$$|f_k(x)| \leqslant \frac{2}{k^2}, \tag{2}$$

$$g_k(Tx) - g_k(x) = f_k(x). \tag{3}$$

Let us estimate the integral of $\varphi_k(x)$. To do so, note that $\mu(E_i^k) \geqslant \frac{1-\varepsilon_k}{k^5}$, $1 \leqslant i \leqslant k^5$. Therefore,

$$\int_X \varphi_k(x)\,d\mu \geqslant \sum_{i=k^5-k^3+1}^{k^5} \int_{E_i^k} \varphi_k(x)\,d\mu \geqslant k^3 \cdot \frac{1}{k^2}\,\frac{1-\varepsilon_k}{k^5} = \frac{1-\varepsilon_k}{k^4}.$$

Hence

$$\int_X g_k(x)\,d\mu = k^3 \int_X \varphi_k(x)\,d\mu \geqslant \frac{1-\varepsilon_k}{k}.$$

Set $M_k = \{x\colon\ g_k(x) \neq 0\}$.

$$M_k \subseteq \bigcup_{j=0}^{k^3-1} T^{-j} G_k \subseteq \Big(\bigcup_{i=k^5-2k^3+1}^{k^5} E_i^k\Big) \cup \Big(\bigcup_{j=0}^{k^3-1} \bigcup_{i=k^5-k^3+1}^{k^5} T^{-j}(\tilde{E}_i^k \setminus E_i^k)\Big).$$

We have $\mu(M_k) \leqslant \frac{2}{k^2} + \frac{k^6}{k^8} = \frac{3}{k^2}$. Since $\sum_{k=2}^{\infty} \mu(M_k)$ converges, by the Borel–Cantelli lemma $g(x) = \sum_{k=2}^{\infty} g_k(x)$ converges μ-almost everywhere. But by virtue of (2), $f(x) = \sum_{k=2}^{\infty} f_k(x)$ converges uniformly, and f(x) is continuous. Summing over all k ≥ 2 in (3), we obtain g(Tx) − g(x) = f(x) almost everywhere. In addition, g(x) ≥ 0 and

$$\int_X g(x)\,d\mu = \sum_{k=2}^{\infty} \int_X g_k(x)\,d\mu \geqslant \sum_{k=2}^{\infty} \frac{1-\varepsilon_k}{k} = \infty,$$

since $\varepsilon_k \to 0$. This proves the theorem.

LITERATURE CITED

1. D. V. Anosov, Izv. Akad. Nauk SSSR, Ser. Matem., 37, 1259-1274 (1973).
2. P. R. Halmos, Lectures on Ergodic Theory, Chelsea Publ. Co., New York (1956).

ALGEBRAIC CURVES AND COMMUTING MATRICIAL DIFFERENTIAL OPERATORS

I. M. Krichever

In [1] we have presented the algebraic-geometric construction of the exact solutions of the Zakharov–Shabat equations which are conditionally periodic functions of their arguments. By the Zakharov–Shabat equations [2] we mean nonlinear differential equations which can be represented in the form

$$\left[L_1 - \frac{\partial}{\partial y},\ L_2 - \frac{\partial}{\partial t}\right] = 0. \tag{1}$$

The condition of the commutativity of two operators is equivalent with the presence of a "sufficiently large" (here we do not define this concept more exactly) collection of functions, simultaneously converted by them into zero. In [1] we have considered operators with scalar coefficients and we have proved that for them sufficient collections are the functions Ψ(x, y, t, P), where P is a point of a nonsingular complex curve given by its analytic properties on $\mathfrak{R}$ and having an essential singularity of a specific form at some fixed point. In the present note we consider functions which have essential singularities in l points. This brings us to oper-

G. M. Krzhizhanovskii Energetic Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 10, No. 2, pp. 75-76, April-June, 1976. Original article submitted December 22, 1975.

ators whose coefficients are $l \times l$ matrices. Without examining physically interesting examples (see [2]), we shall show that the solutions of the corresponding equations, constructed with respect to such functions, can be explicitly written in terms of Riemann's θ functions.

1. Let $\mathfrak{R}$ be a nonsingular complex curve of genus g with the distinguished points $P_1, \ldots, P_l$. We consider the functions $\Psi(x, y, t, P)$, $P \in \mathfrak{R}$, satisfying the conditions:

1. $\Psi(x, y, t, P)$ is meromorphic on $\mathfrak{R}$ outside the points P_j, the divisor D of its poles does not depend on x, y, t, is nonspecial and has degree $g + l - 1$.

2. In the neighborhood of P_j it has the form

$$\exp(k_j x + Q_j(k_j) y + R_j(k_j) t) \cdot \left(\xi_0^j + \sum_{s=1}^{\infty} \xi_s^j (x, y, t) z_j^s\right).$$

Here $z_j = 1/k_j$ is a local parameter in the neighborhood of P_j, $Q_j(k) = c_j^n k^n + \ldots + c_0^j$, $R_j(k) = b_m^j k^m + \ldots + b_0^j$ are polynomials.

As mentioned in [4], for $l = 1$ these conditions, together with the normalization $\xi_0 = 1$, determine uniquely Ψ. Similarly, for $l > 0$ the normalization $\xi_{i0} = \delta_{ij}$ determines uniquely the functions $\Psi_i(x, y, t, P)$.

THEOREM 1. There exist unique operators

$$L_1 = \sum_{\alpha=0}^{n} u_\alpha(x, y, t) \frac{d^\alpha}{dx^\alpha} \quad u \quad L_2 = \sum_{\beta=0}^{m} v_\beta(x, y, t) \frac{d^\beta}{dx^\beta}$$

such that $L_1\Phi = \frac{\partial}{\partial y}\Phi$, $L_2\Phi = \frac{\partial}{\partial t}\Phi$, where Φ is a vector whose i-th component is $\Psi_i(x, y, t, P)$.

The matrices $u_\alpha(x, y, t)$ are determined from the systems of equations

$$\sum_{\alpha=s}^{n} u_\alpha \sum_{\beta=s}^{\alpha} C_\alpha^\beta \xi_{\beta-s}^{(\alpha-\beta)} = \sum_{\gamma=s}^{n} \xi_{\gamma-s} c_\gamma.$$

The element ξ_s^{ij} of the matrix ξ is equal to the coefficient of z_j^s of the expansion of $\Psi_i(x, y, t, P)$ in the neighborhood of P_j. The matrix c_γ is equal to $c_\gamma^j \delta_{ij}$. One can find similarly the matrices $v_\beta(x, y, t, P)$.

COROLLARY 1. The operators L_1 and L_2 satisfy Eq. (1).

If on the curve $\mathfrak{R}$ there exists a meromorphic function E(p), having poles at the points P_j (the ring of these functions will be denoted by $\Lambda(\mathfrak{R}, P_1, \ldots, P_l)$), whose Laurent series expansion at P_j has principal part $Q_j(k_j)$, then $\Phi(x, y, t, P)$ can be represented in the form $\Phi_0(x, t, P)\exp(E(P)y)$.

COROLLARY 2. Under the assumptions made, we have $L_1\Phi_0 = E\Phi_0$, $L_2\Phi_0 = \frac{\partial}{\partial t}\Phi_0$ and so $[L_2, L_1] = \frac{\partial L_1}{\partial t}$.

Now, if there exists $H(P) \in \Lambda(\mathfrak{R}, P_1, \ldots, P_l)$, equivalent to $R_j(k_j)$ in P_j, then $\Phi(x, y, t, P) = \Phi_0(x, P)\exp(E(P)y + H(P)t)$.

COROLLARY 3. The function Φ_0 satisfies the equalities $L_1\Phi_0 = E\Phi_0$ and $L_2\Phi_0 = H\Phi_0$, while the operators satisfy the equation $[L_1, L_2] = 0$.

Thus, each divisor of degree $l + g - 1$ gives a homomorphism λ_D of the ring $\Lambda(\mathfrak{R}, P_1, \ldots, P_l)$ into the ring of linear differential operators with $l \times l$ matrix coefficients.

Remark. We note that the constructed solutions of the Eqs. (1), as well as λ_D, depend only on the class of the divisor D since going over to an equivalent divisor D' reduces to the multiplication of Φ by a constant matrix.

THEOREM 2. If in the commutative ring Λ of linear differential operators with matricial coefficients there exist two operators with relatively prime orders and with nonsingular leading coefficients, then there exist a curve $\mathfrak{R}$, points $P_1, \ldots, P_l$, divisor D such that λ_D gives an isomorphism between $\Lambda(\mathfrak{R}, P_1, \ldots, P_l)$ and Λ.

2. As a local parameter z_j in the neighborhood of P_j we select the function $\int_{P_0}^{P} \omega_2$, where P_0 is a fixed point, ω_2 is a normalized differential of the second kind with second-order poles at the points $P_1, \ldots, P_l$. We

denote by $2\pi i U$ the vector of its b periods (for all necessary information and missing definitions we refer to [5]) and by $2\pi i V$ and $2\pi i W$ the b periods of the differentials $\omega(Q)$ and $\omega(R)$, equivalent in P_j with $d(Q_j(1/z_j))$ and $d(R_j(1/z_j))$, respectively. We also introduce the vectors $2\pi i U^{kj}$, which are the b periods of the differentials having a unique singularity at P_j of the form $k!\frac{dz_j}{z_j^{k+1}}$.

We consider the functions $\chi_s^{ij}(x, y, t)$, given by the formulas

$$\left[\sum\prod_{k=1}^{s}\frac{1}{(k\alpha_k)!}\frac{\partial^{\alpha_1+\ldots+\alpha_s}}{\partial^{\alpha_1}\eta_{1j}\ldots\partial^{\alpha_s}\eta_{sj}}\ln\theta\left(Ux'+Vy+Wt+\sum_{k,j}U^{kj}\eta_{kj}+Z_i\right)\right]\Bigg|_{x'=\eta_{kj}=0}^{x'=x,\ \eta_{kj}=0}.$$

The summation is taken with respect to all the collections $\alpha_1, \ldots, \alpha_s$ such that $\sum_{k=1}^{s} k\alpha_k = s$. The vectors Z_i correspond by Abel's substitution to the divisors $p_1+\ldots+p_{g-1}+p_i$, $1\leqslant i\leqslant l$, where $D=\sum_{s=1}^{l+g-1}p_s$.

Explicit expressions for the matrices $\xi_s(x, y, t)$, in terms of which the solutions of the Zakharov–Shabat equations are expressed, are given by the following theorem.

THEOREM 3. We have the equality $\xi_s = \tilde{\xi}_0^{-1}\tilde{\xi}_s$, where the elements of the matrices $\tilde{\xi}_s$ are given by the equality

$$\sum_{s=0}^{\infty}\tilde{\xi}_s^{ij}z^s = \exp\left(\sum_{s=0}^{\infty}\chi_s^{ij}(x, y, t)\,z^s\right).$$

In particular, for the Kadomtsev–Petviashvili equation given in [1] we obtain that its solution is given by the formula

$$u(x, y, t) = c - 2\frac{\partial}{\partial x}\xi_1(x, y, t) = c + 2\frac{\partial^2}{\partial x^2}\ln\theta(Ux+Vy+Wt+Z).$$

LITERATURE CITED

1. I. M. Krichever, Dokl. Akad. Nauk SSSR, 227, No. 2 (1976).
2. V. E. Zakharov and A. B. Shabat, Funktsional'. Analiz i Ego Prilozhen., 8, No. 3, 35-43 (1974).
3. B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, Usp. Matem. Nauk, 31, No. 1, 55-136 (1976).
4. N. I. Akhiezer, Dokl. Akad. Nauk SSSR, 142, No. 2, 263-266 (1961).
5. É. I. Zverovich, Usp. Matem. Nauk, 26, No. 1, 68-113 (1971).