (1.1) is a complex linear space, which for relatively prime n and m, by the corollary we have proved, is isomorphic to a fiber space over the variety of the moduli of curves with l indicated points at which there exist functions with poles of orders n and m whose fiber consists of the Jacobians of the curves.

By the Jacobian of a curve we mean a g-dimensional complex torus formed by the equivalence classes of divisors of fixed degree (see [12]). We shall discuss this in more detail in Sec. 4.

In the case $n = 2$ and $l = 1$, this result was obtained in [16].

We now ask: When does the curve $\mathfrak{R}$ corresponding to the commutative ring A have a fixed genus? The genus of $\mathfrak{R}$ in terms of the ring $A(\mathfrak{R}, P)$ is given as follows. Suppose that n is the minimal possible multiplicity of a pole of functions belonging to $A(\mathfrak{R}, P)$, and μ_i, $i = 1, \ldots, n-1$, are the minimal numbers for which there exists a function in this ring with a pole multiplicity $\mu_i n + i$. Then $g = \mu_1 + \ldots + \mu_{n-1}$.

LEMMA 2.5. The genus of the curve $\mathfrak{R}$ is no greater than g if and only if in the ring A there exist operators of orders n, $\mu_i n + 1$, where $\mu_1 + \ldots + \mu_{n-1} = g$.

For almost all points P of the curve $\mathfrak{R}$ we have $n = g + 1$. All other points are called Weierstrass points. For almost all curves of fixed genus g at the Weierstrass points we have $n = g$, $\mu_1 = 2$, $\mu_i = 1$, $i > 2$.

COROLLARY. The fiber space $\hat{\hat{M}}$ over the finite-sheeted covering $\hat{M} \to M$ of the variety of moduli of curves of genus g corresponding to the fixation on the curve of a Weierstrass point whose fiber is the Jacobian of the curve is isomorphic to the space of solutions of the system of equations

$$[L, L_i] = 0, \qquad i = 1, \ldots, g-1, \tag{2.2}$$

where L and L_i are differential operators with scalar coefficients of orders g, 2g + 1, g + i (i > 2), respectively.

The differential equations (2.2) on the complex linear space of solutions of the equations $[L, L_1] = 0$ give us the algebraic equations describing the variety $\hat{\hat{M}}$.

We must make a careful analysis of the possibilities of such an approach for the solution of the problem of unirationality of the complete variety of moduli of curves of genus g, which is extremely important in algebraic geometry.

3. ALGEBRAIC OPERATORS AND ZAKHAROV–SHABAT EQUATIONS

As we noted in the introduction, the possibilities of the inverse problem are considerably broader than those of the direct problem, and its solution is always possible when there exists for the operator the set of "spectral data" listed at the beginning of the preceding section.

In this paper we shall confine ourselves to operators of the form

$$\sum_{\alpha=0}^{n} u_\alpha(x, y)\frac{\partial^\alpha}{\partial x^\alpha} - \frac{\partial}{\partial y} = L - \frac{\partial}{\partial y}. \tag{3.1}$$

Definition. An operator $L - \partial/\partial y$ is called algebraic if there exist a curve $\mathfrak{R}$ of genus g with indicated points $P_1, \ldots, P_l$, an effective divisor D of degree $g + l - 1$, and a vector function $\psi(x, y, P)$, $P \in \mathfrak{R}$, such that

1) $$\left(L - \frac{\partial}{\partial y}\right)\psi(x, y, P) = 0;$$

2) outside of the points P_j it is meromorphic, and for a divisor of its poles D_ψ we have $D + D_\psi \geqslant 0$;

3) in a neighborhood of P_j the function $\psi(x, y, P)\exp(-k_j(P)x - Q_j(k_j(P))y)$ is analytic and its value at P_j is equal to a vector with a single nonzero j-th coordinate, which is equal to 1. Here $z_j(P) = k_j^{-1}(P)$ is a local parameter, and the $Q_j(k)$ are polynomials of degree n.

Remark. This definition of algebraicity corresponds to the properties of operators whose coefficients are solutions of the general type of Novikov equations. It is not difficult to give a definition of the analogs of separatrix solutions of these equations (see the corollary to Theorem 1.3).

We consider the problem of reconstructing an algebraic operator from its "spectral data."

We set the polynomials $q_j(k)$ appearing in the definition of the space $\Lambda(q, D)$ equal to $q_j(k) = kx + Q_j(k)y$; then, as a corollary to Theorem 2.1, we obtain the following.

COROLLARY. For fixed local parameters $z_j(P)$ and polynomials $Q_j(k)$ the conditions 2 and 3 uniquely define the function $\psi(x, y, P)$.

THEOREM 3.1. There exists a unique operator $L = \sum_{\alpha=0}^{n} u_\alpha(x, y)\frac{\partial^\alpha}{\partial x^\alpha}$ such that

$$\left(L - \frac{\partial}{\partial y}\right)\psi(x, y, P) = 0,$$

where n is the maximal degree of the $Q_j(k)$.

Proof. As in the proof of Theorem 2.1, we construct for the vector function $\psi(x, y, P)$ a formal series with matrix coefficients, which has the form

$$\Psi(x, y, k) = \left(\sum_{s=0}^{\infty} \xi_s(x, y) k^{-s}\right) e^{kx + Q(k)y}, \tag{3.2}$$

where Q(k) is a polynomial with matrix coefficients,

$$Q(k) = \sum_{j=1}^{l} Q_j(k)\delta_{ij} = \sum_{m=0}^{n} \tilde{Q}_m k^m.$$

By the normalization condition $\xi_0(x, y)$ is a unit matrix.

LEMMA 3.2. For any series of the form (3.2) there exists a unique operator of the form (3.1) such that

$$\left(L - \frac{\partial}{\partial y}\right)\Psi(x, y, k) \equiv 0 \quad (\mathrm{mod}\, O(k^{-1}) e^{kx + Q(k)y}).$$

Proof. The required congruence is equivalent to having the following equation satisfied:

$$\sum_{\alpha=0}^{n} u_\alpha \sum_{l=0}^{\alpha} C_\alpha^l \frac{\partial^{\alpha-l}}{\partial x^{\alpha-l}} \xi_{s+l} = \sum_{m=0}^{n} \xi_{s+m} \tilde{Q}_m, \quad s = -n, \ldots, 0. \tag{3.3}$$

From these equations we can successively find the matrices $u_\alpha(x, y)$.

Continuing the proof as in Theorem 2.1, we find that for the constructed operator $(L - \partial/\partial y)\psi(x, y, P) = 0$.

We introduce an equivalence relation between the sets of local parameters $z_j(P)$ and the polynomials $Q_j(k)$. We shall say that $(z_j(P), Q_j(k))$ and $(z_j'(P), Q_j'(k))$ are equivalent if when we set $k_j(P) = a_{-1}k_j(P) + a_0 + a_1 k_j^{-1}(P) + \ldots$ we find that

$$Q_j(k) \equiv Q_j'(a_{-1}k + a_0 + \ldots) \pmod{k^{-1}}.$$

COROLLARY. The set of algebraic operators which have a fixed set of "spectral data" $\mathfrak{R}, P_1, \ldots, P_l, D$, is in one-to-one correspondence with the equivalence classes of the sets $(z_j(P), Q_j(k))$.

We make use of an already proved theorem for the construction of the solutions of Zakharov–Shabat equations.

Let the operators $L_1 = \sum_{\alpha=0}^{n} u_\alpha(x, y, t) \frac{\partial^\alpha}{\partial x^\alpha}$ and $L_2 = \sum_{\beta=0}^{m} v_\beta(x, y, t) \frac{\partial^\beta}{\partial x^\beta}$ be such as to satisfy the commutativity condition

$$0 = \left[L_1 - \frac{\partial}{\partial t},\ L_2 - \frac{\partial}{\partial y}\right] \Leftrightarrow [L_1, L_2] = \frac{\partial L_1}{\partial y} - \frac{\partial L_2}{\partial t}. \tag{3.4}$$

We assume that $n \geqslant m$; then, since the right-hand side of the last equation contains an operator of order less than or equal to $n - 1$ (it should be borne in mind that all the operators under consideration have constant diagonal matrices as their leading coefficients), it follows that L_1 and L_2 are commutative to within operators of order $n - 1$.

We can easily verify that the restricted commutativity of the operators L_1 and L_2 is sufficient for carrying out the proof of Corollary 3 to Theorem 1.2; we have the following.

LEMMA 3.3. A system of nonlinear equations in the coefficients of the operators L_1 and L_2, which is equivalent to Eq. (3.4), will be equivalent to a family of systems of equations in the coefficients of the operator L_1 alone, parametrized by sets of complex numbers A_s^i, $0 \leqslant s \leqslant m$, $i = 1, \ldots, l$.

The corresponding equations in the matrix elements $u_\alpha^{ij}(x, y, t)$ are called Zakharov–Shabat equations.

For every nonsingular complex curve $\mathfrak{R}$ of genus g with fixed local parameters $z_j(P)$, in a neighborhood of the indicated points P_j we construct (setting the polynomials $q_j(k)$ equal to $q_j(k) = kx + Q_j(k)y + R_j(k)t$), for every divisor $D \geqslant 0$ of degree $g + l - 1$, a function $\psi(x, y, t, P) \in \Lambda(x, y, t, D)$ normalized as usual at the points P_j.

Regarding y and t in turn as parameters and making use of Theorem 3.1, we find the following.

COROLLARY. There exist unique operators

$$L_1 = \sum_{\alpha=0}^{n} u_\alpha(x, y, \cdot) \frac{\partial^\alpha}{\partial x^\alpha}, \quad L_2 = \sum_{\beta=0}^{m} v_\beta(x, y, t) \frac{\partial^\beta}{\partial x^\beta}$$

such that $(L_1 - \partial/\partial t)\psi(x, y, t, P) = 0$, $(L_2 - \partial/\partial y)\psi(x, y, t, P) = 0$, where n and m are the degrees of the polynomials $R_j(k)$ and $Q_j(k)$, respectively. The operators L_1 and L_2 depend only on the class of the divisor D.

Since $\left[L_1 - \frac{\partial}{\partial t}, L_2 - \frac{\partial}{\partial y}\right]\psi(x, y, t, P) = 0$, it follows that $\left[L_1 - \frac{\partial}{\partial t}, L_2 - \frac{\partial}{\partial y}\right] = 0$.

COROLLARY. The coefficients of the operator L_1 constructed for the set $(\mathfrak{R}, P_1, \ldots, P_l, D, z_j(P), Q_j(k), R_j(k))$ are solutions of the Zakharov–Shabat equations.

Remark. The set of constants parametrizing a Zakharov–Shabat equation is defined as follows. If $z_j'(P)$ is a local parameter in which the polynomial $R_j'(k)$ equivalent to $R_j(k)$ is equal to k^n, then the A_s^j are equal to the coefficients of the polynomial $Q_j'(k) \sim Q_j(k)$.

Let us consider the solutions of Zakharov–Shabat equations which are not dependent on one of the variables y, t (e.g., y), i.e., solutions of equations equivalent to the opera-

$$\left[L_1 - \frac{\partial}{\partial t}, L_2\right] = 0. \tag{3.5}$$

The coefficients of L_1 and L_2 depend on x and t. Among such equations is the Korteweg–de Vries equation.

Suppose that on the curve $\mathfrak{R}$ there exists a function E(P) with poles of multiplicity n at the points $P_1, \ldots, P_l$. As local parameters we take the functions $z_j(P) = \sqrt[n]{E(P)}$. By Theorem 3.1, the set of polynomials $R_j(k)$ defines for each class of divisors (D) an algebraic differential operator $(L_1 - \partial/\partial t)$ and its eigenfunction $\psi(x, t, P)$. Regarding t as a parameter, we see that, as in Theorem 2.2, to the function E(P) there corresponds under the homomorphism λ an operator $L_2 = \sum_{\beta=0}^{m} v_\beta(x,t) \frac{\partial^\beta}{\partial x^\beta}$ such that $L_2\psi(x, t, P) = E(P)\psi(x, t, P)$.

COROLLARY. The coefficients of the operators L_1 and L_2 constructed for $(\mathfrak{R}, E(P), D, R_j(k))$, satisfy Eqs. (3.5).

4. EXPLICIT FORMULAS AND EXAMPLES

It follows from Eqs. (3.3) [or (1.4)] that the matrix elements of the coefficients of algebraic operators are rational functions of the matrix elements ξ_s^{ij}.

To find the matrices ξ_s, we express in terms of the Riemann θ-function the generators of the linear space $\Lambda(q, D)$, after which we obtain formulas for the coefficients of the expansion of the corresponding functions in a neighborhood of the points P_j.

We fix on the nonsingular algebraic curve $\mathfrak{R}$ of genus g a basis of cycles

$$a_1, \ldots, a_g, \; b_1, \ldots, b_g$$

with a matrix of intersections $a_i \circ a_j = b_i \circ b_j = 0, a_i \circ b_j = \delta_{ij}$. We introduce a basis of holomorphic differentials ω_i on $\mathfrak{R}$, normalized by the conditions $\oint_{a_i} \omega_k = \delta_{ik}$. We denote by B the matrix of b-periods, $B_{ik} = \oint_{b_i} \omega_k$. It is known that this is symmetric and has a positive definite imaginary part.

The integral combinations of vectors in C^g with coordinates δ_{ik} and B_{ik} form a grid determining the complex torus $J(\mathfrak{R})$ which we call the Jacobi manifold of the curve.

Let P_0 be an indicated point on $\mathfrak{R}$; then the mapping $\omega: \mathfrak{R} \to J(\mathfrak{R})$ is defined. The coordinates of the vector $\omega_k(P)$ are equal to $\int_{P_0}^{P} \omega_k$.

For the matrix of b-periods, as for any matrix with a positive definite imaginary part, we can construct an entire function of g complex variables:

$$\theta(u_1, \ldots, u_g) = \sum_{k \in Z^g} \exp\{\pi i (Bk, k) + 2\pi i (k, u)\},$$

where $(k, u) = k_1u_1 + \ldots + k_gu_g$.

This function has the following readily verifiable properties:

$$\begin{gathered} \theta(u_1, \ldots, u_j + 1, \ldots, u_g) = \theta(u_1, \ldots, u_j, \ldots, u_g), \\ \theta(u_1 + B_{1k}, \ldots, u_g + B_{gk}) = \exp\{-\pi i B_{kk} - 2\pi i u_k\} \theta(u_1, \ldots, u_g). \end{gathered} \tag{4.1}$$

In addition, for any nonspecial effective divisor $D = \sum_{j=1}^{g} P_j$ of degree g, there exists a vector W(D) such that the function $\theta(\omega(P) + W(D))$, defined on $\Re$, cut along the cycles a_i, b_j, has exactly g zeros, which coincide with the points P_j (see [18]);

$$W_k(D) = -\sum_{j=1}^{g} \omega_k(P_j) + \frac{1}{2} - \frac{1}{2} B_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{g} \oint_{a_j} \left(\int_{P_0}^{t} \omega_k \right) \omega_j, \quad t \in a_j.$$

For any set of polynomials $q_1(k), \ldots, q_l(k)$ there exists a unique Abelian differential of the second kind (see [12]) ω (for the sake of simplicity, we do not write the index q for this) which has a singularity at the indicated point P_j on $\Re$ of the form $dq_j(z_j^{-1})$ in the local parameter z_j and is normalized by the conditions $\oint_{a_i} \omega = 0$.

LEMMA 4.1. Let $\tilde{D}$ be an arbitrary effective nonspecial divisor of degree g; then the function

$$\psi(q, P) = \exp\left(\int_{P_0}^{P} \omega \right) \frac{\theta(\omega(P) + W(\tilde{D}) + V)}{\theta(\omega(P) + W(\tilde{D}))}, \tag{4.2}$$

where $V = (V_1, \ldots, V_g)$ and $V_j = \frac{1}{2\pi i} \oint_{b_j} \omega$, is a generator of the one-dimensional space $\Lambda(q, \tilde{D})$.

Proof of the lemma can be obtained by simply verifying the properties of the function $\psi(q, P)$. It follows directly from properties (4.1) that the right-hand side of Eq. (4.2) correctly defines a function on $\Re$; i.e., its values as we go around the cycles a_i, b_j remain unchanged. In a neighborhood of P_j, the function $\psi(q, P)$ has an essential singularity of the required kind, and a divisor of its poles coincides with $\tilde{D}$. (It should be noted that formulas of this kind were first obtained by Its [19].)

For the eigenfunctions of operators whose coefficients satisfy the Zakharov–Shabat equations, the polynomials $q_j(k)$ have the form $q_j(k) = k(x - x_0) + Q_j(k)(y - y_0) + R_j(k) \cdot (t - t_0)$. The corresponding differential ω and its periods are equal to

$$\omega = \omega_2 (x - x_0) + \omega_Q (y - y_0) + \omega_R (t - t_0),$$
$$V = U_1 (x - x_0) + U_2 (y - y_0) + U_3 (t - t_0),$$

where ω_2, ω_Q, and ω_R are normalized differentials with singularities at P_j of the form $-dz/z^2$, $d(Q(1/z))$, $d(R(1/z))$, and $2\pi i U_1$, $2\pi i U_2$, $2\pi i U_3$ are the vectors of their b-periods.

As a local parameter $z_j(P)$ we take the function $\left(\int_{P_0}^{P} \omega_2 \right)^{-1}$. In this, the function $\omega(P)$ can be expanded in a series:

$$\omega(P) = \omega(P_j) + \mathcal{V}_{1j} z_j(P) + \ldots \mathcal{V}_{nj} z_j^n(P) + \ldots,$$

where $2\pi i \mathcal{V}_{nj}$ is the vector of the b-periods of the normalized Abelian differential with a unique singularity at P_j of the form dz/z^{n+1} (see [12]).

COROLLARY. The coefficient $\xi_s^j(x, y_1, t_1)$ of z_j^s in the expansion in the neighborhood of P_j of the functions

$$\psi(x, y, t, P)\, \psi^{-1}(x_0, y_1, t_1, P)\, e^{-k(x-x_0)}$$

is equal to

$$\frac{1}{s!} \frac{d^s}{dz^s} \left. \frac{\theta\left(\sum_{m=0}^{s} \mathcal{V}_{mj} z^m + U_1(x - x_0) + U_2(y - y_0) + U_3(t - t_0) + W(\tilde{D}) \right)}{\theta\left(\sum_{m=0}^{s} \mathcal{V}_{mj} z^m + U_2(y_1 - y_0) + U_3(t_1 - t_0) + W(\tilde{D}) \right)} \right|_{z=0}. \tag{4.3}$$

We construct the matrix $\tilde{\xi}_s = \tilde{\xi}_s^{ij}(x, y, t)$ for the effective divisor D of degree $g + l - 1$, $D = P_1 + \ldots + P_{g+l-1}$, substituting for $\tilde{D}$ in formula (4.3) the divisors $D_i = P_1 + \ldots + P_{g-1} + P_{g+i}$.

LEMMA 4.2. The matrices ξ_s, which define from Eqs. (3.3) the coefficients of the algebraic operators, are equal to

$$\xi_s = \tilde{\xi}_0^{-1}\tilde{\xi}_s. \tag{4.4}$$

Proof. The functions $\tilde{\psi}_i(x, y, t, P)$, given by formulas (4.2), into which the divisors D_i have been substituted instead of $\tilde{D}$, form a basis for the space $\Lambda(x, y, t, D)$. The vector function $\tilde{\psi}(x, y, t, P)$ with such coordinates differs from the vector function $\psi(x, y, t, P)$ appearing in the preceding section only in the normalization at the points P_j. Therefore,

$$\psi(x, y, t, P) = \tilde{\xi}_0^{-1}\tilde{\psi}(x, y, t, P),$$

which proves Eq. (4.4).

The vectors U_1, U_2, U_3 give us rectilinear windings on the Jacobian torus of the curve $\mathfrak{R}$.

COROLLARY 1. The Zakharov–Shabat equations constructed in Sec. 3 are conditionally periodic functions of their arguments.

In order to obtain the formulas for the matrices $\xi_s(x)$ determining the solutions of the Novikov equations, it is sufficient to set $U_2 = U_3 = 0$ in (4.3).

COROLLARY 2. Almost all solutions of the Novikov equations are conditionally periodic functions.

Now let us take a few simple examples.

Example 1. For operators with scalar coefficients, Eqs. (3.4) are nontrivial, beginning with n = 3, m = 2;

$$L_2 = v_2\frac{\partial^2}{\partial x^2} + v_0(x, y, t), \quad L_1 = u_3\frac{\partial^3}{\partial x^3} + u_1(x, y, t)\frac{\partial}{\partial x} + u_0(x, y, t).$$

The coefficient $v_0(x, y, t)$ is equal to $(2/3)u_1(x, y, t) + h$. The corresponding Zakharov–Shabat equations have the form

$$\alpha\frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial x^2} - 2\frac{\partial u_0}{\partial x}, \quad \beta\frac{\partial u_0}{\partial y} - \frac{2\alpha}{3}\frac{\partial u_1}{\partial t} = -\frac{\partial^2 u_0}{\partial x^2} + \frac{2}{3}\frac{\partial^3 u_1}{\partial x^3} + \frac{2}{3}u_1\frac{\partial u_1}{\partial x},$$

where $\alpha = 1/v_2$, $\beta = 1/u_3$.

Eliminating $u_0(x, y, t)$ from this system, we find that the coefficient $v = v_0(x, y, t)$ satisfies the Kadomtsev–Petviashvili equation (see [20])

$$\frac{3}{4}\beta^2\frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x}\left\{\alpha\frac{\partial v}{\partial t} + hv_x + \frac{1}{4}(v_{xxx} + 6vv_x)\right\} = 0.$$

In order to obtain the solutions of this equation, we must set $Q(k) = v_2k^2 + c$, $R(k) = u_3k^3 + c_1k + c_2$. From Eqs. (3.3) it follows that

$$v_0(x, y, t) = -2\frac{\partial}{\partial x}\xi_1(x, y, t) + c, \quad u_1(x, y, t) = -3\frac{\partial}{\partial x}\xi_1(x, y, t) + c_1,$$

and therefore, $h = c - (2/3)c_1$.

By Lemma 4.2, the coefficient $\xi_1(x, y, t)$ is equal to

$$\frac{d}{dz}\ln\frac{\theta(\mathcal{V}_2 z + U_1(x - x_0) + U_2(y - y_0) + U_3(t - t_0) + W(D))}{\theta(\mathcal{V}_2 z + U_2(y - y_0) + U_3(t - t_0) + W(D))}\bigg|_{z=0}.$$

From the definition of the vectors $\mathcal{V}_2$ and U_1 it follows that $\mathcal{V}_2 = -U_1$, and therefore, we finally find that the functions

$$c + 2\frac{\partial^2}{\partial x^2}\ln\theta(U_1(x - x_0) + U_2(y - y_0) + U_3(t - t_0) + W) \tag{4.5}$$

are solutions of the Kadomtsev–Petviashvili equation.

In those cases when on the curve $\mathfrak{R}$ there exist functions with singularities of second or third order at the point P, which are equivalent to the equations $U_2 = 0$, $U_3 = 0$, the function (4.5) will satisfy either the Korteweg–de Vries equation

$$\alpha \frac{\partial v(x, t)}{\partial t} + hv_x + \frac{1}{4}(v_{xxx} + 6vv_x) = 0,$$

or the equation

$$\frac{3}{4}\beta^2 \frac{\partial^2 v}{\partial y^2} + h\frac{\partial^2 v}{\partial x^2} + \frac{1}{4}\frac{\partial^4 v}{\partial x^4} + \frac{3}{2}\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial x}\right) = 0,$$

which for $h = -\frac{3}{4}\beta^2 = \pm 1$ is a variant of the equation of a nonlinear string (see [21]).

(If in (4.5) we set $U_2 = 0$, we arrive at the Matveev–Its formula [15].)

Example 2. For the first-order matrix operators

$$L_1 = u_1 \frac{d}{dx} + u_0(x, y, t), \quad u_1^{ij} = c_i\delta_{ij}, \quad L_2 = v_1\frac{d}{dx} + v_0(x, y, t), \quad v_1^{ij} = d_i\delta_{ij}$$

Eqs. (3.4) are nontrivial beginning with $l = 3$.

From Eqs. (3.3) it follows that $u_0 = [u_1, \xi_1]$, $v_0 = [v_1, \xi_1]$. In [2] it was noted that for the additional symmetry conditions $\xi_1 = \xi_1^+$ after a relatively simple substitution, Eqs. (3.4) for $l = 3$ reduce to equations describing the resonance interaction of three waves in a nonlinear medium.

In order to obtain the solutions of these equations by our scheme, we must have existing on $\mathfrak{R}$ the antiinvolutions T_1, T_2, such that $T_1 \circ T_2 = T_2 \circ T_1$, $T_1(P_1) = P_1$, $T_2(P_2) = P_2$, $T_1(P_2) = P_3$, $T_2(P_3) = P_1$. If the divisor D is invariant with respect to T_i, then, as we can readily convince ourselves, $\xi_1 = \xi_1^-$.

In this case, general formula (4.3) can be simplified. We select the cycles a_j, b_j in such a way that when T_i is applied, we have $T_i^* a_j = -a_j$, $T_i^* b_j = b_j$; then $\mathscr{V}_{2j} = \mathscr{V}$ for all j, $U_1 = 3\mathscr{V}$, $U_2 = c\mathscr{V}$, $U_3 = d\mathscr{V}$; $c = \operatorname{Sp} u_1$, $d = \operatorname{Sp} v_1$.

The coefficients of the matrix ξ_1^{ij} are given by the formula

$$\frac{1}{3}\frac{d}{dx}\frac{\theta(\omega(P_j) + (3(x - x_0) + c(y - y_0) + d(t - t_0))\mathscr{V} + W(D_i))}{\theta(\omega(P_j) + (c(y - y_0) + d(t - t_0))\mathscr{V} + W(D_i))},$$

which enables us to find ξ_1 from Eq. (4.4).

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CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS OF GROUPS OF AUTOMORPHISMS OF BRUHAT—TITS TREES

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UDC 519.46

INTRODUCTION

Bruhat and Tits [1] constructed, for an arbitrary semisimple algebraic group $\mathcal{G}$ over a local field, an object replacing the symmetric space of a real semisimple Lie group. This so-called building is a semisimplicial complex of dimension equal to the relative rank ν of $\mathcal{G}$, on which $\mathcal{G}$ acts by automorphisms. For $\nu = 1$ it happens that the group of all automorphisms of this building (analogous to the group of isometries of a symmetric group) represents a new and interesting family of locally compact groups.* We shall give a series of facts about these groups.

*For $\nu \geqslant 2$ this group actually coincides with the original semisimple group $\mathcal{G}$, and $\mathcal{G}$ can be reconstructed from its building, up to isogeny (this result was found by Tits). For $\nu = 1$ a given building $\mathcal{J}$ is connected with infinitely many groups over various fields, and Aut $\mathcal{J}$ is substantially larger than any of them.

VNIPIIStromsyr'e. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 11, No. 1, pp. 32-42, January-March, 1977. Original article submitted April 26, 1976.