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Knot polynomials and Vassiliev's invariants

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Summary. A fundamental relationship is established between Jones' knot invariants and Vassiliev's knot invariants. Since Vassiliev's knot invariants have a firm grounding in classical topology, one obtains as a result a first step in understanding the Jones polynomial by topological methods.

Introduction

The main result in this paper is to establish a deep connection between two seminal papers about knots and links in 3-space. The first paper is by Jones [J2]. The second and more recent is the work of Vassiliev [V].

Jones' paper, published in 1986, has received considerable attention. It led to the discovery of vast new families of Laurent polynomial invariants of knots and links in 3-space. It is not even impossible that, taken together, they constitute a complete set of knot and link invariants. These are related in many ways to areas of mathematics and physics in which knotting or linking had not previously been thought to play any role, for example to the study of type II_1 factors in von Newmann algebras, and to the study of exactly solvable models in statistical mechanics.

The Jones polynomial and its relatives are computable from a knot diagram or from a closed braid representative of a knot, the computation involving in an essential way the computation of related invariants of links. The computation is not difficult for simple examples, however its complexity grows exponentially with crossing number or with braid index. The invariants are multiplicative under connected sum. At this writing they are best understood as combinatorial objects associated to a knot diagram, and the various known proofs of their topological invariance offer little insight into the underlying topology. An outstanding open

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problem about them is to find a topological interpretation. Thus we are in a situation where we could have at hand a definitive solution to the knot problem, without having any real understanding of what it means.

The Vassiliev knot invariants have both similarities with and differences from the Jones invariants. Unlike the Jones invariants, they rest on firm topological foundations, and in fact introduce a new and exciting way to look at knots which both generalizes the classical approach and adds new insight. In classical knot theory, one studies the topology of a single knot (or more generally a single link) in 3-space. For example, the Alexander polynomial of a knot, one of the most useful knot invariants derived from the classical theory, describes aspects of the integral homology of the universal abelian covering space of the complement of a knot. Vassiliev broadens this classical approach. His first change is a very natural one in the history of mathematics: instead of thinking of a knot as the image of an embedding of S^1 in R^3 he changes the point of view and focusses on the embedding itself. He then broadens his approach by introducing the space $\mathscr{M}$ of all smooth maps from S^1 to $\mathbb{R}^3$, allowing one to study not just a single knot but also the way in which distinct knots (thought of as imbeddings of S^1 into $\mathbb{R}^3$) fit together in $\mathscr{M}$. The *discriminant* Σ of $\mathscr{M}$ is defined to be the set of maps which are not embeddings. The components of $\mathscr{M}\backslash\Sigma$ are clearly in one-one correspondence with knot types. Thinking of a numerical knot invariant as a function on the components of $\mathscr{M}\backslash\Sigma$, one is led to study the cohomology of $\mathscr{M}\backslash\Sigma$.

Vassiliev introduces a system of subgroups of $\tilde{H}^0(\mathscr{M}\backslash\Sigma)$:

$$0 = G_1 \subset G_2 \subset G_3 \subset \cdots \subset \tilde{H}^0(\mathscr{M}\backslash\Sigma),$$

where $\tilde{H}^*$ is reduced cohomology with integer coefficients and G_i is free abelian of finite rank. Note: for our purposes it will be more convenient to work with cohomology over the rationals. This will not change the underlying mathematics, so from now on we assume we are working with rational coefficients. The evaluation of an element in G_i/G_{i-1} on the component of $\mathscr{M}\backslash\Sigma$ corresponding to an oriented knot type $\mathbf{K}$ gives us a rational number $v_i(\mathbf{K})$ associated to $\mathbf{K}$. This is a *Vassiliev invariant of order i*. Let l_i be the rank of G_i/G_{i-1}. There are l_i linearly independent such invariants for each i. The first few values of l_i are $l_1 = 0$, $l_2 = l_3 = 1$ and $l_4 = 3$. Notice that the zero-dimensional cohomology classes of the space $\mathscr{M}\backslash\Sigma$ distinguish knot types in $\mathbb{R}^3$. It is not known whether the Vassiliev subgroups are close enough to the full group to determine a complete system of knot invariants. Vassiliev conjectures (see 6.1 of [V]) that they do. At the very least, his approach gives a framework in which one can think about the problem.

One computes Vassiliev's invariants, like the Jones invariants, from a knot diagram via crossing changes to the unknot, however the computation involves in an essential way the computation of related invariants for special types of knotted graphs rather than of links. The combinatorics of the computation are much more difficult than the corresponding computation for the Jones polynomial. For example, the reader who studies [V] can anticipate considerable difficulty in reproducing the calculations given there, even though they are restricted to knots which have diagrams with $\leqq 7$ crossings and to order $\leqq 4$. It is not clear from the work in [V] whether non-trivial Vassiliev invariants of order $i \geqq 5$ exist.

Our main result is very easy to describe. We state it now, in the special case of the one-variable Jones polynomial, which is a Laurent polynomial in the variable t:

Theorem. *Let* $\mathbf{K}$ *be a knot and let* $J_t(\mathbf{K})$ *be its Jones polynomial. Let* $U_x(\mathbf{K})$ *be obtained from* $J_t(\mathbf{K})$ *by replacing the variable* t by e^x. *Express* $U_x(\mathbf{K})$ *as a power series in* x:

$$U_x(\mathbf{K}) = \sum_{i=0}^{\infty} u_i(\mathbf{K})x^i.$$

Then $u_0(\mathbf{K}) = 1$ *and each* $u_i(\mathbf{K}), i \geqq 1$ *is a Vassiliev invariant of order* i.

We will also prove corresponding assertions for the HOMFLY and Kauffman polynomials ([FHLMOY] and [K1], [K2]), and work out some interesting consequences about the relationships between these invariants and Vassiliev's.

As an immediate corollary, we obtain a result which seems to be inaccessible by the combinatorial methods in [V]:

Corollary. *Non-trivial Vassiliev invariants of every order exist.*

We now describe, in very general terms, the meaning of $v_i(\mathbf{K})$. If $\Phi \in \mathcal{M}$ determines a knot type $\mathbf{K}$, there is a path Φ_t in $\mathcal{M}$ which joins $\mathbf{K}$ to the unknot $\mathbf{O}$ and Φ to a representative of $\mathbf{O}$. The path may be chosen so that for all but a finite number of values of t the map Φ_t is an embedding. We may further assume that where Φ_t fails to be an embedding it has a single transverse double point. See Fig. 1. Let $\mathcal{M}_1 \subset \Sigma$ be the subspace of maps in Σ which have at least one transverse double point. Iteratively, let $\mathcal{M}_k$ be the space of all maps $\Phi \in \mathcal{M}$ which have k transverse double points (and possibly other singularities too), so that $\mathcal{M} \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \cdots$. Call a map $\Phi \in \mathcal{M}_k$ a *k-embedding* if its only singularities are the k transverse double points. Let Σ_k be the subspace of all maps $\Phi \in \mathcal{M}_k$ which are not k-embeddings. The ith Vassiliev invariant $v_i(\mathbf{K})$ is an invariant of $\mathbf{K}$ which takes into account topological information about the embedded graph types associated to the cells in $\mathcal{M}_1 \backslash \Sigma_1, \mathcal{M}_2 \backslash \Sigma_2, \ldots, \mathcal{M}_i \backslash \Sigma_i$, in the collection of paths from $\mathbf{K}$ to the unknot $\mathbf{O}$.

Here is a guide to the paper. In §1 we review Vassiliev's results. Our review culminates in §1.5 and §1.6 with a description of the combinatorial scheme given in [V] for the computation of the invariants.

In §2 we introduce the knotted graphs which are an essential part of the picture and describe their "configurations". We then show that Vassiliev's invariants are determined (much like the Jones invariants) by a set of axioms and initial data. The axioms are very simple, but the initial data is the heart of the matter. Without explaining how to compute it, we give the initial data in the cases $i = 2, 3$ and 4. We then illustrate by an example how one uses the axioms and the initial data to compute one of Vassiliev's invariants.

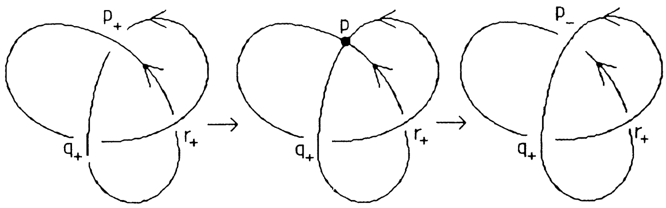


Fig. 1

In §3 we treat the construction of the initial data. We prove that the determination of the initial data is equivalent to solving a particular system of homogeneous linear equations in a very large number of unknowns. The unknowns are the Vassiliev invariants of a special set of knotted graphs. The equations express relationships between the invariants of these graphs. Roughly speaking, it will be seen that the axioms and initial data of Vassiliev's invariants incorporate in a single setting the crossing change formulas and initial data which determine the HOMFLY and Kauffman polynomials.

To begin to construct the initial data, one sets up a table $\mathbf{A}(i)$ which Vassiliev calls an "actuality table of order i". The table $\mathbf{A}(i)$ has a row $\mathbf{R}(i,j)$ for each j with $i \geqq j \geqq 2$. The top row $\mathbf{R}(i,i)$ consists of a list of certain combinatorial patterns known as "admissible $[i]$-configurations". The row $\mathbf{R}(i,j)$ for each $j = 2, \ldots, i-1$ consists of a list of immersions in $\mathscr{M}$. Each of them "respects" the associated "admissible $[j]$-configuration". These immersions are also associated with unknown indices which are put together into a column vector $\mathbb{X}_j$. We show in §3 that the construction of the initial data for a Vassiliev invariant of order i in terms of constructing an actuality table of order i is equivalent to solving a system of linear equations:

$$\begin{cases} \mathbb{M}_i \mathbb{X}_i = 0, \\ \mathbb{M}_j \mathbb{X}_j = \mathbb{N}_{i,j}, \quad 1 \leqq j \leqq i-1. \end{cases}$$

Here the integer matrices $\mathbb{M}_i$ and $\mathbb{M}_j$ for $1 \leqq j \leqq i-1$ are all of the same nature and depend only on certain combinatorial patterns. The components of the column vector $\mathbb{N}_{i,j}$ are linear functions (depending on the choice of the immersions in the table) of components of $\mathbb{X}_{j+1}, \ldots, \mathbb{X}_i$ with integer coefficients. Thus, our system is actually a much larger system of homogeneous linear equations in the combined set of variables:

$$\mathbb{M}^i \mathbb{X}^i = 0,$$

where $\mathbb{X}^i$ is a column vector made up out of the previously defined column vectors $(\mathbb{X}_i, \mathbb{X}_{i-1}, \ldots, \mathbb{X}_1)$. The dimension of the space of solutions of this combined system of homogeneous linear equations is $l_1 + \ldots + l_i$, where l_j is the number of linearly independent Vassiliev invariant of order $j \leqq i$.

In §4 we use the knowledge gained in §2 and §3, to study knot polynomials in the Vassiliev setting. In Theorems 4.1 and 4.8 we derive generalized versions of the theorem stated above for the HOMFLY and Kauffman polynomials. Corollary 4.2 is the result we stated above. Corollaries 4.3, 4.4 and 4.7 and Example 4.5 are other consequences of the main theorems.

In §5 we prove that the unknotting number of a knot cannot be a Vassiliev invariant. This result seemed surprising to us initially, because the unknotting number of a knot has such a natural interpretation in the Vassiliev setting, as the fewest number of passages across the discriminant Σ in a path in $\mathscr{M}$ from the given knot $\mathbf{K}$ to the unknot $\mathbf{O}$. However, it leaves open the possibility that unknotting number is the limit of a sequence of Vassiliev invariants. The question of whether every numerical knot invariant is obtained as the limit of such a sequence is clearly of deep importance.

Remarks.

1. The second author has proved, in a separate paper [L], that the axioms and initial data actually determine much more, i.e. every knot invariant which arises in

the setting of quantum groups. A different proof has also been given by the first author in [B].

2. Ted Stanford [S] has used the axiomatic approach which is developed in Section 2 of this paper to define Vassiliev-like invariants for links and for rather general types of embedded graphs. His invariants reduce to Vassiliev invariants in the special case of knots. His methods, however, are very different from those used by Vassiliev in [V]. They involve a direct application of the axiomatic approach which we introduce here in Section 3 and use generalized Reidemeister moves and other techniques from PL topology.

3. Approaching the knot problem from yet another direction (via Witten's formulation and perturbative Chern–Simons theory), Bar-Natan has proved in [BN] that associated to every irreducible representation of a simple Lie algebra there is a solution to the system of equations $\mathbb{M}_i\mathbb{X}_i = 0$ which were described above. He was not, however, able to prove that his solutions can always be extended to solutions to the full system $\mathbb{M}^i\mathbb{X}^i = 0$. Thus his results fall short of giving knot invariants.

4. M. Kontsevich has recently proved the very striking result that every solution to the system of equations $\mathbb{M}_i\mathbb{X}_i = 0$ extends to solutions to the full system $\mathbb{M}^i\mathbb{X}^i = 0$, thereby proving that Bar-Natan's work does indeed give knot invariants. His methods, however, do not yield a constructive procedure for filling in the actuality tables. Thus, at this writing, the method we use to prove Corollary 4.2 is essentially the only known way (apart from a difficult case-by-case computation for $i = 2, 3, 4, 5, \ldots.$) to construct actuality tables. A fundamental open problem in the area is to prove Kontsevich's theorem by direct combinatorial methods.

1 A review of Vassiliev's work

1.1 Basic ideas

The space $\mathcal{M}$ which is the basic object of interest is infinite-dimensional, and Vassiliev's work begins with the construction of certain finite-dimensional approximations to $\mathcal{M}$. The first step is to pass from maps with domain S^1 to maps with domain $\mathbb{R}^1$. Admissible maps are then required to have a fixed assymptotic direction at infinity. Let $\Gamma^d \in \mathcal{M}$ be the space of maps from $\mathbb{R}^1$ to $\mathbb{R}^3$ given by

$$t \mapsto (p_1(t), p_2(t), p_3(t)),$$

where the $p_i(t)$'s are polynomials of the form

$$t^{d+1} + a_1 t^d + \ldots + a_d t,$$

with d even.Thus the maps in Γ^d tend assymptotically to (1, 1, 1) and (− 1, − 1, − 1) as t tends to infinity. The space Γ^d may be identified with Euclidean space of dimension $3d$. Deforming Γ^d by a small perturbation, if necesssary, we may assume that the resulting subspace (which we call a *generic* Γ^d) is in general position with respect to Σ. It will still be a $3d$-dimensional affine space.

Notice that for fixed d, the maps in Γ^d necessarily yield finitely many knot types because a polynomial only has a finite number of maxima and minima. Notice also that we cannot embed Γ^d in Γ^{d+1} in the obvious way because the coefficient of t^{d+1} is 1 for a polynomial in Γ^d but is not 1 in general for a polynomial in Γ^{d+1}. So, instead, we reparametrize the real line by setting $t = s^3 + s$ to obtain an embedding $\Gamma^d \to \Gamma^{3d+2}$. Thus if $\{d_n\}$ is a sequence of integers determined by the recursive formula

$$d_{n+1} = 3d_n + 2,$$

which begins with an even integer, then we have a sequence $\{\Gamma^{d_n}\}$ of subspaces of $\mathscr{M}$ and embeddings $I_n : \Gamma^{d_n} \to \Gamma^{d_{n+1}}$. The Weierstrass approximation theorem then implies that

$$\tilde{H}^0(\mathscr{M}\backslash\Sigma) \cong \underleftarrow{\lim}\ \tilde{H}^0(\Gamma^{d_n}\backslash\Gamma^{d_n} \cap \Sigma),$$

where $\tilde{H}^*$ means reduced cohomology with rational coefficients.

If we have cohomology classes $v^{(n)} \in \tilde{H}^0(\Gamma^{d_n}\backslash\Gamma^{d_n} \cap \Sigma)$ which stabilize when $n \to \infty$, then $v = \underleftarrow{\lim}\ v^{(n)}$ is a cohomology class in $\tilde{H}^0(\mathscr{M}\backslash\Sigma)$. Since $\tilde{H}^*$ is cohomology with rational coefficients, we thus obtain a rational knot invariant $v(\mathbf{K})$, with $v(\mathbf{O}) = 0$.

The fact that Γ^d is homeomorphic to R^{3d} allows us to apply the Alexander duality theorem, so:

$$\tilde{H}^0(\Gamma^d\backslash\Gamma^d \cap \Sigma) \cong \bar{H}_{3d-1}(\Gamma^d \cap \Sigma),$$

where $\bar{H}_*$ means closed homology, i.e. the homology of the one point compactification of the space, modulo the compactifying point. Thus we are led to the study of the closed homology $\bar{H}_*(\Gamma^d \cap \Sigma)$, for generic Γ^d.

1.2 The configuration of the discriminant

The discriminant contains maps with very complicated types of singularities. The set of ideas which we now review help Vassiliev to order them in a systematic fashion. Let A be a finite sequence of $\# A$ positive integers $\{a_1, \ldots, a_{\# A}\}$ with $a_1 \geqq a_2 \geqq \cdots \geqq a_{\# A} \geqq 2$. Let $|A| = a_1 + a_2 + \cdots + a_{\# A}$. An *A-configuration* is a family of $|A|$ distinct points on the real line $\mathbb{R}^1$, partitioned into groups whose cardinalities are $a_1, a_2, \ldots, a_{\# A}$. Let ζ be a non-negative integer. An (A, ζ)-*configuration* is an A-configuration, together with an additional family of ζ distinct points on $\mathbb{R}^1$, some of which may coincide with points in the A-configuration.

The *complexity* $k = k(A, \zeta)$ of an (A, ζ)-configuration is the number $|A| + \zeta - \# A$. It is well-defined on equivalence classes. Three cases will be of special interest later:

1. $A = \{2, 2, \ldots, 2\}$, $\# A = i$ and $\zeta = 0$. An example is the "[4]-configuration" in Fig. 2.
2. $A = \{3, 2, \ldots, 2\}$, $\# A = i$ and $\zeta = 0$. An example is the "$\langle 4 \rangle$-configuration" in Fig. 2.
3. $A = \{2, 2, \ldots, 2\}$, $\# A = i - 1$, $\zeta = 1$ and the only additional point does not coincide with any of $2(i-1)$ points constituting the corresponding A-configuration. An example is the "[3]*-configuration" in Fig. 2.

These types of (A, ζ)-configurations are called *non-complicated* (A, ζ)-configurations.

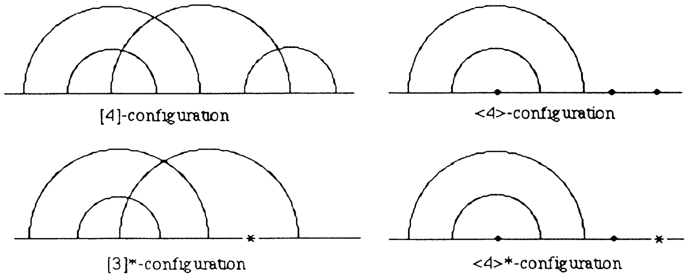


Fig. 2

Let J be an (A, ζ)-configuration. A map $\theta: \mathbb{R}^1 \to \mathbb{R}^3$ *respects* J if it has singularities (where derivatives vanish) at each of the ζ distinguished points of J, and if in addition the points in each of the $\# A$ groups in the underlying A-configuration are mapped by θ to a single point in $\mathbb{R}^3$, so that θ has $\# A$ multiple points of order $a_1, a_2, \ldots, a_{\# A}$. For example, if $K = \theta(\mathbb{R}^1)$ is a knot, and if $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ is a projection which defines an i-crossing diagram for K, then $\pi\theta$ will respect a non-complicated (A, ζ)-configuration of type 1.

Two (A, ζ)-configurations are *equivalent* if one can be sent into the other by an orientation-preserving diffeomorphism of $\mathbb{R}^1$ which sends points in one configuration to points in the other configuration. We shall regard equivalent configurations as being identical. The space of (A, ζ)-configurations which are equivalent to a given one is an open cell whose dimension is equal to the number of geometrically distinct points in its definition. It is at most $|A| + \zeta$.

Recall that Γ^d has dimension $3d$. Let $\mathcal{M}(\Gamma^d, J)$ be the set of maps in Γ^d which respect J. They are determined by $3k$ linear equations, where k is the complexity $k(A, \zeta)$. For a generic Γ^d we may assume that these $3k$ equations are linearly independent for almost all (A, ζ)-configurations equivalent to J. Notice that the space of all (A, ζ)-configurations equivalent to J has dimension at most $|A| + \zeta \leqq 2k$. From this observation and the weak transversality theorem in singularity theory, we have the following lemma:

Lemma 1.1 (Lemma 2.1.2 of [V]) *For a generic Γ^d and any (A, ζ)-configuration J, the following hold:*

(i) *For almost all $J' \sim J$* the set *$\mathcal{M}(\Gamma^d, J') \subset \Gamma^d$ has codimension $3k$. In particular, if $k > d$, it is empty.*

(ii) *Assume $k \leqq d$. Then, in the set of all $J' \sim J$:*
(a) *$\{J'; \mathcal{M}(\Gamma^d, J') = \emptyset\}$ has codimension $\geqq 3d - 3k + 1$;*
(b) *$\{J';$ codim $(\mathcal{M}(\Gamma^d, J')) = 3k - i, i \geqq 1\}$ has codimension $\geqq i(3d - 3k + i + 1)$. In particular, if $k \leqq (3d + 1)/5$, the codimension of $\mathcal{M}(\Gamma^d, J)$ for any J of complexity k* is *$3k$.*

(iii) *Assume $k > d$. Then $\{J' \sim J;$ dim $(\mathcal{M}(\Gamma^d, J')) \geqq 0\}$ is of codimension $\geqq (s + 1)(3k - 3d + s)$. In particular, the set of all J'* such that *$\mathcal{M}(\Gamma^d, J') \neq \emptyset$ is of codimension $\geqq 3k - 3d$. It is empty when $k > 3d$.*

In view of Lemma 1.1, we may take $k \leqq (3d + 1)/5$ as our stable range.

1.3 The resolvent of the discriminant and a spectral sequence

A *resolvant* Ω for $\Gamma^d \cap \Sigma$ is an auxiliary space whose closed homology coincides with that of $\Gamma^d \cap \Sigma$. Let $\tau:(t, t') \mapsto (t', t)$ be an involution on $\mathbb{R}^2$. Let $\Psi = \mathbb{R}^2/\tau$. The following lemma allows us to construct a resolvent.

Lemma 1.2 (Lemma 2.3.1 of [V]) *For any integer $n > 0$, there exists an integer $N > 0$ and an embedding $\lambda: \Psi \to \mathbb{R}^N$ with the property that no set of n distinct points in Ψ are mapped into an $(n-2)$-dimensional affine plane in $\mathbb{R}^N$.*

Let L be a finite set of points in $\mathbb{R}^1$. A *generating family* for L is a non-ordered family $\{(t_1 \mp t'_1), \ldots, (t_r \mp t'_r)\}$ of distinct, non-ordered pairs of points of $\mathbb{R}^1$ such that for all maps $\theta: \mathbb{R}^1 \to \mathbb{R}^3$, the following statements are equivalent:

(i) θ identifies all the points in L, and
(ii) For any $j = 1, 2, \ldots, r$, θ identifies t_j and t'_j.

For example, if $L = \{1, 2, 3\}$, then $T = \{(1, 2), (2, 3)\}$ is a generating family of L.

A pair (T, Υ), where $T = \{(t_1, t'_1), \ldots, (t_r, t'_r)\}$ and Υ is a certain set of points $\upsilon_1, \ldots, \upsilon_\zeta$ in $\mathbb{R}^1$, is a *generating family* of an (A, ζ)-configuration if T is the union of generating families for each of the $\# A$ sets constituting the underlying A-configuration, while is the set of c points which make the A-configuration into an (A, ζ)-configuration.

There is a unique maximal generating family of cardinality

$$|L|(|L| - 1)/2$$

consisting of all pairs of distinct points in L. Similarly, for an (A, ζ)-configuration J there is a unique maximal generating family of cardinality

$$\sum_i \frac{a_i(a_i - 1)}{2} + \zeta.$$

Now, suppose the (A, ζ)-configuration J is of complexity k. Let

$$(T, \Upsilon) = \{(t_1, t'_1), \ldots, (t_r, t'_r), \upsilon_1, \ldots, \upsilon_\zeta\}$$

be a generating family of J. For a generic Γ^d, we know that $\mathcal{M}(\Gamma^d, J) = \emptyset$ when $k > 3d$, by Lemma 1.1(iii). So we always assume $k \leqq 3d$. Then

$$r + \zeta \leqq \sum_i \frac{a_i(a_i - 1)}{2} + \zeta \leqq (k+1)\frac{\sum_i (a_i - 1) + \zeta}{2}$$
$$= \frac{(k+1)k}{2} \leqq \frac{(3d+1)3d}{2}.$$

Let us fix the embedding $\lambda: \Psi \to \mathbb{R}^N$ in Lemma 1.2 for $n = 3d(3d+1)$. By Lemma 1.2 the $r + \zeta$ points $\lambda(t_1, t'_1), \ldots, \lambda(t_r, t'_r), \lambda(\upsilon_1, \upsilon_1), \ldots, \lambda(\upsilon_\zeta, \upsilon_\zeta)$ in $\mathbb{R}^N$ span an $(r + \zeta - 1)$-dimensional simplex in $\mathbb{R}^N$, which we call a *standard simplex* associated with (T, Υ). The choice of n ensures that two standard simplices in $\mathbb{R}^N$ either have no common interior point or are identical.

Let S_J be the standard simplex associated with the maximal generating family of J. Then, all other standard simplices associated with other (non-maximal) generating families of J are sub-simplices of S_J. The *resolvent* of $\Gamma^d \cap \Sigma$ is defined to be:

$$\Omega = \bigcup \mathcal{M}(\Gamma^d, J) \times S_J \subset \Gamma^d \cap \Sigma \times \mathbb{R}^N,$$

where the union is taken over all (A, ζ)-configurations J having complexity $\leqq 3d$.

Here is another interpretation of the resolvent Ω. Consider the map $\pi: \Omega \to \mathbb{R}^N$ induced by the projection $\Gamma^d \times \mathbb{R}^N \to \mathbb{R}^N$. The image $\pi(\Omega)$ is a simplicial complex. For any point x in $\pi(\Omega)$, let s be the standard simplex such that x is in the interior of s. The vertices of s correspond to points in Υ. They generate an (A, ζ)-configuration J. The pre-image $\pi^{-1}(x)$ is the affine space $\mathcal{M}(\Gamma^d, J)$.

Theorem 1.3 (Theorem 2.3.5 of [V]) *The map $\Omega \to \Gamma^d \cap \Sigma$ induced by the projection $\Gamma^d \times \mathbb{R}^N \to \Gamma^d$ is proper and induces an isomorphism between the closed homology of Ω and $\Gamma^d \cap \Sigma$.*

1.4 A Spectral Sequence

Let Ω be the resolvent of $\Gamma^d \cap \Sigma$ for a generic Γ^d. Let Ω_i be the subset of Ω consisting of all elements in $\Omega \subset \Gamma^d \times \mathbb{R}^N$ whose projection on Γ^d are maps respecting (A, ζ)-configurations with complexity $\leqq i$. Another way to describe Ω_i is as the union of $\pi^{-1}(s)$ over all standard simplices s whose vertices, thought of as points in Υ, generate (A, ζ)-configurations with complexity $\leqq i$. Then we get an increasing filtration

$$\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_{3d-1} \subset \Omega_{3d} = \Omega.$$

Consider thelogy spectral sequence $\{E^r_{p,q}(d)\}$ generated by this filtration. It converges to $\bar{H}_*(\Omega)$. We can transform it into a cohomology sequence by renaming the term $E^r_{p,q}(d)$ as $E_r^{-p,3d-1-q}(d)$. For the purpose of constructing knot invariants, we are primarily interested in $\bar{H}_{3d-1}(\Omega)$, and the sequence $\{E_r^{-i,i}(d)\}$ converges to it.

Theorem 1.4 (See [V, Corollary 2.5.1.]) *For generic Γ^d and $\Gamma^{d'}$, we have $E_r^{-i,i}(d) = E_r^{-i,i}(d')$ for all $r = 1, 2, \ldots, \infty$ and all even d, d' with $d' > d \geqq (5i-1)/3$.*

Let us denote by $E_\infty^{-i,i}$ the stabilized limit of $\{E_r^{-i,i}(d)\}$ for $i = 1, 2, \ldots$. An element in $E_\infty^{-i,i}$ determines a knot invariant in the following way. In the stable range $i \leqq (3d+1)/5$, we identify $E_\infty^{-i,i}$ with $E_\infty^{-i,i}(d)$. An element in $E_\infty^{-i,i}(d)$ can be lifted to $\bar{H}_{3d-1}(\Omega)$. The lifts corresponding to different d's are also stabilized when $d \to \infty$. Thus by the discussion in §1.1 and Theorem 1.4, the stabilized limits of these lifts gives us a knot invariant. This knot invariant is defined to have the *order i*.

Examples: Vassiliev invariants of order 1 and 2

Let J be an arbitrary (A, ζ)-configuration. A *J-block* is the subset B_J of Ω defined by

$$B_J = \bigcup \mathcal{M}(\Gamma^d, J') \times S_{J'},$$

where the union is taken over all (A, ζ)-configurations J' equivalent to J. Then B_J is the total space of the locally trivial fibration whose base is the space of all (A, ζ)-configurations equivalent to J and whose fibre is $\mathcal{M}(\Gamma^d, J) \times S_J$. So, B_J is an open cell whose dimension is determined by J. For non-complicated (A, ζ)-configurations J, the dimensions of B_J are listed below:

1. If J is of type 1, then $\dim B_J = 3d - 1$.
2. If J is of type 2, then $\dim B_J = 3d - 1$.
3. If J is of type 3, then $\dim B_J = 3d - 2$.

We will soon see that we only need to consider non-complicated J-blocks of types 1 and 2. Let us work this out, in the simplest examples. Assume that we are in the stable range: $i \leqq (3d + 1)/5$. By definition, $E_1^{-i,i} = \bar{H}_{3d-1}(\Omega_i \backslash \Omega_{i-1})$.The simplest case is $i = 1$, that is the group:

$$E_1^{-1,1} = \bar{H}_{3d-1}(\Omega_1).$$

By definition, Ω_1 is homeomorphic to the $(3d - 1)$-dimensional half sphere

$$\{(x_1, \ldots, x_{3d-1}) \in \mathbb{R}^{3d-1};\ x_1 \geqq 0\}.$$

In fact, Ω_1 is the total space of a locally trivial fibration whose base is the space of all (A, ζ)-configurations with complexity 1 and whose fibre over an (A, ζ)-configuration J with complexity 1 is the $(3d - 3)$-dimensional affine space $\mathcal{M}(\Gamma^d, J)$. So Ω_1 is the union of two J-blocks B_{J_1} and B_{J_2} where J_1 is an (A, ζ)-configuration with $A = \{2\}$, $\zeta = 0$ and J_2 is an (A, ζ)-configuration with $A = \emptyset$, $\zeta = 1$. We have $\dim B_{J_1} = 3d - 1$, $\dim B_{J_2} = 3d - 2$ and $\partial B_{J_1} = B_{J_2}$. Thus we conclude that:

$$E_1^{-1,1} = 0 = E_\infty^{-1,1}$$

so that the first order Vassiliev invariant is zero for every knot.

Next let us try to figure out the case $i = 2$.

$$E_1^{-2,2} = \bar{H}_{3d-1}(\Omega_2 \backslash \Omega_1).$$

Let J be an (A, ζ)-configuration with complexity 2. If $\dim B_J = 3d - 1$, than either $A = \{2, 2\}$ and $\zeta = 0$ or $A = \{3\}$ and $\zeta = 0$, i.e. J must be of non-complicated type 1 or 2. Let us consider the boundary of these $(3d - 1)$-cells.

For an (A, ζ)-configuration J with $A = \{3\}$ and $\zeta = 0$, S_J is a 2-simplex. It has 3 edges corresponding to the 3 non-maximal generating families of J, each having two pairs of points. They can be specified by distinguishing one of the 3 points in J. In ∂B_J there are only three $(3d - 2)$-cells, corresponding to the three edges of S_J. This part of ∂B_J can be written as

$$\bigcup \mathcal{M}(\Gamma^d, J') \times \partial S_{J'},$$

where J' runs over all (A, ζ)-configurations equivalent to J. The other part of ∂B_J comes from the boundary of the space of all (A, ζ)-configurations equivalent to J. This part of the boundary can be obtained by successfully shrinking all finite segments of $\mathbb{R}^1$ which are bounded by neighboring points in J, to obtain two $(3d - 3)$-cells.

For an (A, ζ)-configuration J with $A = \{2, 2\}$ and $\zeta = 0$, S_J is a 1-simplex. The part of ∂B_J corresponding to

$$\bigcup \mathcal{M}(\Gamma^d, J') \times \partial S_{J'}$$

consists of $(3d-2)$-cells in Ω_1. Just as in the previous case, the other part of ∂B_J can be obtained by shrinking neighboring points in J successively. If two neighboring points belong to the same pair in J, the (A, ζ)-configuration resulting from shrinking the interval between them will have $A = \{2\}$ and $\zeta = 1$ where the singular point corresponds to the shrunken segment. If two neighboring points belong to different pairs in J, the resulting (A, ζ)-configuration has $A = \{3\}$ and $\zeta = 0$. Moreover, the point corresponding to the shrunken segment is distinguished so that it determines a non-maximal generating family for the triple.

In view of the above discussion, it is not hard to see that $E_1^{-2,2} \cong \mathbb{Z}$ is generated by a linear combination of $(3d-1)$-dimensional J-blocks, where the J's are non-complicated (A, ζ)-configurations of complexity 2.

The next term $E_2^{-2,2}$ in the spectral sequence is the (co)homology of

$$E_1^{-3,2} \to E_1^{-2,2} \to E_1^{-1,2}.$$

We have $E_1^{-1,2} = \bar{H}_{3d-2}(\Omega_1) = 0$ and $E_1^{-3,2} = \bar{H}_{3d}(\Omega_3 \backslash \Omega_2) = 0$. So $E_2^{-2,2} = E_1^{-2,2} \cong \mathbb{Z}$. Using the fact that $E_1^{p+q} = 0$ if $p + q < 0$ (Theorem 2.5 in [V]), we can get $E_\infty^{-2,2} = E_2^{-2,2} \cong \mathbb{Z}$. Thus there is one integer knot invariant of order 2.

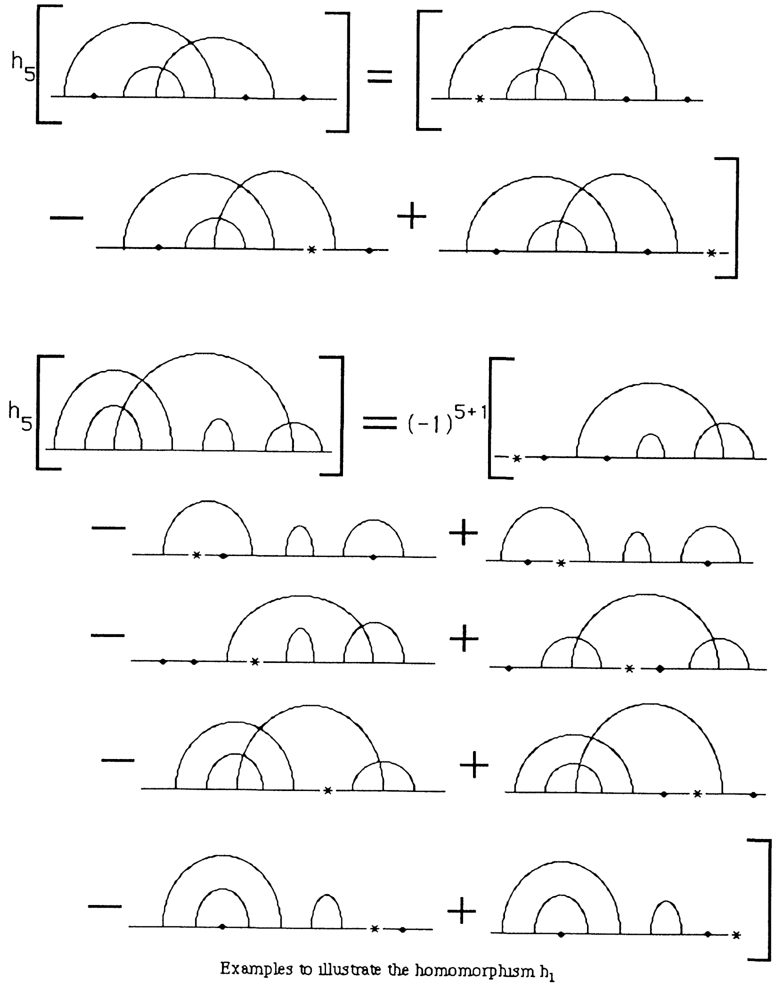
Notice that an element in $E_1^{-2,2} = \bar{H}_{3d-1}(\Omega_2 \backslash \Omega_1)$ is a relative homology class. Let $\gamma_1 \in E_1^{-2,2}$ be thought of as a linear combination of $(3d-1)$-dimensional cells (top dimensional J-blocks) in Ω_2. Then $\partial\gamma_1$ is a linear combination of $(3d-2)$-dimensional cells in Ω_1. To get a cycle in Ω_2, we should have some linear combination of components of $\Omega_1 \backslash \partial\gamma_1$, say γ_2, such that $\partial(\gamma_1 + \gamma_2) = 0$. We can always accomplish this in the following way. If a component of $\Omega_1 \backslash \partial\gamma_1$ meets $\partial\Omega_1$, then the coefficient of that component in γ_2 is zero. If two components of $\Omega_1 \backslash \partial\gamma_1$ are separated by a cell in $\partial\gamma_1$, their coefficients in γ_2 will differ by the coefficient of that cell in $\partial\gamma_1$, multiplied by ± 1 (the incidence coefficient). The standard homological argument shows that this procedure is well-defined. Thus, the cycle $\gamma_1 + \gamma_2$ gives us a homology class $[\gamma_1 + \gamma_2]$ in $\bar{H}_{3d-1}(\Omega)$. The inverse limit of $[\gamma_1 + \gamma_2]$ is a knot invariant of order 2.

1.5 The group $E_1^{-i,i}$

We now generalize the previous examples, and describe how to figure out the group $E_1^{-i,i}$ for arbitrary i. The group $E_1^{-i,i}$ is the kernel of an explicitly defined homomorphism $h_i: \mathscr{X}_i \to \mathscr{Y}_i$ between free abelian groups $\mathscr{X}_i$ and $\mathscr{Y}_i$ of finite ranks.

The group $\mathscr{X}_i$ is freely generated by all possible $[i]$-configurations and $\langle i \rangle$-configurations, where an $[i]$-configuration is a non-complicated (A, ζ)-configuration of type 1 with $\# A = i$ and an $\langle i \rangle$-configuration is a non-complicated (A, ζ)-configurations of type 2 with $\# A = i - 1$. The group $\mathscr{Y}_i$ is freely generated by $\langle i \rangle^*$-configurations and $[i-1]^*$-configurations. See Figure 2 for examples. The former are $< i >$-configurations with one of the three points in the triple distinguished. An $[i-1]^*$-configuration is a non-complicated (A, ζ)-configuration of type 3 with $\# A = i - 1$.

Let $\{t_1, \ldots, t_n\}$ be n distinct points in an (A, ζ)-configuration, ordered by the orientation of $\mathbb{R}^1$, so that $t_1 < t_2 < \cdots < t_n$. The homomorphism $h_i: \mathscr{X}_i \to \mathscr{Y}_i$ will be defined on generators of $\mathscr{X}_i$. It can then be extended linearly to all of $\mathscr{X}_i$.



Examples to illustrate the homomorphism h_1

Fig. 3

The reader may wish to consult the examples in Fig. 3 as an aid in understanding the definition.

- If β is an $\langle i \rangle$-configuration, then $h_i(\beta)$ is a linear combination of the three $\langle i \rangle^*$-configurations which belong to the same underlying $\langle i \rangle$-configuration, the coefficients being -1, $+1$, -1 according as the distinguished point is the first one in the "missing triple", the second one or the third.
- If α is an $[i]$-configuration, then $h_i(\alpha)$ is a linear combination of the $2i-1$ configurations which are obtained by shrinking, one after the other, the segments on $\mathbb{R}^1$ which are bounded by neighboring points in the array $\{t_1, \ldots, t_{2i}\}$. If two neighboring points belong to the same pair in α, then the finite segment they bound shrinks to a singular point and the resulting configuration will be an $[i-1]^*$-configuration. If neighboring points belong to different pairs in α, the

finite segment they bound shrinks to a distinguished point in a triple and the resulting configuration is an $\langle i\rangle^*$-configuration. The element $h_i(\alpha)$ will be a linear combination of these configurations, with coefficients which are determined in the following manner. If a configuration in $h_i(\alpha)$ is obtained from shrinking the finite segment bounded by the t_{m-1} and t_m, its coefficient will be $(-1)^{i+m-1}$.

Thus we have defined the homomorphism $h_i:\mathscr{X}_i \to \mathscr{Y}_i$.

Theorem 1.5 (See §2.5 of [V]) *The group* $E_1^{-i,i}$ *is equal to the kernel of* h_i *for all* $i \geqq 2$.

As in the case $i = 2$, an element $\gamma \in E_1^{-i,i}$ is a linear combination of $(3d-1)$-dimensional J-blocks consisting of $[i]$- or $\langle i\rangle$-configurations in Ω_i. The boundary of γ_1 is a linear combination of $(3d-2)$-cells in Ω_{i-1}. For γ_1 to survive as a part of a cycle in $\Omega_i \subset \Omega$, we should find a linear combination of components of $\Omega_{i-1}\backslash\partial\gamma_1$, say γ_2, such that $\partial(\gamma_1+\gamma_2)$ is a linear combination of $(3d-2)$-cells in Ω_{i-2}. Keep going in this way until we obtain a cycle $\gamma_1+\gamma_2+\cdots+\gamma_i$ in $\Omega_i \subset \Omega$. The inverse limit of the homology class $[\gamma_1+\gamma_2+\cdots+\gamma_i]$ will be a knot invariant of order i.

1.6 Actuality tables and extended actuality tables

The actual computation of knot invariants of order i cannot begin until one has constructed an *actuality table* of order i. The careful reader will see that this table was used implicitly in the case $i = 2$. The actuality table is computed as a subset of an *extended actuality table* , which we now describe.

First, we describe the figures in the extended tables. The top row of the extended table contains a list of all the $[i]$- and $\langle i\rangle$-configurations. The figures in the jth row, $j = i-1, i-2, \ldots, 2$ is a list of immersions in $\mathscr{M}$ (defined via pictures of their images in $\mathbb{R}^3$) whose configurations of self-intersections are in one-one correspondence with all possible $[j]$- and $\langle j\rangle$-configurations, subject to the following restriction: Let Φ be an immersion whose image represents some $[j]$- or $\langle j\rangle$-configuration. Suppose there is a double point x in $\Phi(\mathbb{R}^1)$ which has the property that $\Phi^{-1}(x)$ consists of a pair of points $t_p, t_{p+1} \in \mathbb{R}^1$ which are not separated on $\mathbb{R}^1$ by any other points t_s in the underlying set of points on $\mathbb{R}^1$ which are mapped to multiple points by Φ. Then Φ must be chosen so that the loop $\Phi([t_p, t_{p+1}])$ in $\mathbb{R}^3$ bounds a disk whose interior has empty intersection with $\Phi(\mathbb{R}^1)$. Also, if we identify $\mathbb{R}^1 \cup \{\infty\}$ (resp. $\mathbb{R}^3 \cup \{\infty\}$) with S^1 (resp. S^3) and if t_p, t_{p+1} are neighboring points on S^1, then the loop $\hat{\Phi}([t_p, \infty] \cup [\infty, t_{p+1}])$ in S^3 bounds a diswhose interior has empty intersection with $\hat{\Phi}(S^1)$. Here $\hat{\Phi}:S^1 \to S^3$ is the natural extension of $\Phi:\mathbb{R}^1 \to R^3$.

We also need to attach indices to each picture. The indices in the top row are determined by an element $\gamma_1 \in E_1^{-i,i} = \mathrm{kernel}(h_i)$, in the following manner. Since $\mathrm{kernel}(h_i) \subset \mathscr{X}_i$, we see that γ_1 is a linear combination $\sum(A_r\alpha_r + B_s\beta_s)$ of $[i]$-configurations α_r and $\langle i\rangle$-configurations β_s. The index of α_r is A_r and the index of β_r is B_s.

The method for computing the indices of the immersions in rows $i-1, i-2, \ldots, 2$ is described in §4.4–4.6 of [V]. Let $m = i-j+1$, so that row i corresponds to $m = 1$. Assume, inductively, that we have determined rows $i, i-1, \ldots, i-m+2$ of an extended actuality table. The figures, together with their indices in rows $i, i-1, \ldots, i-m+2$ give rise to a chain $\gamma_1 + \cdots + \gamma_{m-1}$ in Ω_i whose boundary in Ω_{i-m+2} is zero. We need to consider its boundary in Ω_{i-m+1}. The latter can be thought of as an element $d^*(\gamma_1 + \cdots + \gamma_{m-1}) \in \mathscr{Y}_{i-m+1}$ given in the following way.

First, it can be shown that $d^*(\gamma_1 + \cdots + \gamma_{m-1})$ contains no $[i-m+1]^*$-configurations. Our task is to figure out the coefficient of each $\langle i-m+1\rangle^*$-configuration in $d^*(\gamma_1 + \cdots + \gamma_{m-1})$. We describe how to determine the coefficient of a particular $\langle i-m+1\rangle^*$-configuration J^* in $d^*(\gamma_1 + \cdots + \gamma_{m-1})$. We make the initial assumption that the indices of the immersions chosen for the mth row are all zero. Let $\{x, y, z\}$ be the triple in J^* with z the distinguished point. Let J be the $\langle i-m+1\rangle$-configuration obtained from J^* by forgetting that there is a distinguished point. We split the point z in J into a pair of neighboring points z' and z'', and pair these two points with x and y to obtain a $[i-m+1]$-configuration. There are two ways to pair these four points, i.e. pair x with z' and y with z'' or pair x with z'' and y with z'. Thus we obtain two $[i-m+1]$-configurations J' and J''. If Ψ is the immersion which we selected to respect J, then a corresponding pair of resolutions, denoted Ψ' and Ψ'', of Ψ respect J' and J''. In general, these will not coincide with the immersions Φ' and Φ'' we chose in the table to respect J' and J''. However since we have assumed temporarily that the indices of Φ' and Φ'' are zero, we can compute the indices of Ψ' and Ψ'' by using crossing changes to change Ψ' and Ψ'' to Φ' and Φ'' respectively, referring our computation to the indices we know in row $i, i-1, \ldots, i-m+2$. In this way we compute the indices of Ψ' and Ψ''. Then, the coefficient of the element we selected, i.e. J^*, in $\mathscr{Y}_{i-m+1}$ will be

$$\varepsilon[\mathrm{Ind}_i(\Psi') + \mathrm{Ind}_i(\Psi'')]$$

where the sign $\varepsilon = \pm 1$ is the coefficient of J^* in $h_m(J') = h_m(J'')$. The coefficients of the other $\langle i-m+1\rangle^*$-configurations in $d^*(\gamma_1 + \cdots + \gamma_{m-1})$ are computed similarly.

Let $J_1, \ldots, J_p$ be the list of all $\langle i-m+1\rangle^*$-configurations in $\mathscr{Y}_{i-m+1}$. We have associated to each J_q a coefficient C_q. Then $d^*(\gamma_1 + \cdots + \gamma_{m-1}) = \sum C_q J_q$.

The last step will be to correct the assumption that the indices of the immersions chosen in row m are all zero. To do this, we need to find an element $\gamma_m \in \mathscr{X}_{i-m+1}$ such that:

$$h_{i-m+1}(\gamma_m) + d^*(\gamma_1 + \cdots + \gamma_{m-1}) = 0.$$

Then, the indices in the mth row will be the coefficients in γ_m of the corresponding $[i-m+1]$- and $\langle i-m+1\rangle$-configurations.

In this way we have described (inductively) Vassiliev's construction of an extended actuality table. If we can carry out the procedure down to the bottom row ($m = i-1$ as the row $m = i$ can be completed automatically), then $\gamma_1 + \cdots + \gamma_{i-1} + \gamma_i$ will be a cycle of dimension $3d-1$. This gives us a homology class

$$[\gamma_1 + \cdots + \gamma_i] \in \bar{H}_{3d-1}(\Omega).$$

Its stabilized limit is a knot invariant of order i.

Thus, there is a non-trivial knot invariant of order i whenever we can complete the construction of an extended actuality table whose indices in the top row are not all zero. Finally, an actuality table is just the subset of an extended actuality table corresponding to all $[j]$-configurations, $j = i, i-1, \ldots, 2$.

2 An axiomatic description of Vassiliev's work

In this section we will translate Vassiliev's recipe for constructing his invariants into a set of axioms and initial data, which allow one to begin with a knot diagram and compute $v_i(\mathbf{K})$. Our translation will seem natural to knot theorists because it points up the similarities (and differences) between Vassiliev's work and related computations for other knot polynomials, and sets the stage for the direct connection we will later make between the two. The main result is Theorem 2.4.

In our review of Vassiliev's work in §1 above we saw that it was important to the technical work that Vassiliev worked with maps $\Phi: \mathbb{R}^1 \to \mathbb{R}^3$ rather than maps $\hat{\Phi}: S^1 \to \mathbb{R}^3$, for in the latter case the Weierstrass Approximation theorem would not be valid. We will see that, having his invariants in hand, we will be able to return to maps with domain S^1. This will make the calculations we need to do much more efficient. We will do this carefully, as there are some subtle points having to do with the signs of the invariants. The two cases will be distinguished notationally by the addition of a circumflex in the circular case, as above.

Assign a fixed orientation to the circle S^1 and to 3-space $\mathbb{R}^3$ and let $\hat{\Phi}^j$: $S^1 \to \mathbb{R}^3$ be an immersion whose singularities are restricted to j transverse double points. Call such an immersion *j-generic*. In the case $j = 0$ the image $\hat{K}^0 = \hat{\Phi}^0(S^1)$ is a knot. More generally $\hat{K}^j = \hat{\Phi}^j(S^1)$ is an oriented knotted graph. Each vertex will have valence four. Such graphs have appeared elsewhere in the literature which relate to the Jones polynomial, for example in the work of Yamada [Y], who calls the graphs in question *flat vertex* graphs because they may be obtained from a knot diagram by collapsing some of the vertices to double points. Thus, for each vertex v of $\hat{K}^j$, there is a neighborhood B_v of v in 3-space and a proper 2-disk $P_v \subset B_v$ such that $K^j \cap B_v \subset P_v$. In Vassiliev's work such graphs arise when two $(j-1)$-generic immersions are related by a single passage through the jth level of the discriminant. At the instant of passage one obtains a j-generic immersion $\hat{\Phi}^j$ whose image $\hat{\Phi}^j(S^1)$ is a flat vertex graph.

We shall pass freely from the concept of a j-generic immersion to its associated flat vertex graph whenever it improves clarity to do so, e.g. when we are looking at diagrams. An *immersion* will always mean a j-generic immersion and a *graph* will always mean a flat vertex graph, unless it is stated otherwise. We will add the qualifying adjectives only if we wish to stress the special nature of our immersions and graphs.

We need an equivalence relation between immersions. Two immersions $\hat{\Phi}_0^j$ and $\hat{\Phi}_1^j$ are *equivalent* if there is an isotopy taking $\hat{\Phi}_0^j$ to $\hat{\Phi}_1^j$ with each $\hat{\Phi}_t^j$ a j-generic immersion, also during the isotopy the order of the edges which meet at a vertex is to be preserved. This implies that two graphs $\hat{K}_0^j$ and $\hat{K}_1^j$ are *equivalent* if there is an isotopic deformation h_t of $\mathbb{R}^3$ with h_0 = identity, $h_1(\hat{K}_0^j) = \hat{K}_1^j$, such that each $h_t(\hat{K}_0^j)$ is a flat vertex graph. This equivalence relation (which is also the one considered in [Y]) is not the usual notion of isotopy of graphs in 3-space when $j > 0$. For example, Fig. 4 shows two flat vertex graphs which are isotopic as ordinary graphs but are not equivalent in the sense considered here. We shall not distinguish between equivalent immersions, or between equivalent graphs. The equivalence class of $\hat{\Phi}^j$ will be denoted $\pounds\hat{\mathbf{\Phi}}^j$. The equivalence class of $\hat{K}^j$ will be denoted $\hat{\mathbf{K}}^j$.

An *invariant* of an immersion is a numerical invariant which is unchanged by the equivalence relation which we just defined. In the special case $j = 0$, it is a numerical *knot invariant*, in the usual sense.

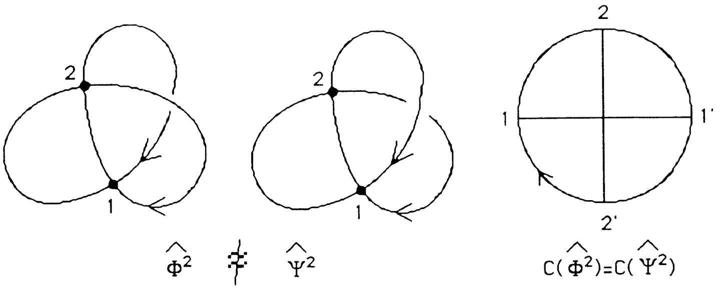


Fig. 4

Here is a non-numerical invariant which one may associate to our immersions. Regard the domain S^1 of $\hat{\Phi}^j$ as an oriented planar circle which bounds a disc δ in the plane. Choosing points t_r, t_s which are pre-images on S^1 of a double point of $\hat{\Phi}$, join t_r to t_s by an arc in δ. Do this for every double point of $\hat{\Phi}^j$. The resulting pattern of arcs in δ, denoted $\hat{\alpha}^j$, is the (circular) $[j]$ *-configuration* which $\hat{\Phi}^j$ *respects*. Equivalent immersions respect the same configuration. We shall write $\hat{\alpha}^j = C(\hat{\mathbf{\Phi}}^j)$ when we wish to stress that $\hat{\mathbf{\Phi}}^j$ respects $\hat{\alpha}^j$.

Let $\mathbf{\Phi}^j$ be the equivalence class of Φ^j and let $\hat{\mathbf{\Phi}}^j$ be the corresponding circular equivalence class. The Vassiliev invariants $v_i(\mathbf{K})$ of knots which were introduced in [V] and reviewed in §1 of this paper were shown there to generalize in a natural way to invariants $\mathrm{Ind}_i(\mathbf{\Phi}^j)$, $0 \leqq j \leqq i$, $i = 1, 2, 3, \ldots$ The first step in our work is to prove that $\mathrm{Ind}_i(\mathbf{\Phi}^j)$ is well-defined on circular graphs, i.e. is independent of the choice of the point at infinity.

Lemma 2.1 *Supppose that two immersions Φ_1^j and Φ_2^j determine the same circular equivalence class $\hat{\mathbf{\Phi}}^j$. Then:*

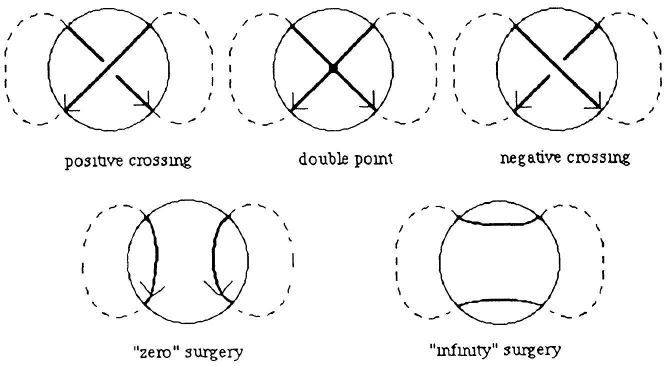
$$\mathrm{Ind}_i(\mathbf{\Phi}_1^j) = \mathrm{Ind}_i(\mathbf{\Phi}_2^j).$$

Proof. A careful reading of [V] (see §0.2 and §0.3 and especially §3.3) reveals that the invariants $\mathrm{Ind}_i(\mathbf{\Phi}^j)$ satisfy a recursive formula. To describe it let $\Phi_{x_+}^j$ and $\Phi_{x_-}^j$ be immersions which coincide everywhere except near a single crossing point in a defining diagram, where they differ in the manner indicated in Fig. 5. Let Φ_x^{j+1} be the immersion which is obtained from $\Phi_{x_+}^j$ and $\Phi_{x_-}^j$ by replacing the crossing by a double point. The recursive formula is:

$$\text{(2.1)} \qquad \mathrm{Ind}_i(\mathbf{\Phi}_{x_+}^j) - \mathrm{Ind}_i(\mathbf{\Phi}_{x_-}^j) = (-1)^{j+\Delta x}\mathrm{Ind}_i(\mathbf{\Phi}_x^{j+1}), \quad 0 \leqq j \leqq i-1.$$

The quantity Δx in (2.1) is defined as follows: Let α^j be the $[j]$ -configuration which $\mathbf{\Phi}_{x_+}^j$ and $\mathbf{\Phi}_{x_-}^j$ respect. and let α^{j+1} be the $[j+1]$ -configuration which Φ_x^{j+1} respects. Then α^{j+1} is obtained from α^j by adding one new arc, say τ, and Δx is the number of arc endpoints in α^{j+1} which lie between the two endpoints of the new arc τ.

There is a one-to-one correspondence between the knot types of embeddings of $\mathbb{R}^1$ and of S^1 in $\mathbb{R}^3$. Thus the invariant $v_i(\hat{\mathbf{\Phi}})$ is independent of the choice of the point at infinity. Assume, inductively, that $\mathrm{Ind}_i(\hat{\mathbf{\Phi}}^j)$ is well-defined, independently of the choice of the point at infinity. Then (2.1) shows that the same is true for



Crossing Changes

Fig. 5

$\mathrm{Ind}_i(\hat{\Phi}^{j+1})$, if the coefficient $(-1)^{j+\Delta x}$ in (2.1) does not depend upon the choice of the initial point. Now, the total number $2(j+1)$ of points in the underlying point set for the $[j+1]$-configuration of $\hat{\alpha}^{j+1}$ is even. Fixing any two points, there will be an even number of points which remain. Thus the parity of Δx is independent of which way around the circle we do the counting. Thus $\mathrm{Ind}_i(\hat{\Phi}^{j+1})$ is well-defined. □

We now define two numerical invariants which are determined by a circular $[j]$-configuration which an immersion $\hat{\Phi}^j$ respects. The *intersection number* $|C(\hat{\Phi}^j)|$ is the minimum number of intersections between the arcs in $C(\hat{\Phi}^j)$ when they are deformed so that two arcs intersect at most once and all intersections are transverse double points. For example, in Fig. 4 we have $|C(\hat{\Phi}^2)| = 1$. The *sign* of $C(\hat{\Phi}^j)$ is

$$S(C(\hat{\Phi}^j)) = (-1)^{j(j-1)/2 + |C(\hat{\Phi}^j)|}.$$

The invariant we will consider here agrees with Vassiliev's invariant $\mathrm{Ind}_i(\hat{\Phi}^j)$ up to sign. It is:

$$v_i(\hat{\Phi}^j) = S(C(\hat{\Phi}^j))(\mathrm{Ind}_i(\hat{\Phi}^j)). \tag{2.2}$$

Notice that in the case $j = 0$ we have $\mathrm{Ind}_i(\hat{\Phi}^0) = v_i(\hat{\Phi}^0)$ by definition. Since the sign of $C(\hat{\Phi}^0) = 1$ trivially, we see that (2.2) makes sense for every $j \geqq 0$.

Lemma 2.2 *The numerical invariant $v_i(\hat{\Phi}^j)$ satisfies the following simplified version of the recursion formula* (2.1) :

$$v_i(\hat{\Phi}^j_{x_+}) - v_i(\hat{\Phi}^j_{x_-}) = v_i(\hat{\Phi}^{j+1}_x) \quad \textit{for all } j \geqq 0.$$

Proof. In view of Lemma 2.1, we may replace each immersion Φ^j in (2.1) by the corresponding immersion $\hat{\Phi}^j$. Since $|C(\hat{\Phi}^1)| = 0$ (because there is only one arc in a circular [1] -configuration, so that no other arcs can intersect it) we see that $S(C(\hat{\Phi}^1)) = 1$. Thus (2.2) shows that $\mathrm{Ind}_i(\hat{\Phi}^1) = v_i(\hat{\Phi}^1)$, therefore our formula holds when $j = 0$. As for the case when $j > 0$, we see that (2.1) and (2.2) imply:

$$v_i(\hat{\Phi}^j_{x_+}) - v_i(\hat{\Phi}^j_{x_-}) = (-1)^{j+\Delta x} S(C(\hat{\Phi}^j)) S(C(\hat{\Phi}^{j+1})) v_i(\hat{\Phi}^{j+1}_x)$$

Using (2.2), we thus see that our formula will hold if:

$$(2.3)\qquad j + \Delta x + (j)(j-1)/2 + |C(\hat{\Phi}^j)| + (j+1)(j)/2 + |C(\hat{\Phi}^{j+1})|$$

is congruent to zero mod 2. Since

$$|C(\hat{\Phi}^{j+1})| = |C(\hat{\Phi}^j)| + \Delta x (mod\, 2),$$

our expression reduces to

$$j + j^2 + 2|C(\hat{\Phi}^j)| + 2\Delta x$$

and the latter is congruent to zero mod 2. □

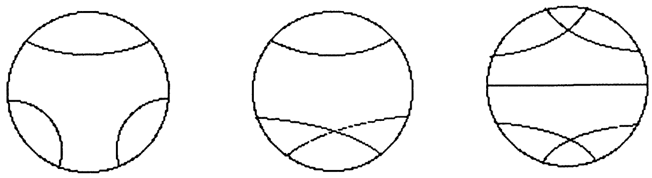
To continue, let $\hat{\alpha}^j$ be a circular $[j]$-configuration and let $\tau_1, \ldots, \tau_j$ be the arcs in δ which define $\hat{\alpha}^j$. An arc τ_p is *separating* if the endpoints of τ_p are not separated by the endpoints of τ_q for any $q \neq p$. The configuration $\hat{\alpha}^j$ is *inadmissible* if it contains a separating arc. See Fig. 6 for examples. Let $\hat{\Phi}^j$ be a j-generic immersion with $\hat{\alpha}^j = C(\hat{\Phi}^j)$. We say that $\hat{\Phi}^j$ is a *good model* for $\hat{\alpha}^j$ if the following holds: Among all separating arcs in $\hat{\alpha}^j$ there is at least one, say τ, with the property: the endpoints of τ separate S^1 into subarcs μ_1 and μ_2, and their images $\hat{\Phi}^j(\mu_1)$ and $\hat{\Phi}^j(\mu_2)$ are geometrically unlinked. See Fig. 7 for an example.

Lemma 2.3 (i) Given any circular $[j]$ -configuration $\hat{\alpha}^j$ we may find a immersion $\hat{\Phi}^j$ which respects $\hat{\alpha}^j$.

(ii) If $\hat{\alpha}^j$ is inadmissible, the immersion constructed in (i) may be chosen so that it is a good model for $\hat{\alpha}^j$.

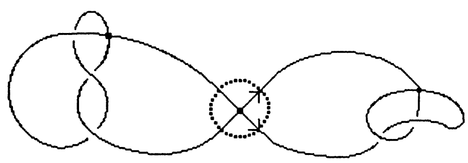
(iii) Any two immersions which respect the same circular $[j]$ -configuration may be changed to equivalent immersions by a series of crossing changes.

Proof. (i) Fig. 8 shows a general method for construction a immersion which respects a given circular $[j]$-configuration: replace each arc τ in the configuration by a loop in the manner indicated there. If two arcs τ, μ intersect in δ, then one of the loops must pass above the other in order not to create unwanted double points.



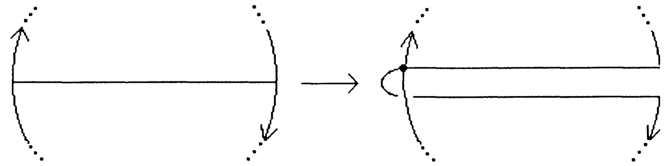
Examples of inadmissible circular [j]-configurations

Fig. 6



Example of a good model respecting an inadmissible configuration

Fig. 7



Construction of an immersion respecting a given circular [j]-configuration

Fig. 8

(ii) The method of Fig. 8 yields a good model when the configuration is inadmissible.

(iii) If $\hat{\Phi}^j$ and $\hat{\Psi}^j$ both respect $\hat{\alpha}^j$, we may modify $\hat{\Phi}^j$ and $\hat{\Psi}^j$ by a series of crossing changes to immersions which have unknotted and unlinked edges and which still respect $\hat{\alpha}^j$. Since there are only finitely many such immersions, and since all are clearly equivalent under crossing changes, the assertion follows. □

Theorem 2.4 *The invariants* $v_i(\hat{\mathbf{\Phi}}^j)$, $i \geqq 1$ and $j \geqq 0$ *are determined by axioms and initial data*:

(2.4) $v_i(\hat{\mathbf{\Phi}}^j_{x_+}) - v_i(\hat{\mathbf{\Phi}}^j_{x_-}) = v_i(\hat{\mathbf{\Phi}}^{j+1}_x)$.

(2.5) $v_i(\hat{\mathbf{\Phi}}^i)$ *depends only on* $\hat{\alpha}^i = C(\hat{\mathbf{\Phi}}^i)$. *Equivalently*, $v_i(\hat{\mathbf{\Phi}}^j) = 0$ *if* $j > i$.

(2.6) $v_i(\mathbf{O}) = 0$ *for all i, where* **O** *denotes an unknotted circle.*

(2.7) $v_i(\hat{\mathbf{\Phi}}^j) = 0$ *if* $\hat{\mathbf{\Phi}}^j$ *is a good model which respects an inadmissible* $[j]$-*configuration.*

(2.8) *The remaining initial data is in the form of a table* $\mathbf{A}(i)$ *which Vassiliev calls an actuality table. It gives the values of* $v_i(\hat{\mathbf{\Phi}}^j)$ *on a basic set of model immersions, one respecting each admissible* $[j]$-*configuration with* $2 \leqq j \leqq i$.

Proof. We have two tasks: to prove that (2.4)–(2.8) are valid, and to prove that they suffice for the computation of $v_i(\hat{\mathbf{\Phi}}^j)$. We consider the second task first.

If $j = 0$ and if $\hat{\mathbf{\Phi}}^0$ is the unknot, then by (2.6) we are done. If not, we choose a series of crossings $x_1, \ldots, x_n$ on our diagram for $\hat{\mathbf{\Phi}}^0$ such that after switching all of them we obtain the unknot. Similarly, if $j > 0$, we determine the $[j]$-configuration $\hat{\alpha}^j$ which $\hat{\mathbf{\Phi}}^j$ respects. If $\hat{\alpha}^j$ is inadmissible and if $\hat{\mathbf{\Phi}}^j$ is a good model, we are done, by (2.7). If not, we choose a series of crossing changes $x_1, \ldots, x_n$ in our diagram for $\hat{\mathbf{\Phi}}^j$ such that after switching all of them we obtain a good model for an immersion respecting $\hat{\alpha}^j$. This is possible by Lemma 2.3. Finally, if $\hat{\alpha}^j$ is admissible, and if $\hat{\mathbf{\Phi}}^j$ agrees with the model immersion in the actuality table which was chosen to respect $\hat{\alpha}^j$ we are done. If not, we choose a series of crossing changes which modify $\hat{\mathbf{\Phi}}^j$ to the given model. Since the terms on the left hand side in (2.4) each have one less double point than the term on the right, one may then apply (2.4) repeatedly to modify the immersions with $j + 1$ double points until they agree with the model immersion in the actuality table. In so-doing, new immersions with $j + 2$ double points enter into the equation. Iterating the process, we modify each immersion with fewer than $i - 1$ double points to one of the models in the table, and then evaluate its index. The process ends because by (2.5) the index of a immersion with exactly i double points depends only on its configuration, and so it can be determined from the top row of the table, no matter what its embedding may be.

It remains to prove that (2.4)–(2.8) are implicit in Vassiliev's work and our modifications of it. We have already shown (Lemma 2.2) that (2.4) holds. The first assertion in (2.5) is to be found at lines 10–11 on page 29 of [V], with details given in the latter part of [V]. Clearly the first assertion in (2.5) implies the second. The second assertion in (2.5) together with (2.4) implies the first assertion in (2.5). Assertion (2.6) is a consequence of the assumption that our invariants are elements of the reduced cohomology group $\tilde{H}^0(\mathscr{M}\backslash\Sigma)$.

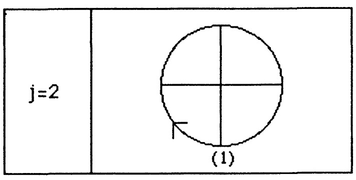
With regard to (2.7), our first observation is that by Lemma (2.3) we may always find a good model respecting an inadmissible configuration. So, assume that $\hat{\mathbf{\Phi}}^j$ is a good model which respects $\hat{\alpha}^j$ and let x be the double point which splits $\hat{\mathbf{\Phi}}^j$ into unlinked subimmersions. Change $\hat{\mathbf{\Phi}}^j_x$ to immersions $\hat{\mathbf{\Phi}}^{j-1}_{x_+}$ and $\hat{\mathbf{\Phi}}^{j-1}_{x_-}$ by replacing x by positive and negative crossings. From Figure 1.4 it is clear that $\hat{\mathbf{\Phi}}^{j-1}_{x_+}$ and $\hat{\mathbf{\Phi}}^{j-1}_{x_-}$ must be equivalent immersions, since the crossing at x is "nugatory", so (2.4) implies (2.7).

Only one thing remains: to show that Vassiliev's work allows one to construct a table $\mathbf{A}(i)$. Trivially, we can always fill in the table with zeros, but we are hoping there are non-trivial tables. An algorithm which will yield a non-trivial table whenever one exists is given in §4.3–4.6 of [V]. It is summarized in §1.5–1.6 of our review in this paper. In the next section we will begin to investigate it in detail, and to translate it into a new form. As we shall see, it is the heart of the difficulty in understanding Vassiliev's invariants. □

Example. Figures 9–11 are actuality tables $\mathbf{A}(i)$ for $i = 2, 3, 4$. Without justifying them, we illustrate how to use the table in Fig. 10 to compute $v_3(\mathbf{T})$ when $\mathbf{T}$ is the trefoil knot. Refer to Fig. 11. Label the three crossings p, q and r and use the subscripts x_+, x_-, x to denote a positive crossing, negative crossing or double point respectively at x, where $x \in \{p, q, r\}$. Thus our copy of the trefoil knot is $\mathbf{T}_{p_+q_+r_+}$. We compute its third order Vassiliev invariant:

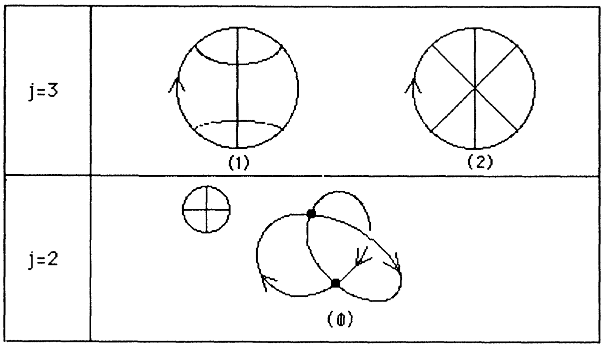
$$\begin{aligned} v_3(\mathbf{T}_{p_+q_+r_+}) &= v_3(\mathbf{T}_{p_-q_+r_+}) + v_3(\mathbf{T}^1_{pq_+r_+}) \\ &= 0 + v_3(\mathbf{T}^1_{pq_+r_+}) \\ &= v_3(\mathbf{T}^1_{pq_-r_+}) + v_3(\mathbf{T}^2_{pqr_+}) \\ &= 0 + v_3(\mathbf{T}^2_{pqr_+}) \\ &= v_3(\mathbf{T}^2_{pqr_-}) + v_3(\mathbf{T}^3_{pqr}) \end{aligned}$$

In this calculation the first equality is (2.4). The second equality follows from the fact that $\mathbf{T}_{p_-q_+r_+}$ is the unknot. The graph $\mathbf{T}^1_{pq_+r_+}$ respects an inadmissible [1]-configuration, but it is not a good model, so we change the crossing at q in the third



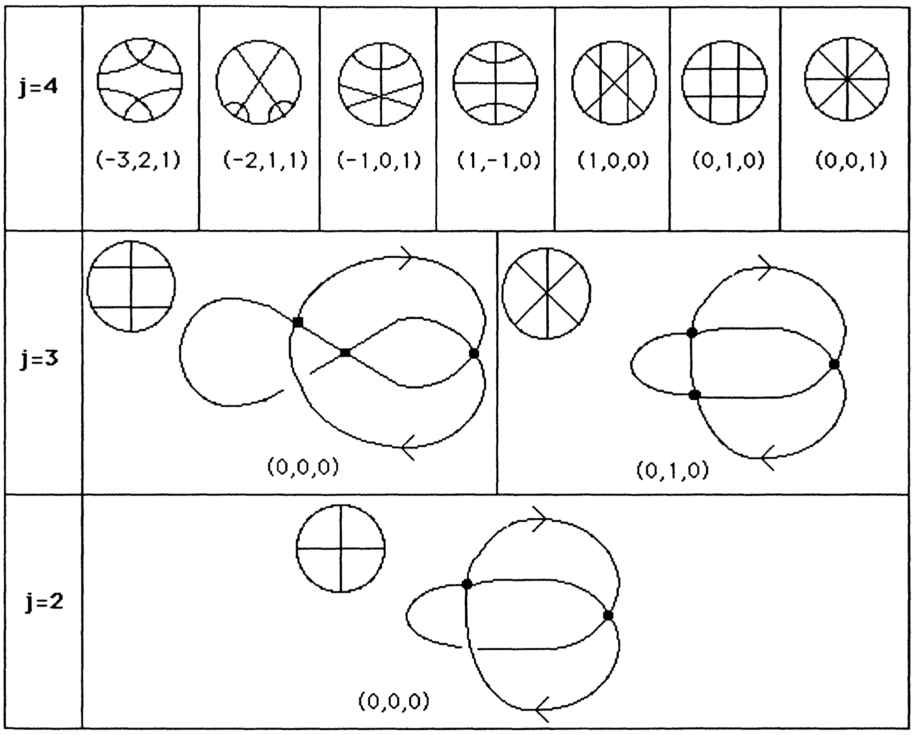
Actuality table for i=2

Fig. 9



Actuality table for i=3

Fig. 10



Actuality table for i=4

Fig. 11

equality, using (2.4) again, to change it to a good model. Its Vassiliev invariant is then zero, as in the fourth equality. Now we look at the table in Fig. 10 and see that $\mathbf{T}^2_{pqr_+}$ is not the same as the model immersion in the table, so we need to switch the crossing at r. This gives the fifth equality. Both $\mathbf{T}^2_{pqr_-}$ and $\mathbf{T}^3_{pqr}$ coincide with the models in the table, so we can use the tables to evaluate their Vassiliev invariants, obtaining:

$$v_3(T_{p_+q_+r_+}) = 2.$$

As an exercise, the reader may wish to verify by a similar calculation, that $v_2(\mathbf{T}_{p_+q_+r_+}) = 1$.

3 The initial data

In this section we consider the problem of computing the actuality table, that is the table of initial data $\mathbf{A}(i)$ which was described in (2.8) of Theorem 2.4. The main result is Theorem 3.7.

Initially we will work with linear immersions Φ^j and $[j]$-configurations α^j, rather than with circular immersions $\hat{\phi}^j$ and configurations $\hat{\alpha}^j$. We will also need to work initially with the index invariant $\mathrm{Ind}_i(\mathbf{\Phi}^j)$ rather than with $v_i(\mathbf{\Phi}^j) = S(C(\mathbf{\Phi}^j))\mathrm{Ind}_i\mathbf{\Phi}^j$. Later, when it is possible to do so, we will return to circular immersions and configurations and to the invariant $v_i(\hat{\mathbf{\Phi}}^j)$.

Recall (see Fig. 2) that:

- An $[i]$-*configuration* is an array of $2i$ points $(t_1, \ldots, t_{2i})$ on $\mathbb{R}^1$, joined up into i pairs by arcs. The symbol α_r will be reserved for a single $[i]$-configuration.
- An $\langle i \rangle$-*configuration* is an $[i-2]$-configuration, augmented by the addition of 3 more points, anywhere in the natural order of the points $t_1, \ldots, t_{2i-4}$, so that there are $2i-1$ points in all. The symbol β_s will be reserved for a single $\langle i \rangle$-configuration.
- An $\langle i \rangle^*$-*configuration* is an $\langle i \rangle$-configuration in which one of the points in the supplementary triple is distinguished.
- An $[i-1]^*$-*configuration* is an $[i-1]$-configuration, augmented by the addition of one distinguished point.

The *underlying point set* in an $[i]$-configuration α (resp. an $\langle i \rangle$-configuration β) is the collection of $2i$ (resp. $2i-1$) points on $\mathbb{R}^1$ which define them. These points have a natural order on $\mathbb{R}^1$. The pairings which define α and β are denoted by:

(3.1) $$\alpha: ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-1}}, t_{\mu_{2i}})),$$

(3.2) $$\beta: ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-s}}, t_{\mu_{2i-4}}), (x, y, z)).$$

We call the arrays in (3.1) and (3.2) the *defining symbols* for α and β. The *triple* in β is the ordered triplet (x, y, z), with $x < y < z$. Note that x, y and z can be located in an arbitrary way in $\mathbb{R}^1 \backslash \{t_1, \ldots, t_{2i-4}\}$, subject to the restriction that $x < y < z$.

We begin with the indices in the top row of $\mathbf{A}(i)$, that is the case $j = i$. Recall that this top row is special, because by (2.5) the indices Ind_i of immersions with exactly i double points only depend upon the $[i]$-configuration which they respect. It was proved by Vassiliev in [V] (see also §1.4 and §1.6 above) that these indices depend upon the kernel of a certain homomorphism h_i between finitely generated abelian groups, and our first goal is to investigate the groups and the homomorphism. Let $\mathscr{X}_{i1}$ (respectively $\mathscr{X}_{i2}$) be the free abelian group with free basis the set of all $[i]$-configurations (respectively $\langle i\rangle$-configurations). Recall that $\mathscr{X}_i = \mathscr{X}_{i1} \oplus \mathscr{X}_{i2}$. Thus a general element of $\mathscr{X}_i$ is a linear combination of $[i]$- and $\langle i\rangle$-configurations. Let $\mathscr{Y}_{i1}$ (resp. $\mathscr{Y}_{i2}$) be the free abelian group with free basis the set of all $\langle i\rangle^*$-configurations (resp. $[i-1]^*$-configurations). Recall that $\mathscr{Y}_i = \mathscr{Y}_{i1} \oplus \mathscr{Y}_{i2}$. We refer the reader to §1.5 and Fig. 3 for the definition of Vassiliev's map $h_i : \mathscr{X}_i \to \mathscr{Y}_i$. Let $\mathscr{H}_i$ be the kernel of h_i. Fix a basis $\xi_1, \ldots, \xi_{k_i}$ for $\mathscr{H}_i$. Then each basis element ξ_k in $\mathscr{H}_i$ has a natural expression as a sum:

$$\xi_k = \sum A_{rk}\alpha_r + \sum B_{sk}\beta_s. \tag{3.3}$$

The *index array* of the $[i]$-configuration α_r is the k_i-tuple of integers:

$$\mathrm{Ind}_i(\alpha_r) = (A_{r1}, A_{r2}, \ldots, A_{rk_i}). \tag{3.4}$$

The kth entry is the kth *index*.

Lemma 3.1 *Let $\xi_k \in \mathscr{H}_i$ be given by* (3.3). *Then $h_i(\beta_s) \in \mathscr{Y}_{i1}$ for all s. Also, if $A_{rk} \neq 0$, then $h_i(\alpha_r) \in \mathscr{Y}_{i1}$.*

Proof. The first claim is true by definition. Let us prove the second claim.

The definition of $h_i | \mathscr{X}_{i1}$ shows that $h_i(\alpha_r)$ projects to $\mathscr{Y}_{i2}$ non-trivially if and only if the defining symbol in (3.1) for α_r contains a pair (t_{μ_1}, t_{μ_1+1}). If this occurs, the distinguished point in $h_i(\alpha_r)$ will be the μ_lth point, so a necessary condition for cancellation between the contributions from two such terms, one in $A_{rk}h_i(\alpha_r)$ and the other in $A_{pk}h_i(\alpha_p)$ is that both contain the same pair (t_{μ_1}, t_{μ_1+1}). Even more, $h_i(\alpha_r)$ and $h_i(\alpha_p)$ must contain the same $[i-1]^*$-configuration, with contributions cancelling. However, if we remove the distinguished point from the image we obtain an $[i-1]$-configuration, and since the two $[i-1]$-configurations must coincide we conclude that $\alpha_r \equiv \alpha_p$ and $A_{rk} = 0$. Since we have assumed that the coefficient A_{rk} in (3.3) is non-zero, it follows that for $h_i(\xi_k)$ to be zero, no such pair (t_{μ_1}, t_{μ_1+1}) occurs in α_r. But then $h_i(\alpha_i) \in \mathscr{Y}_{i1}$. □

Our next lemma underscores the difficulties in understanding the kernel of h_i.

Lemma 3.2 The restrictions $h_i | \mathscr{X}_{i1}$ and $h_i | \mathscr{X}_{i2}$ are one-to-one.

Proof. It is easy to see that $h_i | \mathscr{X}_{i2}$ is one-to-one. For, by definition, $h_i(\beta_s)$ is a linear combination of three $\langle i\rangle^*$-configurations which have identical patterns of paired points and the same triple, but different distinguished points in the triple. There cannot be any cancellation between these three terms because the distinguished points are different. There also cannot be cancellation between $B_s h_i(\beta_s)$ and $C_t h_i(\beta_t)$ unless β_s coincides with β_t and $B_s = -C_t$.

To see that $h_i | \mathscr{X}_{i1}$ is one-to-one, suppose that

$$\varrho = A_1\alpha_1 + \cdots + A_w\alpha_w \tag{3.5}$$

with each $A_r \neq 0$ and $h_i(\varrho) = 0$. It will be convenient to use the notation introduced in (3.1) to describe the $[i]$-configurations α_r in (3.5). Without loss of generality we may assume that the pairs $(t_{\mu_{2j-1}}, t_{\mu_{2j}})$ in this symbol are arranged so that $t_{\mu_1} < t_{\mu_2}$, $t_{\mu_3} < t_{\mu_4}$, $\ldots$, $t_{\mu_{2j-1}} < t_{\mu_{2i}}$, and that the pairs are ordered so that $t_{\mu_1} < t_{\mu_3} < t_{\mu_5} < \cdots < t_{\mu_{2i-1}}$. Let $\mu_2 - \mu_1$, $\mu_4 - \mu_3$, $\ldots$, $\mu_{2i} - \mu_{2i-1}$ be the lengths of the arcs which define the pairings in α. Let the length $\|\alpha_r\|$ of α_r be the length of the shortest arc in α_r. Order the terms in (3.5) so that

$$\|\alpha_1\| \leqq \|\alpha_2\| \leqq \cdots \leqq \|\alpha_q\|.$$

We ask: what are the possibilities for $\|\alpha_1\|$?

The first thing to notice is that $\|\alpha_1\| \geqq 2$. For, if not, we would have an arc of length 1 in α_1, and $h_i(\alpha_1)$ would project to $\mathcal{Y}_{i2}$ non-trivially, contradicting Lemma 3.1. We will complete the proof by showing, by induction on $\|\alpha_1\|$, that in fact no length is possible.

Let β_{pq} denote an $\langle i\rangle^*$-configuration whose qth point is distinguished. Since $h_i(\alpha_p)$ is contained entirely in $\mathcal{Y}_{i1}$, it must be a linear combination of exactly $2i-1$ such configurations, i.e.

$$0 = h_i(\alpha_p) = \sum_{q=1}^{2i-1} (-1)^{q-1}\beta_{pq}. \tag{3.6}$$

Therefore

$$0 = h_i(\varrho) = \sum_{p=1}^{w} A_p \sum_{q=1}^{2i-1} \beta_{pq}. \tag{3.7}$$

Now, there cannot be any cancellation in (3.7) between β_{pq} and β_{pr} if $q \neq r$ because the qth point is distinguished in the former and the rth in the latter. Thus the only way we can have cancellation on the right hand side of (3.7) is if there are p, r, q such that $\alpha_p \neq \alpha_r$, but $\beta_{pq} = \beta_{rq}$. We can say more about such pairs α_p and α_r. Let x be the distinguished point in β_{pq}. Now, $h_i^{-1}(\beta_{pq})$ contains exactly three configurations: the $\langle i\rangle$-configuration β which is obtained from β_{pq} by forgetting that x is a distinguished point, and the two $[i]$-configurations $\pm\beta^{x1}$ and $\pm\beta^{x2}$ which are the resolutions of β at x. Thus α_r may be obtained from α_p by a single transposition of a pair of adjacent points in the underlying point set. That is, the $[i]$-configurations α_p and α_r are identical everywhere except near a single pair of arcs, where they differ in the manner indicated in Fig. 12.

Suppose that $\|\alpha_1\|$ is m. Let $[t_{q-m}, t_q]$ be the arc of length m in α_1. Then the symbol in (3.1) for α_1 must take the form:

$$\alpha_1 : (\dots, (t_{q-m}, t_q), (t_{q-1}, t_{q-1+u}), \dots)$$

where $t_{q-m} < t_{q-1} < t_q < t_{q-1+u}$ because if not there would be an arc of length *less* than m in the defining symbol for α_1. But then, α_1 must be paired with an α_r which is obtained from it by a transposition of t_{q-1} and t_q, as in Fig. 12. However this would produce an arc of length $m-1$ in α_r, contradicting (3.5). Thus $h_i|\mathcal{X}_{i1}$ is one-to-one. □

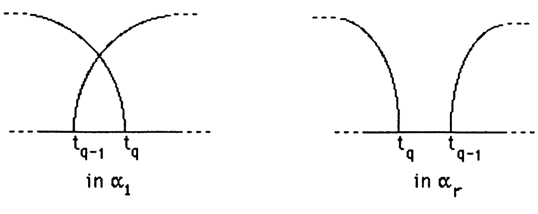


Fig. 12

Recall that in §2 we defined the notions of a separating arc in a $[j]$-configuration, and of admissible and inadmissible circular $[j]$-configurations. We now extend these ideas to $\langle j\rangle$-configurations. An arc τ in a $\langle j\rangle$-configuration will be called *separating* if it is separating in the underlying circular $[j-2]$-configuration, and if all three points in the defining triple are on the same side of τ. A $\langle j\rangle$-configuration β and its associated circular $\langle j\rangle$-configuration $\hat{\beta}$ are *inadmissible* if $\hat{\beta}$ contains a separating arc. Otherwise, both are *admissible.*

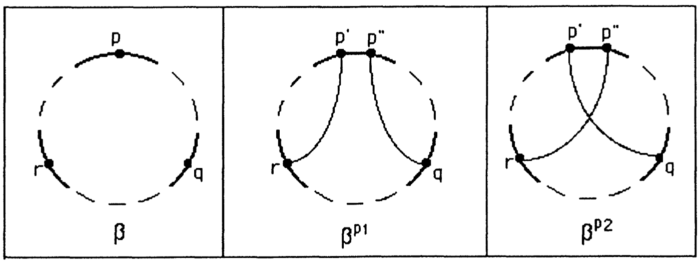
Lemma 3.3 Let $\xi_k = \sum A_{rk}\alpha_r + \sum B_{sk}\beta_s$ be a basis element for $\mathscr{H}_i$. Choose any β_s with defining symbol (3.2). Then B_{sk} is zero if β_s is inadmissible.

Proof. Since $h_i|\mathscr{X}_{i2}$ is one-to-one by Lemma 3.2, the only way that B_{sk} could be non-zero is if each component of its image under h_i cancels against a corresponding component from some α_r. However, if the circular $\langle i\rangle$-configuration $\hat{\beta}_s$ had a separating arc, the index of the associated $[i]$-configuration would be zero by (2.7). Thus no such cancellation could occur. □

Choose any admissible $\langle i\rangle$-configuration β. Then β is defined by an associated $(2i-1)$-tuple, as in (3.2). Choose any point p in the triple (x, y, z). We *resolve* β into $[i]$-configurations β^{p1} and β^{p2}, as follows. Replace the point p by a pair of points $\{p', p''\}$, which are assumed to be arbitrarily close to p on $\mathbb{R}^1$. Then form two $[i]$-configurations by pairing p' and p'' with the other two points in the triple, in the manner illustrated in Fig. 13. There are two ways to do this. The six resolutions of β are the six $[i]$-configurations:

$$\begin{aligned}
\beta^{x1} &= ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-5}}, t_{\mu_{2i-4}}), (x'', y), (x', z)),\\
\beta^{x2} &= ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-5}}, t_{\mu_{2i-4}}), (x'', z), (x', y)),\\
\beta^{y1} &= ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-5}}, t_{\mu_{2i-4}}), (x, y'), (y'', z)),\\
\beta^{y2} &= ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-5}}, t_{\mu_{2i-4}}), (x, y''), (y', z)),\\
\beta^{z1} &= ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-5}}, t_{\mu_{2i-4}}), (x, z''), (y, z')),\\
\beta^{z2} &= ((t_{\mu_1}, t_{\mu_2}), \ldots, (t_{\mu_{2i-5}}, t_{\mu_{2i-4}}), (x, z'), (y, z'')).
\end{aligned} \tag{3.8}$$

Lemma 3.4 *Let* $\xi_k = \sum A_{rk}\alpha_r + \sum B_{sk}\beta_s$ be a basis element for $\mathscr{H}_i$. Choose any admissible β_s. Assume that the defining symbol for β_s is given by (3.2). Then B_{sk} is determined by the indices of the two resolutions of β_s, at any one of the three points in the triple.



Resolutions of a circular <j>-configuration

Fig. 13

Proof. Let $\beta = \beta_s$ and let B be its unknown coefficient in ξ_k. Let (x, y, z) be the triple in β. Choose $p \in \{x, y, z\}$ and let β^{p1} and β^{p2} be the two resolutions of β. Thus each β^{p1} and each β^{p2} is an $[i]$-configuration. Let $\mathrm{Ind}_i(\beta^{p1})$ and $\mathrm{Ind}_i(\beta^{p2})$ be the indices of β^{p1} and β^{p2}. Now, the images of β, β^{p1} and β^{p2} under h_i each have a component in the $\langle i\rangle^*$-configuration β^* which is obtained from β by making the point p the distinguished point. Even more, $h_i^{-1}(\beta^*)$ meets these three basis elements of $\mathscr{X}_i$ and no others. From this it follows that in order for $h_i(\xi_k)$ to be zero, the following relations must hold:

$$
\begin{aligned}
B &= (-1)^{i+1+\#x}[\mathrm{Ind}_i(\beta^{x1}) + \mathrm{Ind}_i(\beta^{x2})] \\
&= -(-1)^{i+1+\#y}[\mathrm{Ind}_i(\beta^{y1}) + \mathrm{Ind}_i(\beta^{y2})] \\
&= (-1)^{i+1+\#z}[\mathrm{Ind}_i(\beta^{z1}) + \mathrm{Ind}_i(\beta^{z2})], \qquad (3.9)
\end{aligned}
$$

where $\#x$, $\#y$ and $\#z$ are the positions of x, y and z among the $2i-1$ points in the ordered underlying point set. The minus sign in front of the middle expression results from the fact (see §1.5) that if β is an $\langle i\rangle$-configuration, then $h_i(\beta)$ is a linear combination of the three $\langle i\rangle^*$-configurations which belong to the same underlying $\langle i\rangle$-configuration, the coefficients being -1, $+1$, -1 according as the distinguished point is the first, the second or the third point in the triple. Since *every* $\langle i\rangle$-configuration β_s arises as $h_i^{-1}(\beta_s^*)$ for some $\langle i\rangle^*$-configuration β_s^*, the assertion of Lemma 3.4 follows. □

Now, we may eliminate B from Eqs. (3.9), changing the three equations into two equations, which only involve the indices of $[i]$-configurations. These equations then go over to two new equations between the associated circular $[i]$-configurations $\hat{\beta}^{xq}$, because by Lemma 2.1 we know that if two $[i]$-configurations define the same circular $[i]$-configuration their indices coincide. With this simplification in place, we may use Eqs. (2.2) to replace the invariant $\mathrm{Ind}_i(\hat{\beta}^{pq})$ by the related invariant $v_i(\hat{\beta}^{pq})$. The modified version of the first equation in (3.9) will then be:

$$(-1)^{\#x+|\beta^{x1}|}[v_i(\hat{\beta}^{x1}) - v_i(\hat{\beta}^{x2})] + (-1)^{\#y+|\beta^{y1}|}[v_i(\hat{\beta}^{y1}) - v_i(\hat{\beta}^{y2})] = 0.$$

The minus sign inside the brackets occurs because $|\hat{\beta}^{p1}| = |\hat{\beta}^{p2}| + 1$ for every $p \in \{x, y, z\}$. To simplify this expression still further, let $\hat{\alpha}$ be the circular $[i-2]$-configuration obtained from the $\hat{\beta}$ by deleting the triple of $\hat{\beta}$. Let τ_{xy}, τ_{yz} and τ_{zx} be the arcs joining x to y, y to z, and z to x respectively. Let $|\tau_{pq}|$ denote the intersection number of τ_{pq} with the arcs in $\hat{\alpha}$. Then:

$$\#x - \#y \equiv |\tau_{xy}| + 1 \pmod 2$$

and

$$
\begin{aligned}
|\hat{\beta}^{x1}| - |\hat{\beta}^{y1}| &\equiv |\tau_{zx}| - |\tau_{yz}| \\
&\equiv |\tau_{zx}| + |\tau_{yz}| \equiv |\tau_{xy}| \pmod 2.
\end{aligned}
$$

Thus after simplification we obtain the following two equations from (3.9)

$$v_i(\hat{\beta}^{x1}) - v_i(\hat{\beta}^{x2}) = v_i(\hat{\beta}^{y1}) - v_i(\hat{\beta}^{y2}), \qquad (3.10)_{xy}$$

$$v_i(\hat{\beta}^{y1}) - v_i(\hat{\beta}^{y2}) = v_i(\hat{\beta}^{z1}) - v_i(\hat{\beta}^{z2}), \qquad (3.10)_{yz}$$

where there is one such set for each $\hat{\beta}$ in the set $\{\hat{\beta}_s; s = 1, \ldots, m\}$ of admissible circular $\langle i \rangle$-configurations.

Regard the system of equations defined by (3.10) as a system of linear equations in unknowns

$$(X_{1i}, \ldots, X_{ni}) = (v_i(\hat{\alpha}_1), \ldots, v_i(\hat{\alpha}_n)).$$

Let

$$\mathbb{X}_i = (X_{1i}, \ldots, X_{ni})^{tr}.$$

Rewrite the system (3.10) in the form:

$$\mathbb{M}_i \mathbb{X}_i = 0 \tag{3.11}$$

where and $\mathbb{M}_i$ is the $2m$ by n matrix of coefficients in the system of equations defined by $(3.10)_{xy}$ and $(3.10)_{yz}$, for $s = 1, \ldots, m$.

Lemma 3.5 *A necessary condition for the existence of non-trivial Vassiliev invariants of order i is that the solution space to the homogeneous system of equations* (3.11) *have positive dimension* $k_i > 0$.

Proof. The equations in (3.9) express each B_{sk} in (3.3) in three possibly distinct ways as a sum of indices A_{rk} of associated $[i]$-configurations β^{pq}. After eliminating B_{sk} we are left with two (in general distinct) linear equations in the unknown indices which are simplified using the related invariants to (3.11).

These equations in the unknown indices are the only relations between the indices of admissible circular $[i]$-configurations, because by Lemma 3.2, the image of an $[i]$-configuration which has non-zero coefficient under h_i projects trivially to $\mathcal{Y}_{i2}$, and we have given a full description of $h_i^{-1}(\beta_s^*)$ as β_s^* ranges over the basis for $\mathcal{Y}_{i1}$. Thus, equations (3.11) are the only relations between the invariants of order i of admissible circular $[i]$-configurations. □

We have seen how to compute the Vassiliev invariants of the configurations in the top row of the actuality table $\mathbf{A}(i)$. We next proceed to modify our calculations in order to determine the Vassiliev invariant of the immersions in the remaining rows. The basic new complication which we must deal with is that (2.5) does not hold if $j < i$. Thus the invariant $v_i(\Phi^j)$ of an immersion which respects our configuration depends not only on the configuration, but also on the choice of the immersion.

The first thing we need to do is to describe *generic* immersions $\hat{\Psi}$ which respect circular $\langle j \rangle$-configurations. Such immersions must contain $j - 2$ transverse double points and one triple point. In order to give precise meaning to the notion of *transverse* near a triple point, we require that the three branches of the image of our immersion of the oriented circle S^1 coincide locally with the positively oriented x, y and z axes near the triple point, as in Fig. 14. In the following discussion, an *immersion* will always mean a generic immersion.

Next we need a notion of equivalence of immersions respecting $\langle j \rangle$-configurations. Call Ψ_0 and Ψ_1 *equivalent* if they are isotopic via an isotopy Ψ_t such that Ψ_t is a generic immersion for each $0 \leqq t \leqq 1$. This implies that Ψ_0 and Ψ_1 respect the same $\langle j \rangle$-configuration.

The next step is to develop a scheme for resolving the triple point into a pair of double points, in such a way that the resolutions of $\hat{\Psi}$ respect the resolutions

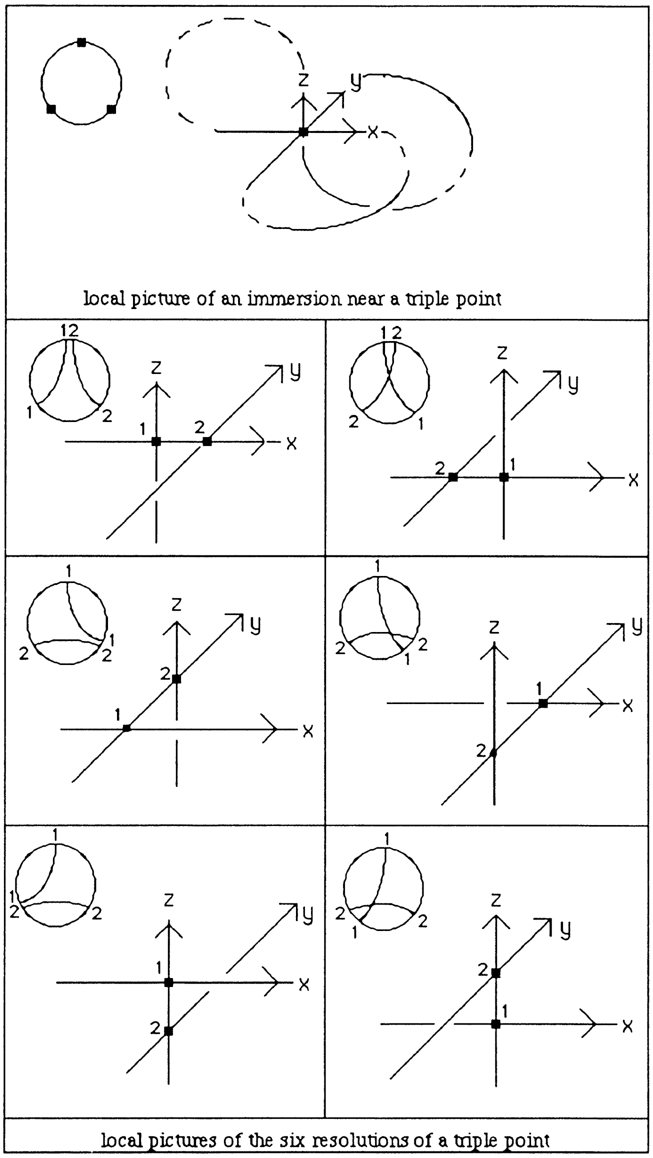


Fig. 14

considered earlier for $\hat{\beta}$. See Fig. 14. Recall that we defined resolutions of $\hat{\beta}$ at p (respectively q and r). The corresponding three resolutions of $\hat{\Psi}$ are: resolve along the x (respectively y and z) axes by sliding the y or z (respectively z or x, x or y) axes along the x (respectively y, z) axes. For each of these three choices there are two possibilities: slide in the positive or in the negative direction. This gives six resolutions of an immersion $\hat{\Psi}$ which respect the corresponding six resolutions of the circular $\langle j \rangle$-configuration $\hat{\beta}$.

Recall that earlier we defined the concept of an inadmissible $\langle j \rangle$-configuration. Our immersion $\hat{\Psi}$ is a *good model* respecting an inadmissible $\hat{\beta}$ if, whenever $\hat{\beta}$ contains a separating arc τ, the two components of the image of S^1 split along the endpoints of τ are geometrically unlinked. Notice that this implies that each of the six resolutions of $\hat{\Psi}$ is a good model respecting an inadmissible $[j]$-configuration. The reason is: the double point which belongs to the separating arc in $\hat{\Psi}$ divides the image of $\hat{\Psi}$ into two subgraphs, say a red one and a blue one, and the changes which occur in passing to the resolutions of $\hat{\Psi}$ are always entirely in the red half or entirely in the blue half, and so cannot introduce any linking between the two.

We can always choose an good model respecting an inadmissible $\hat{\beta}$. Thus, in the following discussion, we will only consider admissible $\hat{\beta}$'s since the Vassiliev invariant of good models respecting inadmissible circular $[j]$-configurations are always zero.

Notice that an admissible circular $\langle j \rangle$-configuration $\hat{\beta}$ may have inadmissible resolutions. This happens exactly when the triple of $\hat{\beta}$ contains an adjacent pair. Thus, we may choose an immersion $\hat{\Psi}$ respecting $\hat{\beta}$ with the property that every resolution of $\hat{\Psi}$ respecting an inadmissible resolution of $\hat{\beta}$ is good. For example, we only need to require that for an adjacent pair in the triple of $\hat{\beta}$, the corresponding loop in the image of the immersion $\hat{\Psi}$ chosen to respecting $\hat{\beta}$ bounds a disk whose interior has no intersection with the image of $\hat{\Psi}$.

We are now ready to proceed with our calculation. Since we know the Vassiliev invariants in the top row $\mathbf{R}(i, i)$ of our table we may proceed inductively, assuming that $\mathbf{R}(i, n)$ has been constructed for every n with $j < n \leqq i$. Let:

- $\hat{\alpha}_1, \ldots, \hat{\alpha}_{n_j}$ be the list of admissible circular $[j]$-configurations;
- $\hat{\Phi}_r$ be a choice of an immersion respecting $\hat{\alpha}_r$, $r = 1, \ldots, n_j$;
- $\hat{\mathbf{\Phi}}_r$ be the equivalence class of $\hat{\Phi}_r$;
- $\hat{\beta}_1, \ldots, \hat{\beta}_{m_j}$ be the list of admissible circular $< j >$ -configurations;
- $\hat{\Psi}_s$ be a choice of an immersion respecting $\hat{\beta}_s$, $s = 1, \ldots, m_j$ such that every resolution of $\hat{\Psi}_s$ respecting an inadmissible resolution of $\hat{\beta}_s$ is good;
- $\hat{\mathbf{\Psi}}_s$ be the equivalence class of $\hat{\Psi}_s$.

The data which is needed to complete $\mathbf{R}(i, j)$ is the set of unknown invariants:

$$(v_i(\hat{\mathbf{\Phi}}_1^j), \ldots, v_i(\hat{\mathbf{\Phi}}_{n_j}^j)).$$

Notice that each resolution of an immersion Ψ which respects a $\langle j \rangle$-configuration is an immersion $\hat{\Psi}^{pq}$ which respects a $[j]$-configuration. The latter has a well-defined index $\mathrm{Ind}_i(\hat{\Psi}^{pq})$. It is an invariant of the equivalence class of the immersion.

Let $\Psi = \Psi_s$. The analogue of (3.9) for immersions is:

$$\begin{aligned} \mathrm{Ind}_i(\mathbf{\Psi}) &= (-1)^{i+1+\#x}[\mathrm{Ind}_i(\mathbf{\Psi}^{x1}) + \mathrm{Ind}_i(\mathbf{\Psi}^{x2})] \\ &= (-1)^{i+\#y}[\mathrm{Ind}_i(\mathbf{\Psi}^{y1}) + \mathrm{Ind}_i(\mathbf{\Psi}^{y2})] \\ &= (-1)^{i+1+\#z}[\mathrm{Ind}_i(\mathbf{\Psi}^{z1}) + \mathrm{Ind}_i(\mathbf{\Psi}^{z2})]. \end{aligned} \tag{3.12}$$

Equations (3.12) determine a system of equations which are the natural generalization of $(3.10)_{xy}$ and $(3.10)_{yz}$ for immersions:

$$v_i(\hat{\mathbf{\Psi}}^{x1}) - v_i(\hat{\mathbf{\Psi}}^{x2}) = v_i(\hat{\mathbf{\Psi}}^{y1}) - v_i(\hat{\mathbf{\Psi}}^{y2}), \tag{3.13$_{xy}$}$$

$$v_i(\hat{\mathbf{\Psi}}^{y1}) - v_i(\hat{\mathbf{\Psi}}^{y2}) = v_i(\hat{\mathbf{\Psi}}^{z1}) - v_i(\hat{\mathbf{\Psi}}^{z2}). \tag{3.13$_{yz}$}$$

In principal, we should now be able to go over to the analogue of (3.11), however there is a complication. Earlier, we chose a model immersion $\hat{\Phi}_r$ as our representative immersion which respects $\hat{\alpha}_r$. Unfortunately, the immersions which occur in equations (3.13) may not agree with these chosen models. To handle this problem, we now recall that by Lemma (2.3), part (iii), we know that $\hat{\Psi}^{pq}$ coincides with $\hat{\Phi}_r$ up to crossing changes. Therefore we may use (2.4), together with the data in $\mathbf{R}(i, n)$ for $j + 1 \leqq n \leqq i$, to compute the difference between the Vassiliev invariants of $\hat{\mathbf{\Psi}}^{pq}$ and $\hat{\mathbf{\Phi}}_r$. Set

$$(3.14) \qquad N_s^{pq} = v_i(\hat{\mathbf{\Phi}}_r) - v_i(\hat{\mathbf{\Psi}}_s^{pq}).$$

Then N_s^{pq} is a linear function of the Vassiliev invariants in $\mathbf{R}(i, n)$ for $j + 1 \leqq n \leqq i$ with integer coefficients. Let $X_s^{pq} = X_{rj}$ denote the unknown Vassiliev invariant of $\hat{\mathbf{\Phi}}_r$, so that $X_s^{pq} \in \{X_{1j}, \ldots, X_{n_j j}\}$.

If $\hat{\beta}_s^{pq}$ is inadmissible, $\hat{\Psi}_s^{pq}$ is a good model for an immersion respecting $\hat{\beta}_s^{pq}$ by our choice of $\hat{\Psi}_s$. In this case, we may set $X_s^{pq} = 0$ as well as $N_s^{pq} = 0$. Then, we have

$$(3.15) \qquad v_i(\hat{\mathbf{\Psi}}_s^{pq}) = X_s^{pq} - N_s^{pq}$$

for any resolution $\hat{\beta}_s^{pq}$.

Substituting (3.15) into $(3.13)_{xy}$ and $(3.13)_{yz}$, we obtain the required analogues of $(3.10)_{xy}$ and $(3.10)_{yz}$ for the case $j < i$:

$$(3.16)_{xy} \qquad [X_s^{x_s 1} - X_s^{x_s 2}] - [X_s^{y_s 1} - X_s^{y_s 2}] = [N_s^{x_s 1} - N_s^{x_s 2}] - [N_s^{y_s 1} - N_s^{y_s 2}],$$

$$(3.16)_{yz} \qquad [X_s^{y_s 1} - X_s^{y_s 2}] - [X_s^{z_s 1} - X_s^{z_s 2}] = [N_s^{y_s 1} - N_s^{y_s 2}] - [N_s^{z_s 1} - N_s^{z_s 2}].$$

The right hand sides of $(3.16)_{xy}$ and $(3.16)_{yz}$ are known integers. Comparing $(3.10)_{xy}$ and $(3.10)_{yz}$ with $(3.16)_{xy}$ and $(3.16)_{yz}$, we see that we have replaced the homogeneous system of equations in (3.11) with a corresponding inhomogeneous system. Thus we have proved:

Lemma 3.6 *Choose an assignment of Vassiliev invariants to the top row $\mathbf{R}(i, i)$ of the actuality table, i.e. a solution to the homogeneous system of equations* (3.11). *Then this assignment extends to an assignment of Vassiliev invariants to $\mathbf{R}(i, j)$, $1 \leqq j \leqq i - 1$, if and only if, corresponding to it there is a solution to the inhomogeneous systems of equations*

$$(3.17) \qquad \mathbb{M}_j \mathbb{X}_j = \mathbb{N}_{i,j}, \quad 1 \leqq j \leqq i - 1$$

where:

- $\mathbb{X}_j$ *is a* 1 *by* n_j *matrix containing the unknown array of invariants* $(X_1, \ldots, X_{n_j})$ *of the immersions chosen to respect the admissible circular* $[j]$*-configurations* $\hat{\alpha}_1, \ldots, \hat{\alpha}_{n_j}$;
- $\mathbb{M}_j$ *is the* n_j *by* $2m_j$ *matrix constructed in the computation of* $\mathbf{R}(j, j)$. *It has a column for each* $\hat{\alpha}_r$ *and two rows for each* $\hat{\beta}_s$;
- $N_{i,j}$ *is the* 1 *by* $2m_j$ *matrix determined from the entries on the right hand side of* $(3.16)_{xy}$ *and* $(3.16)_{yz}$. *These are computed from* (3.14), *using the data in* $\mathbf{R}(i, n)$ *with* $j < n \leqq i$.

We may now put together Lemmas (3.5) and (3.6), to state the main result in this section. First, we put the Eqs. (3.11) and (3.17) together:

$$\begin{cases} \mathbb{M}_i \mathbb{X}_i = 0, \\ \mathbb{M}_j \mathbb{X}_j = \mathbb{N}_{i,j}, \quad 1 \leqq j \leqq i-1. \end{cases}$$

Now observe that that the components of the column vector $\mathbb{N}_{i,j}$ are linear functions of components of $\mathbb{X}_{j+1}, \ldots, \mathbb{X}_i$ with integer coefficients. Thus, our system is actually a much larger system of homogeneous linear equations in the combined set of variables:

$$\mathbb{M}^i \mathbb{X}^i = 0 \tag{3.19}$$

where $\mathbb{X}^i$ is a column vector made up out of the previously defined column vectors $(\mathbb{X}_i, \mathbb{X}_{i-1}, \ldots, \mathbb{X}_1)$.

Theorem 3.7 *Consider the space of solutions of the combined system of homogeneous linear equations* (3.19). *There are non-trivial Vassiliev invariants of order i only if the solution space to this system has positive dimension. The number of linearly independent Vassiliev invariant of order* $\leqq i$ *is the codimension of the subspace of those solutions such that* $\mathbb{X}_i = 0$.

Proof. The unknowns in this system are the components of the vectors $\mathbb{X}_i, \mathbb{X}_{i-1}, \ldots, \mathbb{X}_1$. The vector $\mathbb{X}_j$, for $j = 1, \ldots, i$, has an entry for each admissible circular $[j]$-configuration. A solution to this system of equations with $\mathbb{X}_i \neq 0$ gives us an actuality table for a Vassiliev invariants of order i and hence determines a Vassiliev invariant of order i. On the other hand, the invariants in an actuality table $\mathbf{A}(i)$ certainly satisfy this system of equations. Moreover, the system of equations is consistent with respect to the order meaning that if we set $\mathbb{X}_i = 0$, it reduces to the system of equations corresponding to the order $i-1$. This proves Theorem 3.7. □

Examples are in order, but before we give them we address an important question. In order to set up the system (3.19) it was necessary to make a choice of model immersions representing all $[j]$-configurations with $j \leq i$. Different choices will lead to different equations, because the matrices $\mathbb{N}_{i,j}$ depend upon the choice of the model immersions. Our question is: does the existence of a solution depend upon making a correct choice? The answer is no:

Corollary 3.8 *The existence of a solution to the system of Eqs.* (3.19) *is independent of the choice of the immersions which respect* $[j]$*-configurations.*

Proof. Let us assume that an actuality table $\mathbf{A}(i)$ of order i has been completed. It suffices to prove that if we change one of the immersions in the table, say $\hat{\Phi}$ to $\hat{\Phi}'$ where $\hat{\Phi}$ and $\hat{\Phi}'$ respect the same circular $[j]$-configuration, we can complete a new actuality table $\mathbf{A}'(i)$.

Rows $\mathbf{R}(i,i)$, $\mathbf{R}(i,i-1)$, $\ldots$, $\mathbf{R}(i,j+1)$ in $\mathbf{A}'(i)$ can clearly be chosen to be identical to the corresponding rows in $\mathbf{A}(i)$. Since $\hat{\Phi}$ and $\hat{\Phi}'$ both respect the same $[j]$-configuration, we know from Lemma (2.3), part (iii), that we may find a diagram for $\hat{\Phi}'(\mathbb{S}^1)$ which coincides with that chosen earlier for $\hat{\Phi}(\mathbb{S}^1)$ up to crossing changes. Since rows $i, \ldots, j+1$ have already been filled in for $\mathbf{A}'(i)$, we may then use (2.5) to compute the invariant X' of the new immersion $\hat{\Phi}'$. Suppose that

$$v_i(\hat{\mathbf{\Phi}}) = v_i(\hat{\mathbf{\Phi}}') + \Delta. \tag{3.20}$$

Then Eq. (3.15) will be modified, if it should happen that a particular $\hat{\Phi}_s^{pq}$ coincides with $\hat{\Phi}$, to:

$$v_i(\hat{\Phi}_s^{pq}) = (X_s^{pq} - \Delta) - (N_s^{pq} - \Delta). \tag{3.22}$$

Thus whenever this particular term X_s^{pq} occurs in $(3.16)_{xy}$ or $(3.16)_{yz}$, terms on both sides of the equation will change in corresponding ways, and the equality will still be satisfied.

The entries in row $j-1$ of $\mathbf{A}'(i)$ will be identical with those in $\mathbf{A}(i)$, because the invariants in row j are only used as a reference point in the computation of the invariants of an immersion in row $j-1$. □

Example 3.9

In Figs. 9–11 we exhibited actuality tables for Vassiliev invariants of orders $i = 2$, 3 and 4. We now use Theorem 3.7 to show how the table for $i = 4$ was constructed.

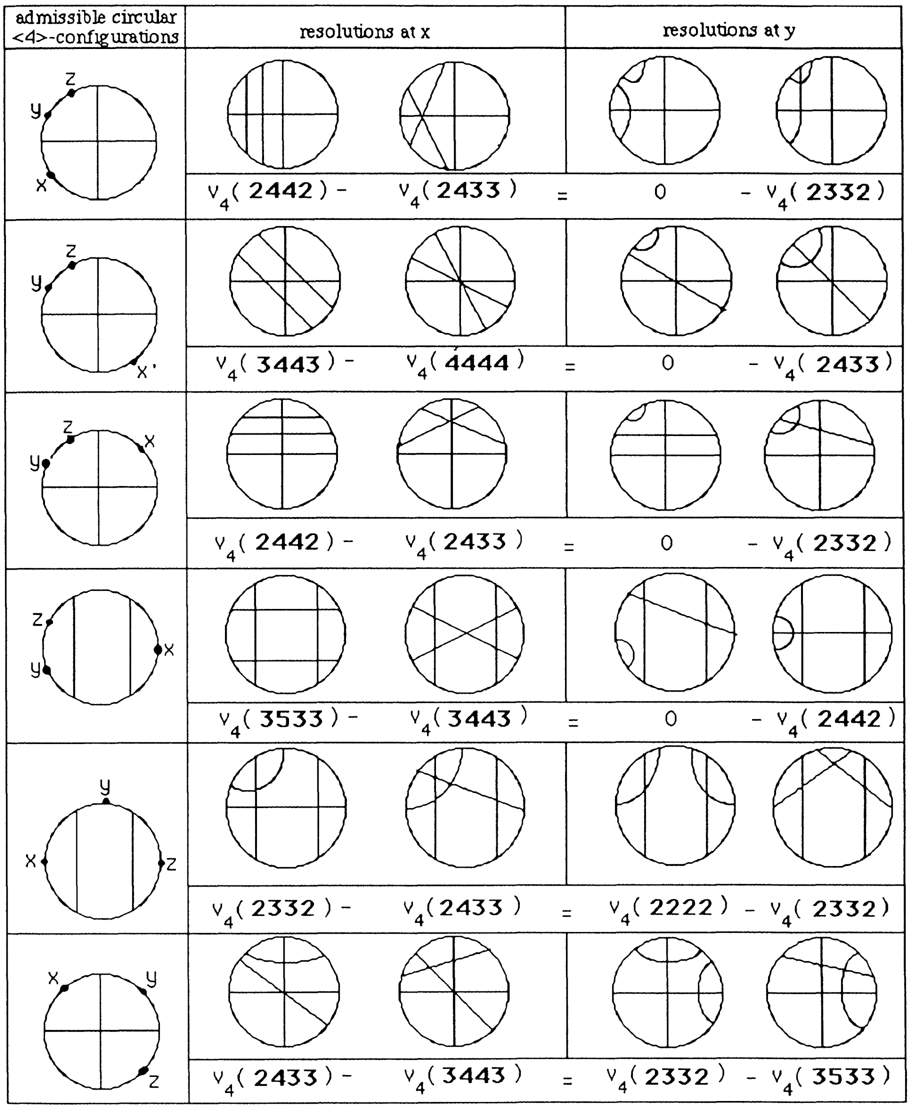
The first step is to list the admissible circular $[j]$-configurations for $2 \leqq j \leqq 4$ and to choose an immersion which respects each for the cases $j = 3, 2$. In this regard, the following notation was useful to us: A circular $[j]$-configuration is determined uniquely by choosing any initial point and recording the lengths of the joining arcs in the order in which they are encountered, travelling around the circle clockwise and measuring each arc by the differences between the order of its initial point and endpoint. Of course, this series of lengths is non-unique, since it depends upon the choice of the initial point. The *name* of a $[j]$-configuration is the smallest number so-obtained, as the initial point is varied. Using this notation and observing that the unknown invariants are in one-to-one correspondence with circular $[j]$-configurations, we may order the entries in the vectors of unknowns by the lexicographically ordered names of their configurations. Let X_{jk} denote the invariant of the immersion which respects the kth $[j]$-configuration, in this lexicographical ordering. Reading the rows from top to bottom, and reading across each row from left to right in Fig. 8 we then note that the configurations in the three rows of the table in Fig. 8 are denoted by the symbols (**2222**, **2332**, **2433**, **2442**, **3443**, **3533**, **4444**), (**232**, **333**), (**22**) respectively.

The indices of the configurations in the top row are determined by solving equations (3.10), so the next step is to set up the equations, which are in 1–1 correspondence with admissible circular $\langle 4 \rangle$-configurations. See Fig. 15. The first thing to notice is that if a $\langle j \rangle$-configuration contains three adjacent dots it will determine two trivial equations, and we have omitted the unique $\langle 4 \rangle$-configuration with this property. Notice also that if there is a pair of adjacent dots or if certain symmetries occur the two equations will reduce to one, and we have only shown one.

Solving the six equations listed below the pictures in Fig. 15, we find that:

$$\begin{aligned}
v_4(\mathbf{2222}) &= -3v_4(\mathbf{3443}) + 2v_4(\mathbf{3533}) + v_4(\mathbf{4444}),\\
v_4(\mathbf{2332}) &= -2v_4(\mathbf{3443}) + v_4(\mathbf{3533}) + v_4(\mathbf{4444}),\\
v_4(\mathbf{2433}) &= -v_4(\mathbf{3443}) + v_4(\mathbf{4444}),\\
v_4(\mathbf{2442}) &= v_4(\mathbf{3443}) - v_4(\mathbf{3533}).
\end{aligned}$$

Thus we have shown that $v_4(\mathbf{3443})$, $v_4(\mathbf{3533})$ and $v_4(\mathbf{4444})$ determine the remaining indices, and may be chosen as a basis for the solution space to (3.10), which has



Equations relating the indices of circular [4]-configurations

Fig. 15

dimension 3. We may choose them in an arbitrary fashion, a natural choice being (1, 0, 0), (0, 1, 0), and (0, 0, 1) respectively. See Fig. 11. With these choices we compute

$$v_4(\mathbf{2222}) = (-3, 2, 1),$$
$$v_4(\mathbf{2332}) = (-2, 1, 1),$$
$$v_4(\mathbf{2433}) = (-1, 0, 1),$$
$$v_4(\mathbf{2442}) = (1, -1, 0).$$

The indices in the top row of the table in Fig. 11 have been filled in. (Remark: It's probably a complete accident that the indices of the numerically largest configurations determine the indices of the remaining ones, but the general problem is sufficiently difficult so that we cannot ignore even such small hints of possible structure.)

We pass to the row associated to $j = 3$. There is a unique circular $\langle 3 \rangle$-configuration, denoted $\mathbf{\Phi}$, depicted in the top left corner of Fig. 16. To its right is an

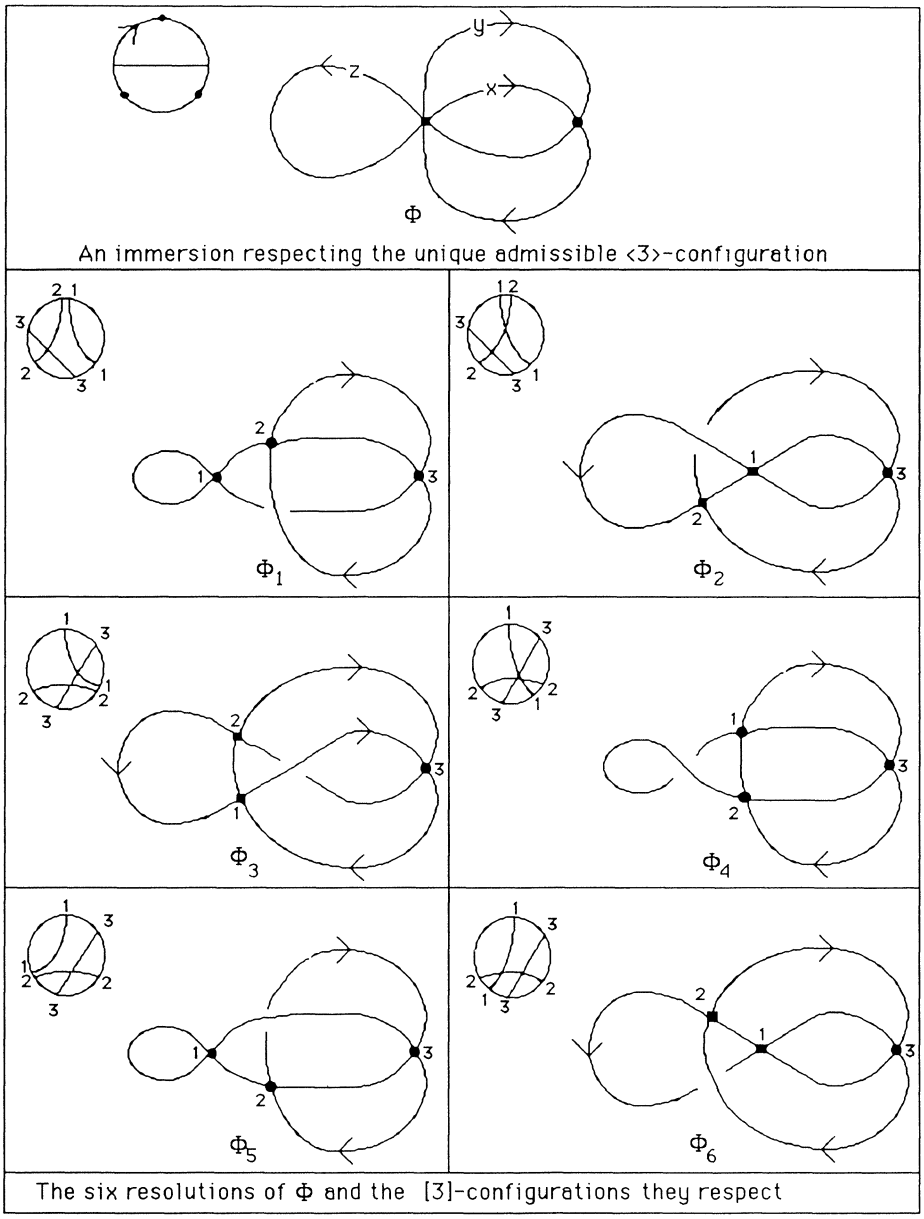


Fig. 16

immersion which we have chosen to respect it. The image of S^1 under the immersion is to be regarded as being embedded in 3-space so that the positive x, y, and z axes are as indicated. Then, referring to Fig. 14 the six resolutions of $\mathbf{\Phi}$ are as illustrated. They are related by equations $(3.13)_{xy}$ and $(3.13)_{yz}$, namely:

$$v_4(\mathbf{\Phi}_1) - v_4(\mathbf{\Phi}_2) = v_4(\mathbf{\Phi}_3) - v_4(\mathbf{\Phi}_4),$$

$$v_4(\mathbf{\Phi}_1) - v_4(\mathbf{\Phi}_2) = v_4(\mathbf{\Phi}_5) - v_4(\mathbf{\Phi}_6).$$

Now, $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_5$ are good immersions which respect inadmissible configurations, so by Theorem 2.4, assertion (2.7), both $v_4(\mathbf{\Phi}_1)$ and $v_4(\mathbf{\Phi}_5)$ must be zero. Thus our two equations reduce to:

$$-v_4(\mathbf{\Phi}_2) = v_4(\mathbf{\Phi}_3) - v_4(\mathbf{\Phi}_4) = -v_4(\mathbf{\Phi}_6).$$

Next we notice, from the pictures in Fig. 17, that $\mathbf{\Phi}_2$ is in fact isotopic to $\mathbf{\Phi}_6$. Identifying $\mathbf{\Phi}_2$ with $\mathbf{\Phi}_6$ our two equations reduce to a single equation, namely:

$$v_4(\mathbf{\Phi}_3) - v_4(\mathbf{\Phi}_4) + v_4(\mathbf{\Phi}_6) = 0.$$

Next, observe that $\mathbf{\Phi}_3$ and $\mathbf{\Phi}_6$ respect the same [3]-configuration. However, from Fig. 18 we see that they only become equivalent after a crossing change. By Theorem 2.7, assertion (2.4) it follows that:

$$v_4(\mathbf{\Phi}_3) - v_4(\mathbf{\Phi}_6) = v_4(\mathbf{3533}) = (0, 1, 0).$$

Thus if we choose $v_4(\mathbf{\Phi}_3) = v_4(\mathbf{\Phi}_4) = (0, 1, 0)$ and $v_4(\mathbf{\Phi}_6) = (0, 0, 0)$ in the three cases all of our consistency conditions will be satisfied. Since the immersions in the middle row of Fig. 11 may be identified with $\mathbf{\Phi}_6$ and $\mathbf{\Phi}_4$, we have extended the solution to the middle row of the actuality table for $i = 4$.

Finally, we see that there are no admissible ⟨2⟩-configurations, so there are no constraints on the entries in the bottom row of the table, and we may take the indices of the unique [2] -configuration to be (0, 0, 0). We have completed an actuality table for the index 4.

Open problem 3.10

Find a constructive proof that every solution to the system of equations (3.11) can be extended to a solution to the system of equations (3.19).

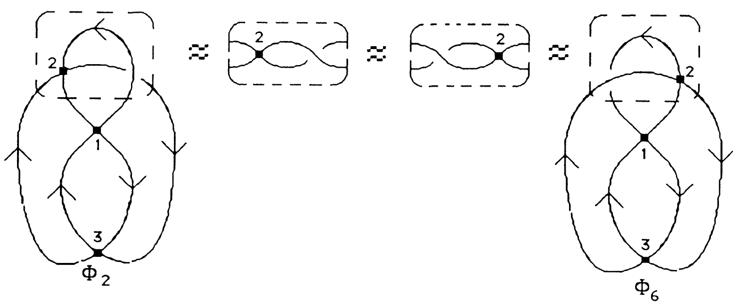


Fig. 17

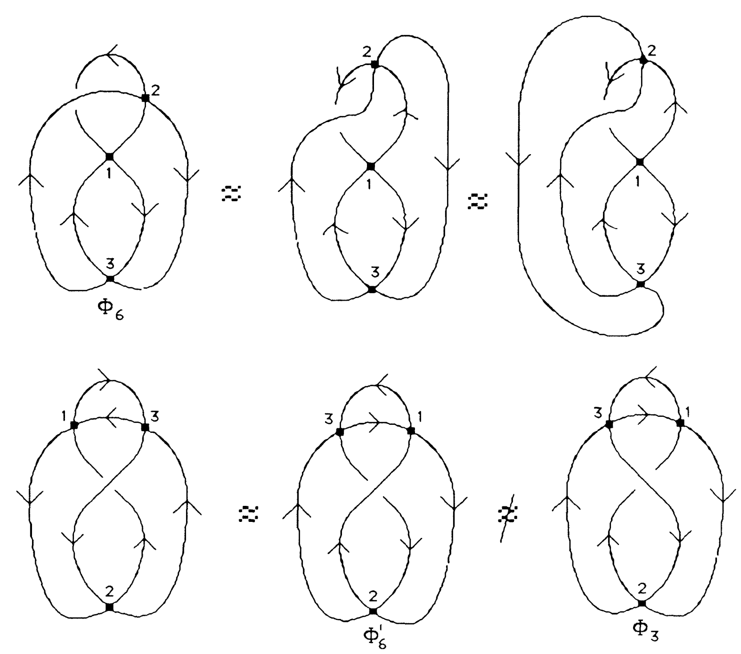


Fig. 18

4 Knot polynomials and Vassiliev invariants

In this section we will establish a connection between the HOMFLY [FHLMOY] and Kauffman [K1] polynomials and Vassiliev invariants, and work out various consequences.

We will no longer need to concern ourselves with the distinctions between immersions of S^1 in $\mathbb{R}^3$ and immersions of $\mathbb{R}^1$ in $\mathbb{R}^3$. Therefore we may simplify our notation and drop all "hats."

The HOMFLY polynomial is a Laurent polynomial in two variables which, like the Vassiliev invariants, is determined by axioms and initial data. The particular model for it which is most appropriate to our work here is an infinite sequence of one-variable specializations which are defined as follows [J3]. Let **O** denote the unknot, and let K_{p_+}, K_{p_-} and K_{p_0} be link diagrams which are identical everywhere except near one crossing, where they are related in the manner indicated in Fig. 5. Then for each knot or link type **K** there is a doubly infinite sequence of Laurent polynomials $\mathscr{H}_{n,t}(\mathbf{K})$ in the variable $t^{\pm 1}$, indexed by the integer n, with $n \neq -1$, which are determined by the crossing-change formula

(4.1) $$t^{(n+1)/2}\mathscr{H}_{n,t}(\mathbf{K}_{p_+}) = t^{-(n+1)/2}\mathscr{H}_{n,t}(\mathbf{K}_{p_-}) + (t^{1/2} - t^{-1/2})\mathscr{H}_{n,t}(\mathbf{K}_{p_0})$$

and the initial data:

(4.2) $$\mathscr{H}_{n,t}(\mathbf{O}) = 1,$$

It is known [J3] that the sequence of polynomials $\mathscr{H}_{n,t}(\mathbf{K})$ determine the 2-variable polynomial of [J2], discovered simultaneously and independently by

Freyd and Yetter, Hoste, Lickorish and Millett and Ocneanu. The 1-variable Jones polynomial of [J2] is obtained by setting $n = -3$.

Theorem 4.1 *Let* $\mathbf{K}$ *be a knot and let* $\mathscr{H}_{n,t}(\mathbf{K})$ *be its nth* HOMFLY *polynomial. Let* $W_{n,x}(\mathbf{K})$ *be obtained from* $\mathscr{H}_{n,t}(\mathbf{K})$ *by replacing the variable t by* e^x. *Using the power series expansion of* e^x to express $W_{n,x}(\mathbf{K})$ *as a power series in x*:

$$W_{n,x}(\mathbf{K}) = \sum_{i=0}^{\infty} w_{n,i}(\mathbf{K})x^i. \tag{4.3}$$

Then $w_{n,0}(\mathbf{K}) = 1$ *and each* $w_{n,i}(\mathbf{K}), i \geqq 1$ is *a Vassiliev invariant of order i.*

We begin our work by proving Theorem 4.1. We will then work out some interesting consequences. See Corollaries 4.2, 4.3, 4.4 and 4.7 and Theorem 4.8, also Example 4.5. At the end of this section we will show how to modify Theorem 4.1 for the Kauffmann polynomial.

Proof of Theorem 4.1 The basic idea of the proof is to show that the coefficients $v_{n,i}(\mathbf{K})$ in the power series $P_{n,t}(\mathbf{K})$ satisfy the axioms for a Vassiliev invariant of order i which we gave in Theorem 2.4.The first step is to use the fact that $\mathscr{H}_{n,t}(\mathbf{K})$ is defined on all knots to construct a related invariant of flat vertex graphs. Motivated by (2.4), we notice that the vertex at p can be resolved into a positive crossing p_+ or a negative crossing p_-, so, recursively, we define:

$$\mathscr{H}_{n,t}(\mathbf{K}_p^j) = \mathscr{H}_{n,t}(\mathbf{K}_{p_+}^{j-1}) - \mathscr{H}_{n,t}(\mathbf{K}_{p_-}^{j-1}) \tag{4.4}$$

Applying this formula first for $j = 1$, then for $j = 2, \ldots$ gives a well-defined invariant which we call the HOMFLY polynomial $\mathscr{H}_{n,t}(\mathbf{K}^j)$. It is a sum of HOMFLY polynomials of 2^j knots.

It also follows that (4.1) yields a crossing-change formula for $\mathscr{H}_{n,t}(\mathbf{K}^j)$, namely:

$$\mathscr{H}_{n,t}(\mathbf{K}_{p_+}^{j-1}) = t^{-(n+1)}\mathscr{H}_{n,t}(\mathbf{K}_{p_-}^{j-1}) + (t^{-n/2} - t^{-(n-2)/2})\mathscr{H}_{n,t}(\mathbf{K}_{p_0}^{j-1}). \tag{4.5}$$

Substituting (4.5) into (4.4) we obtain:

$$\mathscr{H}_{n,t}(\mathbf{K}_p^j) = (t^{-(n+1)} - 1)\mathscr{H}_{n,t}(\mathbf{K}_{p_-}^{j-1}) + (t^{-n/2} - t^{-(n+2)/2})H(\mathbf{K}_{p_0}^{j-1}). \tag{4.6}$$

Suppose, now, that the vertices in $\mathbf{K}^j$ are labeled $1, 2, \ldots, j$. We can use (4.6) to resolve the j crossings. The final result will be to express $\mathscr{H}_{n,t}(\mathbf{K}_{1,2,\ldots,j}^j)$ as a weighted sum of HOMFLY polynomials of 2^j links $\mathbf{K}_\delta$:

$$\mathscr{H}_{n,t}(\mathbf{K}_{1,2,\ldots,j}^j) = \sum_{\delta}(t^{-(n+1)} - 1)^{p_\delta}(t^{-n/2} - t^{-(n+2)/2})^{q_\delta}\mathscr{H}_{n,t}(\mathbf{K}_\delta), \tag{4.7}$$

where:

- $\delta = (\delta_1, \delta_2, \ldots, \delta_j)$, where each δ_q is a minus sign or a zero,
- $p_\delta =$ the number of minus signs in δ,
- $q_\delta =$ the number of zeros in δ,
- $\mathbf{K}_\delta = \mathbf{K}_{1_{\delta 1}, 2_{\delta 2}, \ldots, j_{\delta j}}$.

We now pass to the power series version of (4.7) :

$$W_{n,x}(\mathbf{K}_{1,2,\ldots,j}^j) = \sum_{\delta}(e^{-(n+1)x} - 1)^{p_\delta}(e^{-nx/2} - e^{-(n+2)x/2})^{q_\delta}W_{n,x}(\mathbf{K}_\delta) \tag{4.8}$$

Let $|\mathbf{K}_\delta|$ denote the number of components in the link $\mathbf{K}_\delta$. In [J2] it is shown that $H_{n,1}(\mathbf{K}_\delta) = (n+1)^{|\mathbf{K}_\delta|-1}$, so that each summand $W_{n,t}(\mathbf{K}_\delta)$ in (4.8) has a non-vanishing constant term $(n+1)^{|\mathbf{K}_\delta|-1}$. As for the remaining terms, when we replace e^x by its Taylor series, the first non-vanishing term in the power series expansion of $(e^{-(n+1)x}-1)^{p_\delta}$ will be $(-(n+1)x)^{p_\delta}$, and the first non-vanishing term in the power series expansion of $(e^{-nx/2} - t^{-(n+2)x/2})^{q_\delta}$ will be x^{q_δ}. Using the fact that

$$p_\delta + q_\delta = j,$$

we conclude that the coefficients of the x^j on the right hand of (4.8) is zero if $i < j$. That is, our power series expansion will take the form:

$$(4.9)\qquad W_{n,x}(\mathbf{K}^j_{1,2,\dots,j}) = \sum_{i=0}^{\infty} w_{n,i}(\mathbf{K}^j_{1,2,\dots,j})x^i, \quad \text{where } w_{n,i}(\mathbf{K}^j_{1,2,\dots,j}) = 0 \text{ if } i<j.$$

We are now ready to use Theorem 2.4 to prove that $w_{n,i}(\mathbf{K})$ is a Vassiliev invariant of order i. Property (2.4) is a consequence of (4.4). Property (2.5) coincides with (4.9). Property (2.6) follows from (4.2). It is always possible to satisfy (2.7). So the only thing which remains is to construct the actuality table (2.8). In our situation that is easy. We began with a knot invariant $\mathscr{H}_{n,t}(\mathbf{K})$ and used it to construct a graph invariant, so our graph invariant may be used to construct an actuality table. The table will, of course, be rather special, since it belongs to the HOMFLY polynomial. The indices in the top row of the table are, in fact, given by a very explicit formula which follows directly from the remarks just before the statement of Eq. (4.9) :

$$(4.10)\qquad w_{n,j}(\mathbf{K}^j_{12\dots j}) = \sum_\delta (-1)^{p_\delta}(n+1)^{p_\delta + |K_\delta| - 1}$$

The invariants in (4.10) satisfy Eqs. $(3.10)_{xy}$ and $(3.10)_{yx}$ and the invariants in the remaining rows satisfy $(3.17)_{xy}$ and $(3.17)_{yx}$ because these equations follow from the fact that Vassiliev invariants do not depend on the particular series of crossing changes which take **K** to the unknot **O**. That is clearly the case in our situation because $\mathscr{H}_{n,t}(\mathbf{K})$ is a knot type invariant. □

Corollary 4.2 (i) *For every $i \geqq 2$ there exist knots with non-trivial Vassiliev invariants of order i.*
(ii) *The groups G_i/G_{i-1} have non-trivial rank $l_i \geqq 1$ for every $i \geqq 2$.*

Proof. (i) To prove the first assertion, let **T** be the positive trefoil knot. Its Jones polynomial is $\mathscr{H}_{-3,t}(\mathbf{T}) = t + t^3 - t^4$. Setting $t = e^x$ and expanding e^x in a power series, we see that the coefficient of x^i is $(1 + 3^i - 4^i)/3!$ which is non-zero for every $i \geqq 2$.

(ii) To prove the second assertion, we will show that for every $i \geqq 2$ there is an $[i]$-configuration α such that $w_{-3,i}(\alpha) \neq 0$, i.e., $w_{-3,i} \in G_i \backslash G_{i-1}$. Take $\alpha = (\mathbf{i}, \mathbf{i}, \dots \mathbf{i})$. This configuration is respected by a graph $\mathbf{K}^i$ which is obtained from a standard projection of a type $(2, k)$-torus knot by changing i crossings into double points, where $k = i$ (or $i + 1$) if i is odd (or even). We claim that

$$w_{-3,i}(\mathbf{i}, \mathbf{i}, \dots \mathbf{i}) = \lim_{t\to 1} \frac{\mathscr{H}_{-3,t}(\mathbf{K}^i)}{(t-1)^i} \neq 0$$

if $i \geqq 2$. To see this, notice that a resolution of $\mathbf{K}^i$ which has j negative resolutions and $i - j$ positive resolutions is a type $(2, k - 2j)$ torus knot. Using the formulas in [J2] for the Jones polynomials of torus knots, we compute:

$$\begin{aligned}\mathscr{H}_{-3,t}(\mathbf{K}^i) &= \frac{t^{(k-1)/2}}{1+t}\sum_0^i(-1)^j\binom{i}{j}t^{-j}(1+t+t^2-t^{k+1-2j})\\ &= \frac{(1+t+t^2)t^{(k-1)/2}}{1+t}\sum_0^i(-1)^j\binom{i}{j}t^{-j} - \frac{t^{(3k+1)/2}}{1+t}\sum_0^i(-1)^j\binom{i}{j}t^{-3j}\\ &= \frac{(1+t+t^2)t^{(k-1)/2}}{1+t}(1-t^{-1})^i - \frac{t^{(3k+1)/2}}{1+t}(1-t^{-3})^i\\ &= \frac{(1+t+t^2)t^{(-2i+k-1)/2}}{1+t}(t-1)^i - \frac{t^{(-6i+3k+1)/2}}{1+t}(1+t+t^2)^i(t-1)^i.\end{aligned}$$

Passing to the limit, as above, we obtain:

$$w_{-3,i}(\mathbf{i}, \mathbf{i}, \ldots \mathbf{i}) = \frac{3}{2} - \frac{3^i}{2}.$$

This limit is non-zero for all $i \geqq 2$. □

The astute reader will have noticed that in the proof of Theorem 4.1 we did not need to restrict ourselves to the substitution $t = e^x$. In fact:

Corollary 4.3 *If $t = f(x)$ is any function with the property that $f(x)$ and $1/f(x)$ have convergent power series expansions in some neighborhood of $x = 0$, and if in addition*

$$\lim_{x \to 0} f(x) = 1,$$

then each coefficient $u_{n,i}(\mathbf{K})$ in the power series of expansion of $\mathscr{H}_{n,t}(\mathbf{K})$ determined by $t = f(x)$ is a Vassiliev invariant of order i.

For example, if we had chosen $t = 1 - x$, $t^{-1} = 1 + x + x^2 + \ldots$ the proof goes through unchanged. Let us denote by $\bar{w}_{n,i}(\cdot)$, $i \geqq 1$ the Vassiliev invariant of order i determined by the HOMFLY polynomial using the substitution $t = 1 - x$. Thus:

Corollary 4.4 *Suppose $\mathscr{H}_{n,t}(\mathbf{K})$ is a polynomial (rather than a Laurent polynomial) in t of degree k for a certain knot $\mathbf{K}$. Then, $\bar{w}_{n,i}(\mathbf{K}) = 0$ for $i > k$. Moreover, the coefficients of $\mathscr{H}_{n,t}(\mathbf{K})$ are equal to the values of certain Vassiliev invariants of order $\leqq k$ on $\mathbf{K}$.*

Proof. The first assertion is easy to see. The second assertion is because the coefficients of $\mathscr{H}_{n,t}(\mathbf{K})$ are linear combinations of the coefficients in the new polynomial which is obtained by replacing t by $1 - x$. Since linear combinations of Vassiliev invariants of order i are Vassiliev invariants of order i, the second assertion follows. □

Example 4.5 Earlier, we showed that certain actuality tables for a Vassiliev invariant of order $i \geqq 4$ could be chosen so that all invariants in the rows associated to $j = 2$ and 3 are zero. This implies that, for such a choice of the tables, the trefoil knot **T** has only two non-zero Vassiliev invariants, namely $v_2(\mathbf{T})$ and $v_3(\mathbf{T})$, because (see the sample calculation at the end of §2) the computation of $v_i(\mathbf{T})$ for $i > 3$ does not make use of data in the tables above row 3. This result is a little bit stronger than the result obtained by substituting $1 - x$ for t in the one-variable Jones polynomial of the trefoil, which is $t + t^3 - t^4$, because on sending t to $1 - x$ one obtains a polynomial in x of degree 4, not 3.

The proof of Corollary 4.2 raises a question. The HOMFLY polynomials are indexed by the integers n. Define the *dimension* of the HOMFLY *subspace* of G_i/G_{i-1} to be the number of linearly independent actuality tables of order i which are obtained from the HOMFLY polynomial, as n varies over the integers $n \neq -1$. What is this dimension? Can we identify the subspace in the cases which we know, i.e. $i = 2, 3, 4$?

Corollary 4.6 *The* HOMFLY *subspaces of* G_2 and G_3/G_2 *have dimension* 1. *The* HOMFLY *subspace of* G_4/G_3 *has dimension* 2.

Proof. We use (4.10) to compute the value of $w_{n,i}(\mathbf{K}^i)$ on the basis elements calculated in the examples at the end of §3. For example, for $i = 4$ we need to compute $w_{n,4}(\mathbf{K}^4_{1234})$ on immersions which respect the three [4]- configurations in Figure 3.4 with names **3443**, **3533** and **4444**. There is a very simple rule for determining $|\mathbf{K}_\delta|$, which we now describe. If we use the rule which was given in Fig. 7 for choosing an immersion which respects a given $[j]$-configuration, then if a particular double point is resolved into a negative crossing (respectively zero crossing), as in Fig. 19, we simply delete the corresponding arc (respectively replace it by two parallel arcs). With this rule it is not difficult to compute the sum on the right hand side of (4.10) over the 2^4 possible vectors δ. The results of the computation are:

$$w_{n,2}(\mathbf{22}) = 1 - (n+1)^2$$

$$w_{n,3}(\mathbf{232}) = (n+1) - (n+1)^3$$

$$w_{n,4}(\mathbf{3443}) = 2((n+1)^2 - (n+1)^4)$$

$$w_{n,4}(\mathbf{3533}) = (n+1)^2 - (n+1)^4$$

$$w_{n,4}(\mathbf{4444}) = 1 + (n+1)^2 - 3(n+1)^4$$

Thus, the HOMFLY subspaces of G_2 and G_3/G_2 have dimension 1. Since $w_{n,4}(\mathbf{3443}) = 2w_{n,4}(\mathbf{3533})$, the vectors

$$(w_{n,4}(\mathbf{3443}), w_{n,4}(\mathbf{3533}), w_{n,4}(\mathbf{4444}))$$

span a 2-dimensional subspace in the 3-dimensional space G_4/G_3 as n varies over the integers $n \neq -1$. Thus, the HOMFLY subspace of G_4/G_3 is 2-dimensional, as claimed. □

In [V] Vassiliev proves that if $\mathbf{K}_+$ and $\mathbf{K}_-$ are knots which are related by a crossing switch, then their second order Vassiliev invariants satisfy the crossing-change formula:

$$v_2(\mathbf{K}_+) - v_2(\mathbf{K}_-) = \lambda$$

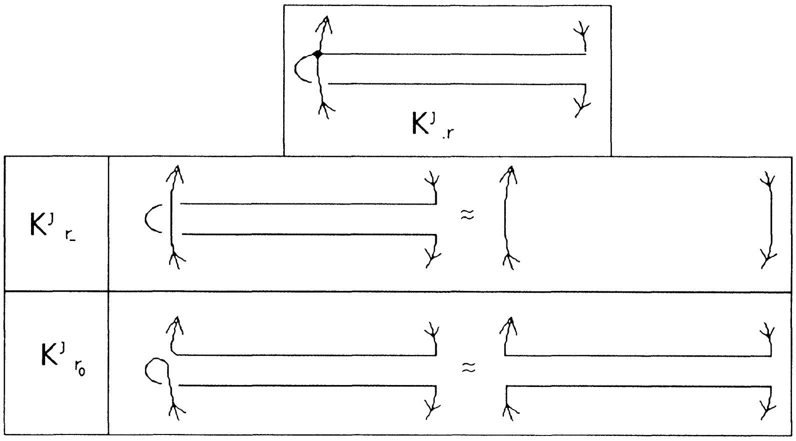


Fig. 19

where λ is the linking number of the 2-component link $\mathbf{K}_0$. As an example, we remarked at the end of §2 that the second order Vassiliev invariant of the trefoil knot is 1, using the actuality table in Fig. 9. We could also have obtained this from Vassiliev's crossing-change formula. Now, we have just shown that, using the axioms (4.1) and (4.2) for the HOMFLY polynomial, the invariant $w_{n,2}(\mathbf{K})$ lies in a subgroup of the cyclic group G_2 which is generated by $1-(n+1)^2$. This means that the second order Vassiliev invariants in the HOMFLY case will satisfy the crossing-change formula:

$$w_{n,2}(\mathbf{K}_+) - w_{n,2}(\mathbf{K}_-) = [1-(n+1)^2]\lambda. \tag{4.11}$$

Using what we learned from the proof of Theorem 4.1 we now generalize this result.

Recall that $\mathscr{H}_{-3,t}(\mathbf{K}) = J_t(\mathbf{K})$ is the one-variable Jones polynomial of a knot. It was proved in [J2] and also in [B-K] that $J_t(K)$ satisfies a crossing-change formula which is analogous to (4.1) but which relates the Jones polynomial of three knots, rather than two knots and a link. The modified crossing-change formula is:

$$J_t(\mathbf{K}_+) = tJ_t(\mathbf{K}_-) + (t^{3\lambda} - t^{3\lambda+1})J_t(\mathbf{K}_\infty) \tag{4.12}$$

where $J_t(\mathbf{K}_\infty)$ is defined in Fig. 5. Let us use the substitution $t = e^x$, to replace the Laurent polynomial $J_t(\mathbf{K})$ by its power series $J_{e^x}(\mathbf{K}) = U_x(\mathbf{K}) = \sum u_i(\mathbf{K})x^i$. Thus $u_i(\mathbf{K}) = w_{-3,i}(\mathbf{K})$. Vassiliev proved in [V] that his first order invariant $v_1(\mathbf{K}) = 0$ for every knot $\mathbf{K}$. (This is the analogue of Jones' observation that the first order derivative of the Jones polynomial of a knot, evaluated at 1, is 0.) Equating like powers of x on both sides of the power series version of (4.12) we obtain:

Corollary 4.7 *Let $\mathbf{K}_+, \mathbf{K}_-, \mathbf{K}_\infty$ be knots which have are defined by immersions which agree everywhere except near a single crossing. Let λ be the linking number of the*

2-component link K_0. Then for every $i \geqq 2$ the following formula holds:

$$u_i(\mathbf{K}_+) - u_i(\mathbf{K}_-) = u_{i-1}(\mathbf{K}_-) - u_{i-1}(\mathbf{K}_\infty) + \frac{1 + (3\lambda)^i - (3\lambda + 1)^i}{i!}$$

$$+ \sum_{j=2}^{i-2} \frac{u_j(\mathbf{K}_-) + u_j(\mathbf{K}_\infty)[(3\lambda)^{i-j} - (3\lambda + 1)^{i-j}]}{(i-j)!} \tag{4.13}$$

Notice that since $u_1(\mathbf{K}) = 0$ for every knot $\mathbf{K}$, and since (from (4.10)) the "Jones subspace" of the space G_2 of second order Vassiliev invariants coincides with G_2, Eq. (4.13) reduces to (4.11) with $n = -3$ in the special case $i = 2$.

Our final result in this section relates to the Kauffman polynomial. The Kauffman polynomial of an oriented link $\mathbf{K}$, like the HOMFLY polynomial, is a Laurent polynomial in two variables which is determined by a doubly-infinite sequence of specializations $Q_{n,t}(\mathbf{K})$, $n \in \{Z\}, n \neq -1$. There is a crossing-change formula which is the analogue of (4.1) and initial data which is the analogue of (4.2), and we begin our discussion by describing them. We assume that $\mathbf{K}$ is defined by a fixed diagram K. The symbol $\bar{K}$ denotes the diagram obtained from K by forgetting the orientation. Choose a crossing $p \in \bar{K}$, and let $\bar{K}_{p_+}$ denote the diagram with this crossing distinguished. Let $\bar{K}_{p_-}$ denote the diagram obtained from $\bar{K}_{p_+}$ by switching the crossing at p and let $\bar{K}_{p_\infty}$ and $\bar{K}_{p_0}$ denote the two possible diagrams obtained from $\bar{K}_{p_\pm}$ by surgery at p. Let $\bar{K}_{\circlearrowright}$ and $\bar{K}_{\circlearrowleft}$ be the diagrams obtained from $\bar{K}$ by adding "curls" as in Fig. 20. Then it is proved in [K2] that a doubly infinite sequence of regular isotopy invariants (see [K1]) of $\bar{K}$ which we denote by the symbols $\bar{Q}_{n,s}(\bar{K})$ is determined by the crossing-change formula:

$$\bar{Q}_{n,t}(\bar{K}_{p_+}) = \bar{Q}_{n,t}(\bar{K}_{p_-}) + (t - t^{-1})[\bar{Q}_{n,t}(\bar{K}_{p_0}) - \bar{Q}_{n,t}(\bar{K}_{p_\infty})], \tag{4.14}$$

the two curl formulas:

$$\bar{Q}_{n,t}(\bar{K}_{\circlearrowright}) = t^{(n+1)}(\bar{Q}_{n,t}(\bar{K})), \tag{4.15}$$

$$\bar{Q}_{n,t}(\bar{K}_{\circlearrowleft}) = t^{-(n+1)}(\bar{Q}_{n,t}(\bar{K})) \tag{4.16}$$

and the initial data:

$$\bar{Q}_{n,t}(\mathbf{O}) = 1 \tag{4.17}$$

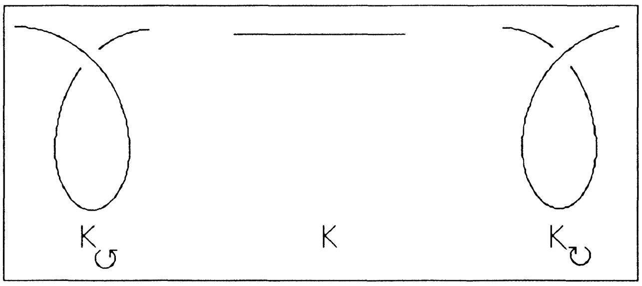


Fig. 20

Reintroducing orientations, and letting $w(K)$ denote the *writhe* or algebraic crossing number of the diagram K, the invariant $Q_{n,t}(\mathbf{K})$ is defined by the formula:

$$Q_{n,t}(\mathbf{K}) = t^{-(n+1)w(K)}\bar{Q}_{n,t}(\bar{K}). \tag{4.18}$$

Let a, b, c, d denote the writhes of K_{p_+}, K_{p_-}, K_{p_0} and K_{p_∞}. Here the orientation of K_{p_∞} is arbitrary. We need to understand how a, b, c *and* d are related. Clearly $a = b + 2 = c + 1$, and this formula holds independently of the connectivity of the diagram. To see how d is related to a, notice that if we delete the point p from the diagram K_{p_+} that diagram will fall apart into two oriented arcs which we denote α and β, together with some number of other components which are unaffected by the change. Whenever α and β belong to the same component of K_{p_+} or not, these two arcs lie on the same component of K_{p_∞} and we may denote that component of K_{p_∞} by either $\alpha \cup \beta^{-1}$ or $\alpha^{-1} \cup \beta$, say the former. Let λ be the algebraic crossing number between β and its complimentary diagram. Then $a = d + 2\lambda + 1$. Thus:

$$\begin{aligned}
Q_{n,t}(\mathbf{K}_{p_+}) &= t^{-(n+1)a}\bar{Q}_{n,t}(\bar{K}_{p_+})\\
&= t^{-(n+1)a}\{\bar{Q}_{n,t}(\bar{K}_{p_-}) + (t - t^{-1})[\bar{Q}_{n,t}(\bar{K}_{p_0}) - \bar{Q}_{n,t}(\bar{K}_{p_\infty})]\}\\
&= t^{-(n+1)(b+2)}\bar{Q}_{n,t}(\bar{K}_{p_-})\\
&\quad + (t - t^{-1})[t^{-(n+1)(c+1)}\bar{Q}_{n,t}(\bar{K}_{p_0}) - t^{-(n+1)(d+2\lambda+1)}\bar{Q}_{n,t}(\bar{K}_{p_\infty})]\\
&= t^{-2(n+1)}Q_{n,t}(\mathbf{K}_{p_-})\\
&\quad + (t - t^{-1})[(t^{-(n+1)}Q_{n,t}(\mathbf{K}_{p_0}) - t^{-(n+1)(2\lambda+1)}Q_{n,t}(\mathbf{K}_{p_\infty})]
\end{aligned}$$

We conclude that for each link diagram K and each crossing p the following formula holds:

$$\begin{aligned}
t^{n+1}Q_{n,t}(\mathbf{K}_{p_+}) &- t^{-(n+1)}Q_{n,t}(\mathbf{K}_{p_-})\\
&= (t - t^{-1})[(Q_{n,t}(\mathbf{K}_{p_0}) - t^{-2\lambda(n+1)}Q_{n,t}(\mathbf{K}_{p_\infty})].
\end{aligned} \tag{4.19}$$

Here λ depends on the diagram, the choice of p, and the choice of orientation on $\mathbf{K}_{p_\infty}$. This is our analogue of the crossing-change formula (4.1) for the Kauffman polynomial.

Theorem 4.8 *Let* $\mathbf{K}$ be a knot and let $Q_{n,t}(\mathbf{K})$ *be its* nth *Kauffman polynomial. Let* $R_{n,x}(\mathbf{K})$ *be obtained from* $\mathcal{Q}_{n,t}(\mathbf{K})$ *by replacing the variable t by* e^x. *Express* $R_{n,x}(\mathbf{K})$ *as a power series in* x:

$$R_{n,x}(\mathbf{K}) = \sum_{i=0}^{\infty} r_{n,i}(\mathbf{K})x^i. \tag{4.20}$$

Then $r_{n,0}(\mathbf{K}) = 1$ *and each* $r_{n,i}(\mathbf{K}), i \geqq 1$ *is a Vassiliev invariant of order i.*

Proof. The proof is almost identical to the proof of Theorem 4.1. Equation (4.19) plays the role of (4.1). Following the ideas used in the proof of Theorem 4.1, we define the Kauffman polynomial of a flat vertex graph $K^j_{12\ldots j}$, inductively. This enables us to resolve the Kauffman polynomial of $K^j_{12\ldots j}$ into a sum of Kauffman polynomials of related links. We can then replace every positive resolution by a sum of terms which come from negative, zero and infinity resolutions, by using (4.19). The key point is that the terms on the left hand side of (4.19) have coefficents which are $\pm t^{\pm(n+1)}$ and that those on the right hand side are divisible by $t - t^{-1}$. This implies that in the power series expansions of the Kauffman polynomial of $K^j_{12\ldots j}$ the coefficient of x^i in the analogue of (4.9) will vanish whenever i is less than j. □

5 A numerical knot invariant which is not a Vassiliev invariant

We began these investigations, in §1, by reviewing the essential features in Vassiliev's study in [V] of the space $\mathscr{K} = \mathscr{M} \backslash \Sigma$ of all knots, and its finite-dimensional approximations $\Gamma^d \backslash \Gamma^d \cap \Sigma$. We showed that the chief object under investigation, the group $\bar{H}_{3d-1}(\Gamma^d \cap \Sigma)$, is naturally isomorphic to $H^0(\Gamma^d \backslash \Gamma^d \cap \Sigma)$, and that:

$$\tilde{H}^0(\mathscr{K}) \cong \varprojlim \tilde{H}^0(\Gamma^{d_n} \backslash \Gamma^{d_n} \cap \Sigma).$$

The invariants which came out of Vassiliev's study are a certain set of invariants of *finite type* in the groups $\bar{H}_{3d-1}(\Gamma^d \cap \Sigma)$. They have been the principal object of investigation in this paper. In view of the fact that $\tilde{H}^0(\mathscr{K})$ clearly classifies knots, a natural question to ask about these invariants is:

Problem. Given any numerical invariant $v: \mathscr{K} \to \mathbb{Q}$, does there exist a sequence of Vassiliev invariants $\{v_i: \mathscr{K} \to \mathbb{Q}, \mathrm{i} = 2, 3, 4 \ldots\}$ such that

$$\lim_{i \to \infty} v_i(\mathbf{K}) = v(\mathbf{K})$$

for every $\mathbf{K} \in \mathscr{K}$?

This question was asked by Vassiliev in his discussion in §6 of [V] of his *stabilization conjecture*. We are unable to answer this difficult question, but we have a small contribution to make which will, perhaps, sharpen the question for the reader by pinpointing, via as example, the essential features of Vassiliev's approximations.

Let $U: \mathbf{K} \to \mathbb{Z}$ be the numerical invariant which takes as its value the unknotting number $U(\mathbf{K})$ of $\mathbf{K} \in \mathscr{K}$. This number may be described, in Vassiliev's setting, as the minimum number of passages across Σ in a path joining $\mathbf{K}$ to the unknot $\mathbf{O}$. We will prove:

Theorem 5.1 *The unknotting number U cannot be a Vassiliev invariant of order i for any integer* $i \geqq 2$.

Proof. Given any knot invariant, e.g. $U(\mathbf{K})$, we may always use axiom (2.4) of Theorem 2.7 to extend it to a numerical invariant $U(\mathbf{K}^j)$ of our special knotted graphs, as in §2 above. The graph invariant so-obtained will of course be a linear combination of unknotting numbers of 2^j associated knots, i.e.:

(5.1) $$U(\mathbf{K}^j) = \sum_{\hat{\varepsilon}} (-1)^{\mathrm{sign}(\hat{\varepsilon})} U(K_{\hat{\varepsilon}}) \quad ,$$

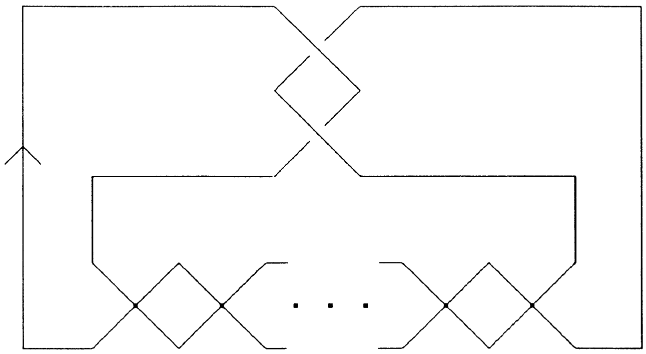


Fig. 21

where

a. the sum is over all $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j)$ whose entries are a sequence of j plus and minus signs,
b. $\mathbf{K}_{\vec{\varepsilon}}$ is the knot obtained from $\mathbf{K}^j$ by resolving the j crossings at $p_1, p_2, \ldots, p_j$, replacing each p_i by a crossing of sign ε_i, and
c. sign($\vec{\varepsilon}$) is the number of negative resolutions in the vector $\vec{\varepsilon}$.

We may also use (2.4), (2.6) and (2.7) to construct a table of initial data for a Vassiliev invariant of order i, however we don't know whether axiom (2.5) is satisfied. Therefore, to prove the theorem, it suffices to show that for each integer i there is a graph $\mathbf{K}^m$ with $m = m(i) > i$ double points, such that $U(\mathbf{K}^m) \neq 0$.

Choose any $m = 4r \neq 0$ and let $\mathbf{K}^m$ be the knotted graph which is depicted in Fig. 21. Each resolution of $\mathbf{K}^m$ is a Whitehead double (with some number of twists) of the unknot, and thus is non-trivial whenever the number of twists is non-zero. The number of twists will be zero exactly when there are an equal number of positive and negative signs in the vector $\vec{\varepsilon}$. Since the non-trivial knots in $\mathbf{K}_{\vec{\varepsilon}}$ each have unknotting number 1, the value of $U(\mathbf{K}^{4r}))$ is thus determined by counting the number of vectors of length j which have s positive signs and $4r - s$ negative signs and adding up the numbers so-obtained, with appropriate signs. Using the fact that the sign of $\vec{\varepsilon}$ and $-\vec{\varepsilon}$ have the same parity, we may pair the terms $\vec{\varepsilon}$ and $-\vec{\varepsilon}$ in our sum, to obtain:

$$U(\mathbf{K}^{4r}) = 2\left(1 - \binom{4r}{1} + \binom{4r}{2} - \binom{4r}{3} + \ldots - \binom{4r}{2r-1}\right). \tag{5.2}$$

Since the terms in this sequence are strictly increasing in absolute value, and since the number of terms is even, we conclude that $U(\mathbf{K}^{4r}) < 0$ for every $r \geqq 1$.

Now choose any $i > 0$. Choose r so that $4r > i$. Then $U(\mathbf{K}^{4r}) \neq 0$, and the theorem is proved. □

Remark. Our proof of the negative result in Theorem 5.1 is in the same spirit as our proof of the positive results in Theorems 4.1 and 4.7. Both proofs serve to highlight the central features of our axiomatic interpretation of the results in [V]:

a. the naturality of (2.4), and

b. the fact that if (2.5) is satisfied then a finite set of data suffices to determine $U(\mathbf{K})$, and
c. the very sharp restrictions which are placed by (2.5).

For these reasons, Stanford's generalization of Vassiliev invariants to links and graphs, which begins with an axiomatic characterization of a family of invariants based on the axioms in this paper, are called *invariants of finite type* in [S].

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